Transformation Groups
https://doi.org/10.1007/s00031-021-09686-5

# On the Signature of the Ricci Curvature on Nilmanifolds

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Received: 19 April 2021 / Accepted: 18 November 2021 / Published online: 27 January 2022
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**Abstract**
We completely describe the signatures of the Ricci curvature of left-invariant Riemannian metrics on arbitrary real nilpotent Lie groups. The main idea in the proof is to exploit a link between the kernel of the Ricci endomorphism and closed orbits in a certain representation of the general linear group, which we prove using the 'real GIT' framework for the Ricci curvature of nilmanifolds.

## 1 Introduction

A classical problem in Riemannian geometry is to determine the possible signatures of the Ricci curvature on a given space. For instance, the Bonnet-Myers theorem [24] states that a complete Riemannian manifold with $\mathrm{Ric} \geq c > 0$ is compact and has finite fundamental group. On the other hand, any smooth manifold of dimension $n \geq 3$ admits a complete metric with $\mathrm{Ric} < 0$ [21].

In this article, we consider the problem under symmetry assumptions. More precisely, given a homogeneous space $M^n = \mathsf{G}/\mathsf{H}$, we are interested in the set

$$\sigma_{\mathrm{Ric}}(\mathsf{G}/\mathsf{H}) := \{\sigma(\mathrm{Ric}\,(g)) : g \ \mathsf{G}\text{-invariant Riemannian metric on } \mathsf{G}/\mathsf{H}\},$$

where $\sigma(\mathrm{Ric}\,(g)) = (s^-, s^0, s^+) \in \mathbb{Z}^3_{\geq 0}$ denotes the signature of the symmetric $(0,2)$-tensor $\mathrm{Ric}\,(g)$. Even under the homogeneity assumption, a complete description of this set turns out to be elusive in most cases. Partial results include Bochner's theorem ($(n,0,0) \notin \sigma_{\mathrm{Ric}}(\mathsf{G}/\mathsf{H})$ for compact $\mathsf{G}$), and the classification of Lie groups

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$\mathsf{G}$ admitting a left-invariant metric with Ric $\geq 0$ [1]. The only semisimple Lie groups (up to covering) for which $\sigma_{\mathrm{Ric}}(\mathsf{G})$ is known are $\mathsf{SL}_2(\mathbb{R})$ and $\mathsf{SU}(2)$ [23]. In fact, it is unknown whether $(6,0,0) \in \sigma_{\mathrm{Ric}}(\mathsf{SL}_2(\mathbb{C}))$. On the other hand, the recent literature suggests that determining whether $(n,0,0) \in \sigma_{\mathrm{Ric}}(\mathsf{G})$ for solvable $\mathsf{G}$ could be out of reach; see [22] and the references therein.

We focus on the case where $\mathsf{G} = \mathsf{N}$ is a connected nilpotent Lie group, with $\mathsf{H}$ trivial by effectiveness. Our main result is a complete description of $\sigma_{\mathrm{Ric}}(\mathsf{N})$ for all such $\mathsf{N}$, in terms of purely Lie-theoretic data:

**Theorem A** *The set of signatures of the Ricci curvature of left-invariant metrics on a connected nilpotent Lie group* $\mathsf{N}$ *with Lie algebra* $(\mathfrak{n}, [\cdot,\cdot])$ *is given by*

$$\sigma_{Ric}(\mathsf{N}) = \bigcup_{r=0}^{\min(a,m)} \left\{ (s^-, s^0, s^+) : s^- \geq u{+}r,\ s^0 \geq a{-}r,\ s^+ \geq z{+}r,\ s^-{+}s^0{+}s^+ = \dim \mathfrak{n} \right\}.$$

*Here,* $\mathfrak{z}(\mathfrak{n})$ *denotes the center of* $\mathfrak{n}$*, and* $u, a, z, m \in \mathbb{Z}_{\geq 0}$ *are defined by*

$$u := \dim \mathfrak{n} - \dim\left([\mathfrak{n},\mathfrak{n}] + \mathfrak{z}(\mathfrak{n})\right), \qquad a := \dim \mathfrak{z}(\mathfrak{n}) - \dim\left(\mathfrak{z}(\mathfrak{n}) \cap [\mathfrak{n},\mathfrak{n}]\right),$$
$$z := \dim\left(\mathfrak{z}(\mathfrak{n}) \cap [\mathfrak{n},\mathfrak{n}]\right), \qquad m := \dim[\mathfrak{n},\mathfrak{n}] - \dim\left(\mathfrak{z}(\mathfrak{n}) \cap [\mathfrak{n},\mathfrak{n}]\right).$$

Recall that any nilpotent Lie algebra may be written as direct sum of ideals

$$\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{n}_1,$$

where here $\mathfrak{a} \simeq \mathbb{R}^a$ is central and $[\mathfrak{n},\mathfrak{n}] \subset \mathfrak{n}_1$. The nilpotent ideal $\mathfrak{n}_1$ has the property that $\mathfrak{z}(\mathfrak{n}_1) \subset [\mathfrak{n}_1,\mathfrak{n}_1]$. Choosing arbitrary direct complements $\mathfrak{u}$ of $[\mathfrak{n},\mathfrak{n}]$ in $\mathfrak{n}_1$, and $\mathfrak{m}$ of $\mathfrak{z}_1 := \mathfrak{z}(\mathfrak{n}) \cap [\mathfrak{n},\mathfrak{n}]$ in $[\mathfrak{n},\mathfrak{n}]$, we have the metric-independent decompositions

$$\mathfrak{n} = \mathfrak{a} \oplus \overbrace{\mathfrak{u} \oplus \underbrace{\mathfrak{m} \oplus \mathfrak{z}_1}_{[\mathfrak{n},\mathfrak{n}]}}^{\mathfrak{n}_1}, \qquad \mathfrak{z}(\mathfrak{n}) = \mathfrak{a} \oplus \mathfrak{z}_1.$$

Given an inner product on $\mathfrak{n}$, the complements $\mathfrak{m}$ and $\mathfrak{u}$ may be chosen so that $\mathfrak{m} \perp \mathfrak{z}_1$ and $\mathfrak{u} \perp [\mathfrak{n},\mathfrak{n}]$. However, we do not necessarily have that $\mathfrak{a} \perp \mathfrak{n}_1$, and the variable $r$ in the statement in some way measures the 'angle' between these subspaces.

Assuming $\mathfrak{a} = 0$ yields the following important particular case:

**Corollary B** *If* $\mathsf{N}$ *is nilpotent and its Lie algebra* $\mathfrak{n}$ *satisfies* $\mathfrak{z}(\mathfrak{n}) \subset [\mathfrak{n},\mathfrak{n}]$*, then*

$$\sigma_{Ric}(\mathsf{N}) = \left\{ (s^-, s^0, s^+) \,:\, s^- \geq u,\ s^0 \geq 0,\ s^+ \geq z,\ s^-{+}s^0{+}s^+ = \dim \mathfrak{n} \right\}.$$

This applies for example when $\mathfrak{n}$ is irreducible, i.e. not a direct sum of proper ideals.

Recall that the Ricci curvature of a left-invariant metric $g$ on a nilpotent Lie group satisfies

$$\mathrm{Ric}\,(X,X) = -\frac{1}{2}\sum_{i,j} g\left([X,X_i],X_j\right)^2 + \frac{1}{4}\sum_{i,j} g\left([X_i,X_j],X\right)^2,$$

see, e.g. [2, (7.33)]. Here $X \in \mathfrak{n}$ is a left-invariant vector field and $\{X_i\}$ is a left-invariant $g$-orthonormal frame. It immediately follows that $\mathrm{Ric}\,|_{\mathfrak{z}(\mathfrak{n})} > 0$, $\mathrm{Ric}\,|_{[\mathfrak{n},\mathfrak{n}]^\perp} < 0$, again assuming $\mathfrak{z}(\mathfrak{n}) \subset [\mathfrak{n},\mathfrak{n}]$ (cf. [4]). Thus, $s^+ \geq z$ and $s^- \geq u$, for all $(s^-, s^0, s^+) \in \sigma_{\mathrm{Ric}}(\mathsf{N})$. Corollary B states that, besides these obvious restrictions, the Ricci curvature can take arbitrary signatures.

The signatures of the Ricci curvature on non-compact connected Lie groups $\mathsf{G}$ have been extensively investigated in the literature. In this case, we sometimes denote it by $\sigma_{\mathrm{Ric}}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of $\mathsf{G}$. Wolf proved in [29] that solvable $\mathfrak{g}$ does not admit non-flat left-invariant metrics with $\mathrm{Ric} \geq 0$. If $\mathfrak{n}$ is nilpotent not abelian, then every $(s^-, s^0, s^+) \in \sigma_{\mathrm{Ric}}(\mathfrak{n})$ satisfies $s^-, s^+ \geq 1$ [23] and $s^- \geq 2$ [5]. A complete characterisation of $\sigma_{\mathrm{Ric}}(\mathfrak{s})$ for $\mathfrak{s}$ unimodular and two-step solvable was obtained by Dotti [8]. In [9], the authors determined $\sigma_{\mathrm{Ric}}(\mathfrak{n})$ for nilpotent Lie algebras admitting a *nice basis* and satisfying $m = 1$ (in the notation of Theorem A) and $\mathfrak{z}(\mathfrak{n}) \subset [\mathfrak{n},\mathfrak{n}]$. They also computed $\sigma_{\mathrm{Ric}}(\mathfrak{n})$ for $\mathfrak{n}$ nilpotent of dimension up to 6, although we believe their classification contains some mistakes; see Section 3. Other previous results in low dimensions include Milnor's article solving the 3-dimensional case [23], work by Kremlev and Nikonorov dealing with the problem in dimension 4 [16, 17], and Kremlev's resolution of the problem for nilpotent Lie algebras of dimension 5 [18]. The particular case of Ricci negative left-invariant metrics has attracted substantial attention in recent years; see for instance [6, 7, 14, 22, 25, 27, 28].

The proof of Theorem A in the case $\mathfrak{z}(\mathfrak{n}) \subset [\mathfrak{n},\mathfrak{n}]$ has two main ingredients. Firstly, by an 'Implicit Function Theorem'-kind of argument it suffices to show that $(u, m, z) \in \sigma_{\mathrm{Ric}}(\mathfrak{n})$. Secondly, in order to show that there exists a metric whose Ricci curvature has an $m$-dimensional radical, we apply a result relating closed orbits in the $\mathsf{Gl}(\mathfrak{n})$-representation space $\Lambda^2(\mathfrak{n}^*) \otimes \mathfrak{n}$, to zeroes of Ricci (Proposition 5.1). The latter relies on the 'moment map' interpretation of the Ricci curvature on nilpotent Lie groups [20], and is inspired by the Kempf-Ness theorem [15] (see also [26] for the real version). Remarkably, it can also be applied to non-reductive subgroups of $\mathsf{Gl}(\mathfrak{n})$, and this is crucial in the proof.

In the general case $\mathfrak{n} = \mathbb{R}^a \oplus \mathfrak{n}_1$, $\mathfrak{z}(\mathfrak{n}_1) \subset [\mathfrak{n}_1, \mathfrak{n}_1]$, the proof goes essentially along the same lines. However, it is slightly more technical as one needs to keep track of the 'angle' between the two summands $\mathbb{R}^a$ and $\mathfrak{n}_1$, which need not be orthogonal. As mentioned above, this is captured by the variable $r$ in the statement.

The article is organised as follows. In Section 2, we review some basic facts about the Ricci curvature of nilmanifolds, and prove the easier inclusion in Theorem A. Section 3 contains a 5-dimensional counterexample to a conjecture stated in [9]. In Section 4, we show that certain orbits are closed in the representation space $\Lambda^2(\mathfrak{n}^*) \otimes \mathfrak{n}$. We then apply these results in Section 5 to produce zeroes of the Ricci curvature. Finally, after establishing key properties of the linearisation of the Ricci curvature map in Section 6, we prove our main result in Section 7.

## 2 The Ricci Curvature of Nilmanifolds

In this section, we review some well-known formulae for the Ricci curvature of left-invariant Riemannian metrics on nilpotent Lie groups.

Let $\mathsf{N}$ be a connected real nilpotent Lie group with Lie algebra $\mathfrak{n}$. Given a left-invariant metric $g$ on $\mathsf{N}$, its Ricci curvature tensor $\mathrm{Ric}_g \in S^2(T^*\mathsf{N})$ is also left-invariant. Hence, both tensors are determined by their value at the identity $e \in \mathsf{N}$:

$$g(e) =: \langle \cdot, \cdot \rangle \in S^2(\mathfrak{n}^*), \qquad \mathrm{Ric}_g(e) =: \mathrm{Ric}_{\langle \cdot, \cdot \rangle} \in S^2(\mathfrak{n}^*).$$

It is well known (see [2, 8, 19]) that for nilpotent $\mathfrak{n}$ the Ricci curvature is given by

$$\begin{aligned} \mathrm{Ric}_{\langle \cdot, \cdot \rangle}(X, Y) = & -\frac{1}{2} \sum_{i,j} \langle \mu(X, X_i), X_j \rangle \, \langle \mu(Y, X_i), X_j \rangle \\ & + \frac{1}{4} \sum_{i,j} \langle \mu(X_i, X_j), X \rangle \, \langle \mu(X_i, X_j), Y \rangle, \qquad X, Y \in \mathfrak{n}, \end{aligned} \tag{1}$$

where $\{X_i\}$ denotes an arbitrary $\langle \cdot, \cdot \rangle$-orthonormal basis for $\mathfrak{n}$ and $\mu \in \Lambda^2(\mathfrak{n}^*) \otimes \mathfrak{n}$ denotes the Lie bracket. In coordinates, using the structure coefficients $\mu(X_i, X_j) = \mu_{ij}^k X_k$ (summation convention over repeated indices being used), one has

$$\mathrm{Ric}_{\langle \cdot, \cdot \rangle}(X_r, X_s) = -\frac{1}{2} \mu_{ri}^j \mu_{si}^j + \frac{1}{4} \mu_{ij}^r \mu_{ij}^s =: \langle \mathrm{Ric}_\mu X_r, X_s \rangle. \tag{2}$$

In other words, the above implicitly defines $\mathrm{Ric}_\mu$, an endomorphism of $\mathfrak{n}$ whose matrix representation in the basis $\{X_i\}$ is also the matrix representation of the bilinear form $\mathrm{Ric}_{\langle \cdot, \cdot \rangle}$ in that basis.

Notice that when changing the metric $\langle \cdot, \cdot \rangle$ for another scalar product $\langle h \cdot, h \cdot \rangle$, $h \in \mathsf{GL}(\mathfrak{n})$, we may take as orthonormal basis $\{h^{-1} X_i\}$, and the corresponding Ricci curvature will satisfy

$$\mathrm{Ric}_{\langle h \cdot, h \cdot \rangle}(h^{-1} X_r, h^{-1} X_s) = -\frac{1}{2} (h \cdot \mu)_{ri}^j (h \cdot \mu)_{si}^j + \frac{1}{4} (h \cdot \mu)_{ij}^r (h \cdot \mu)_{ij}^s =: \left\langle \mathrm{Ric}_{h \cdot \mu} X_r, X_s \right\rangle,$$

The coefficients $(h \cdot \mu)_{ij}^k$ are of course the structure coefficients of $\mu$ with respect to the basis $\{h^{-1} X_i\}$. These coincide with the structure constants of $h \cdot \mu$ with respect to the *original* basis $\{X_i\}$, $(h \cdot \mu)(X_i, X_j) = (h \cdot \mu)_{ij}^k X_k$. Here, $h \cdot \mu$ denotes the standard 'change of basis' action of $\mathsf{Gl}(\mathfrak{n})$ on $\Lambda^2(\mathfrak{n}^*) \otimes \mathfrak{n}$, given by

$$(h \cdot \mu)(\cdot, \cdot) := h \mu (h^{-1} \cdot, h^{-1} \cdot), \qquad h \in \mathsf{GL}(\mathfrak{n}), \quad \mu \in \Lambda^2(\mathfrak{n}^*) \otimes \mathfrak{n}. \tag{3}$$

Observe that we may write the Ricci curvature as a difference

$$\mathrm{Ric}_{\langle \cdot, \cdot \rangle} = -q_{\langle \cdot, \cdot \rangle} + p_{\langle \cdot, \cdot \rangle}, \tag{4}$$

where

$$\begin{aligned} q_{\langle \cdot, \cdot \rangle}(X, Y) &= \frac{1}{2} \langle \mathrm{ad}_\mu X, \mathrm{ad}_\mu Y \rangle, \\ p_{\langle \cdot, \cdot \rangle}(X, Y) &= \frac{1}{4} \sum_{i,j} \langle \mu(X_i, X_j), X \rangle \langle \mu(X_i, X_j), Y \rangle, \qquad X, Y \in \mathfrak{n}, \end{aligned} \tag{5}$$

are positive semi-definite bilinear forms, with radicals $\mathrm{rad}(p_{\langle \cdot, \cdot \rangle}) = \mu(\mathfrak{n}, \mathfrak{n})^\perp$, $\mathrm{rad}(q_{\langle \cdot, \cdot \rangle}) = \mathfrak{z}(\mathfrak{n}, \mu)$. (Recall that the radical of a symmetric bilinear form $b(\cdot, \cdot) \in S^2(W^*)$ is the set $\mathrm{rad}(b) := \{v \in W : b(v, w) = 0, \ \forall w \in W\}$; for a semi-definite form one has $\mathrm{rad}(b) = \{w \in W : b(w, w) = 0\}$.) (4) immediately yields

**Lemma 2.1** *Any scalar product* $\langle\cdot,\cdot\rangle$ *on a nilpotent Lie algebra* $(\mathfrak{n},\mu)$ *satisfies*

$$\mathfrak{z}(\mathfrak{n},\mu)\cap\mu(\mathfrak{n},\mathfrak{n})^{\perp}\subset \mathit{radRic}_{\langle\cdot,\cdot\rangle}.$$

The following lemma allows us to compute $\sigma(\mathrm{Ric}_{\langle\cdot,\cdot\rangle})$ in terms of $\sigma(p_{\langle\cdot,\cdot\rangle})$ and $\sigma(q_{\langle\cdot,\cdot\rangle})$. Recall that the signature $\sigma(b)$ of a symmetric bilinear form $b(\cdot,\cdot)$ on a vector space $V$ is the unique triple $(s^-,s^0,s^+)\in\mathbb{Z}^3_{\geq 0}$ with $s^-+s^0+s^+=\dim V$, and $s^{\pm}$ the maximal dimension of a subspace where $\pm b$ is positive definite.

**Lemma 2.2** *Let* $s(\cdot,\cdot),t(\cdot,\cdot)\in S^2(W^*)$ *be two positive semi-definite symmetric bilinear forms on a finite-dimensional real vector space* $W$*, with* $\sigma(s)=(0,b,c)$*,* $\sigma(t)=(0,c,b)$ *and* $\mathrm{rad}(s)\cap\mathrm{rad}(t)=0$*. Then,* $\sigma(t-s)=(c,0,b)$*.*

*Proof* Using $\dim\mathrm{rad}(s)+\dim\mathrm{rad}(t)=b+c=\dim W$ and $\mathrm{rad}(s)\cap\mathrm{rad}(t)=0$ yields $W=\mathrm{rad}(s)\oplus\mathrm{rad}(t)$. This in turn implies that $(t-s)|_{\mathrm{rad}(s)\times\mathrm{rad}(s)}=t|_{\mathrm{rad}(s)\times\mathrm{rad}(s)}>0$, $(t-s)|_{\mathrm{rad}(t)\times\mathrm{rad}(t)}=-s|_{\mathrm{rad}(t)\times\mathrm{rad}(t)}<0$, and the lemma follows. □

**Lemma 2.3** *Let* $\langle\cdot,\cdot\rangle$ *be a scalar product on a nilpotent Lie algebra* $(\mathfrak{n},\mu)$*, and set*

$$r_\mu:=\dim\mathfrak{z}(\mathfrak{n},\mu)-\dim\big(\mathfrak{z}(\mathfrak{n},\mu)\cap\mu(\mathfrak{n},\mathfrak{n})^{\perp}\big)-\dim\big(\mathfrak{z}(\mathfrak{n},\mu)\cap\mu(\mathfrak{n},\mathfrak{n})\big).$$

*Then,* $(s^-,s^0,s^+):=\sigma(\mathit{Ric}_{\langle\cdot,\cdot\rangle})$ *satisfies*

$$s^-\geq u+r_\mu,\qquad s^0\geq a-r_\mu,\qquad s^+\geq z+r_\mu.$$

*In particular, the inclusion* $\subseteq$ *in Theorem A holds.*

*Proof* Set $\mathfrak{b}:=\mathfrak{z}(\mathfrak{n},\mu)\cap\mu(\mathfrak{n},\mathfrak{n})^{\perp}$, $\mathfrak{z}_1:=\mathfrak{z}(\mathfrak{n},\mu)\cap\mu(\mathfrak{n},\mathfrak{n})$, $r_\mu:=a-\dim\mathfrak{b}\geq 0$. Consider an orthogonal decomposition

$$\mathfrak{n}=\overbrace{\mathfrak{b}\oplus\mathfrak{c}}^{\mu(\mathfrak{n},\mathfrak{n})^{\perp}}\oplus\overbrace{\mathfrak{m}\oplus\mathfrak{z}_1}^{\mu(\mathfrak{n},\mathfrak{n})}. \tag{6}$$

By Lemma 2.1, $s^0=\dim\mathrm{radRic}_{\langle\cdot,\cdot\rangle}\geq\dim\mathfrak{b}=a-r_\mu$.

Regarding $s^-$, we observe that $\mathrm{Ric}_{\langle\cdot,\cdot\rangle}|_{\mathfrak{c}\times\mathfrak{c}}$ is negative definite. Indeed, $\mathfrak{c}\subset\mathrm{rad}(p_{\langle\cdot,\cdot\rangle})$ and $\mathfrak{c}\cap\mathrm{rad}(q_{\langle\cdot,\cdot\rangle})=0$, thus $\mathrm{Ric}_{\langle\cdot,\cdot\rangle}(X,X)=-q_{\langle\cdot,\cdot\rangle}(X,X)<0$ for $X\in\mathfrak{c}\backslash\{0\}$.

Hence,

$$\begin{aligned}s^-\geq\dim\mathfrak{c}&=\dim\mathfrak{n}-\dim\mu(\mathfrak{n},\mathfrak{n})-\dim\mathfrak{b}\\&=\dim\mathfrak{n}-\dim(\mu(\mathfrak{n},\mathfrak{n})+\mathfrak{z}(\mathfrak{n},\mu))+\dim(\mathfrak{z}(\mathfrak{n},\mu))\\&\quad-\dim(\mathfrak{z}\cap\mu(\mathfrak{n},\mathfrak{n}))-\dim\mathfrak{b}\\&=u+a-(a-r_\mu)=u+r_\mu.\end{aligned}$$

To show that $s^+\geq z+r_\mu$, we will prove that there exist two $r_\mu$-dimensional subspaces $W_1\subset\mu(\mathfrak{n},\mathfrak{n})^{\perp}$ and $W_2\subset\mathfrak{m}$ such that the restriction of $\mathrm{Ric}_{\langle\cdot,\cdot\rangle}$ to $W_1\oplus W_2\oplus\mathfrak{z}_1$ has signature $(r_\mu,0,z+r_\mu)$.

We have the decomposition (6) with $\mathfrak{b} \oplus \mathfrak{z}_1 \subset \mathfrak{z}(\mathfrak{n}, \mu)$. Thus, $\mathrm{ad}(\mathfrak{c} + \mathfrak{m}) = \mathrm{ad}\mathfrak{n} \simeq \mathfrak{n}/\mathfrak{z}(\mathfrak{n}, \mu)$. On the other hand, $\mathrm{ad}|_{\mathfrak{c}}$ and $\mathrm{ad}|_{\mathfrak{m}}$ are both injective, since $(\mathfrak{c} \oplus \mathfrak{m}) \cap \mathfrak{z}(\mathfrak{n}, \mu) = 0$. Therefore,

$$\begin{aligned} \dim(\mathrm{ad}\mathfrak{c} \cap \mathrm{ad}\mathfrak{m}) &= \dim \mathrm{ad}\mathfrak{c} + \dim \mathrm{ad}\mathfrak{m} - \dim \mathrm{ad}(\mathfrak{c} + \mathfrak{m}) \\ &= (u + r_\mu) + m - \dim \mathfrak{n}/\mathfrak{z}(\mathfrak{n}, \mu) \\ &= u + r_\mu + m - n + z + a = r_\mu. \end{aligned}$$

This means that there are $r_\mu$-dimensional subspaces $W_1 \subset \mathfrak{c}$, $W_2 \subset \mathfrak{m}$ such that $\mathrm{ad}W_1 = \mathrm{ad}W_2$.

Set now $W := W_1 \oplus W_2 \oplus \mathfrak{z}_1$, $s := q_{\langle\cdot,\cdot\rangle}|_{W\times W}$, $t := p_{\langle\cdot,\cdot\rangle}|_{W\times W}$. We first observe that $\mathrm{rad}(s) = \mathfrak{z}(\mathfrak{n}, \mu) \cap W$ and $\mathrm{rad}(t) = \mu(\mathfrak{n}, \mathfrak{n})^\perp \cap W$, thus $\mathrm{rad}(s) \cap \mathrm{rad}(t) = \mathfrak{b} \cap W = 0$. Also, since $W_1 \subset \mu(\mathfrak{n}, \mathfrak{n})^\perp$ and $W_2 \oplus \mathfrak{z}_1 \subset \mu(\mathfrak{n}, \mathfrak{n})$, we have that $\sigma(t) = (0, r_\mu, z + r_\mu)$. On the other hand,

$$\begin{aligned} \dim \mathrm{rad}(s) &= \dim W \cap \mathfrak{z}(\mathfrak{n}, \mu) = \dim \ker(\mathrm{ad}|_W) = \dim W - \dim \mathrm{ad}(W) \\ &= 2r_\mu + z - r_\mu = z + r_\mu, \end{aligned}$$

since $\mathrm{ad}W = \mathrm{ad}W_1$ is $r_\mu$-dimensional.

Thus, $\sigma(s) = (0, z + r_\mu, r_\mu)$. We may now apply Lemma 2.2 to conclude that the signature of $\mathrm{Ric}_{\langle\cdot,\cdot\rangle}|_{W\times W} = t - s$ is $(r_\mu, 0, z + r_\mu)$, as desired.

Finally, the last claim in the lemma statement will follow from the above after we show that $r_\mu \leq \min\{a, m\}$. It is clear that $r_\mu \leq a$. On the other hand, notice that $a + z = \dim \mathfrak{z}(\mathfrak{n}, \mu)$, from which

$$u = \dim \mathfrak{n} - \dim \mathfrak{z}(\mathfrak{n}, \mu) - \dim \mu(\mathfrak{n}, \mathfrak{n}) + \dim(\mu(\mathfrak{n}, \mathfrak{n}) \cap \mathfrak{z}(\mathfrak{n}, \mu)) = \dim \mathfrak{n} - a - z - m,$$

that is, $\dim \mathfrak{n} = a + z + m + u$. Adding up the three inequalities for the signature, we get

$$\dim \mathfrak{n} = s^- + s^0 + s^+ \geq u + a + z + r_\mu = \dim \mathfrak{n} - m + r_\mu,$$

and this completes the proof. □

## 3 A Counterexample

It has recently been conjectured that the set $\sigma_{\mathrm{Ric}}(\mathfrak{n})$ can be described as follows:

**Conjecture 1** [9] *The set of all possible signatures for the Ricci curvature of left-invariant Riemannian metrics on a nilpotent Lie group with Lie algebra* $\mathfrak{n}$ *can be described in terms of the constants given in Theorem A as*

$$\sigma_{Ric}(\mathfrak{n}) = \bigcup_{r=0}^{\min(a,m)} \left\{(s^-, s^0, s^+) : s^- \geq u + r,\ s^0 \geq a - r,\ s^+ \geq z,\ s^- + s^0 + s^+ = \dim \mathfrak{n}\right\}.$$

It is not hard to see that this conjectural set contains the one stated in Theorem A, and that the inclusion is strict unless $\mathfrak{z}(\mathfrak{n}, \mu) \subset \mu(\mathfrak{n}, \mathfrak{n})$ (if the latter happens then $a = 0$, thus also $r = 0$). Using Lemma 2.3, we can quickly state an explicit counterexample:

*Example 3.1* Consider the 5-dimensional nilpotent Lie algebra $\mathfrak{n}$ with basis $\{X_i\}_{i=1}^5$ and non-zero Lie brackets given by

$$\mu(X_1, X_2) = X_3, \qquad \mu(X_1, X_3) = X_4.$$

(This Lie algebra is denoted by $L_{5,3}$ in [9].) It satisfies $n = 5$, $z = a = m = 1$, $u = 2$. Thus, according to Conjecture 1 we should have $(4, 0, 1) \in \sigma_{\mathrm{Ric}}(\mathfrak{n})$ (setting $r = 1$.) However, from Lemma 2.3 it follows that $(s^-, s^0, s^+) \in \sigma_{\mathrm{Ric}}(\mathfrak{n})$ implies $s^0 + s^+ \geq a + z = 2$, a contradiction.

## 4 Subgroups of $\mathsf{GL}(\mathfrak{n})$ Whose Orbits are Closed

In this section, we will produce closed subgroups of $\mathsf{GL}(\mathfrak{n})$ whose orbits through $\mu \in \Lambda^2(\mathfrak{n}^*) \otimes \mathfrak{n}$ are closed. Let us first set up some notation. Given a Lie algebra $\mathfrak{n}$, a fixed 'background' scalar product $\langle \cdot, \cdot \rangle$ induces scalar products (also denoted by $\langle \cdot, \cdot \rangle$) on $\mathfrak{n}^*$ and on any tensor product. For example, given any orthonormal basis $\{X_i\}$ of $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ with dual basis $\{X^i\}$, the induced scalar products on $\mathrm{End}(\mathfrak{n}) \simeq \mathfrak{n}^* \otimes \mathfrak{n}$ and $\Lambda^2(\mathfrak{n}^*) \otimes \mathfrak{n}$ have orthonormal bases given by $\{X^i \otimes X_j\}$ and $\{(X^i \wedge X^j) \otimes X_k\}$, respectively. Of course, the one on $\mathrm{End}(\mathfrak{n})$ may be alternatively defined by $\langle A, B \rangle := \mathrm{tr} AB^t$, $A, B \in \mathrm{End}(\mathfrak{n})$, where the transpose is defined with respect to $\langle \cdot, \cdot \rangle$.

Let $\left(\mathfrak{n}^{(i)}\right)_{i \geq 0}$ denote the descending central series of a Lie algebra $(\mathfrak{n}, \mu)$:

$$\mathfrak{n}^{(0)} = \mathfrak{n}, \qquad \mathfrak{n}^{(i+1)} := \mu\left(\mathfrak{n}, \mathfrak{n}^{(i)}\right) \quad \text{for } i \geq 0.$$

By definition, $(\mathfrak{n}, \mu)$ is nilpotent if and only if $\mathfrak{n}^{(N)} = 0$ for some $N \in \mathbb{N}$. We will assume this is the case from now on.

Given any direct sum decomposition $\mathfrak{n} = \mathfrak{v}_1 \oplus \mathfrak{v}_2 \oplus \mathfrak{v}_3$ into subspaces $(\mathfrak{v}_i)_{i=1}^3$, consider the following subset of $\mathsf{GL}(\mathfrak{n})$:

$$\mathsf{G}_{(\mathfrak{v}_i)} := \left\{ h \in \mathsf{GL}(\mathfrak{n}) : h\big|_{\mathfrak{v}_1} = \mathrm{Id}_{\mathfrak{v}_1},\ h\big|_{\mathfrak{v}_2} = \mathrm{Id}_{\mathfrak{v}_2},\ h(\mathfrak{v}_3) \subset \mathfrak{v}_2 \oplus \mathfrak{v}_3 \right\}. \tag{7}$$

It is clear that $\mathsf{G}_{(\mathfrak{v}_i)}$ is a closed Lie subgroup of $\mathsf{GL}(\mathfrak{n})$, with Lie algebra

$$\mathfrak{g}_{(\mathfrak{v}_i)} = \left\{ A \in \mathrm{End}(\mathfrak{n}) : A\big|_{\mathfrak{v}_1} = A\big|_{\mathfrak{v}_2} = 0,\ A(\mathfrak{v}_3) \subset \mathfrak{v}_2 \oplus \mathfrak{v}_3 \right\}.$$

*Remark 4.1* The reader interested in understanding the proof of Theorem A in the case $\mathfrak{z}(\mathfrak{n}, \mu) \subset \mu(\mathfrak{n}, \mathfrak{n})$ may assume that $\mathfrak{v}_1 = 0$ throughout this and the following sections.

The next lemma is one of the main ingredients in the proof of Theorem A:

**Lemma 4.2** *Let $\mathfrak{n} = \mathfrak{v}_1 \oplus \mathfrak{v}_2 \oplus \mathfrak{v}_3$ be a nilpotent Lie algebra with Lie bracket $\mu \in \Lambda^2\mathfrak{n}^* \otimes \mathfrak{n}$. Assume that $(\mathfrak{v}_1 \oplus \mathfrak{v}_2) + \mu(\mathfrak{n}, \mathfrak{n}) = \mathfrak{n}$ and $\mathfrak{z}(\mathfrak{n}, \mu) \subset \mathfrak{v}_1 \oplus \mathfrak{v}_2$. Then, the orbit $\mathsf{G}_{(\mathfrak{v}_i)} \cdot \mu$ is closed.*

Its proof requires the following fact about nilpotent Lie algebras.

**Lemma 4.3** *Let $\mathfrak{h}$ be a subalgebra of a nilpotent Lie algebra $\mathfrak{n}$ with $\mathfrak{n} = \mathfrak{h} + \mu(\mathfrak{n}, \mathfrak{n})$. Then, $\mathfrak{h} = \mathfrak{n}$.*

*Proof* By nilpotency, it is enough to prove that $\mathfrak{n} = \mathfrak{h} + \mathfrak{n}^{(r)}$ for all $r \geq 1$. We establish this by induction, the case $r = 1$ being the lemma assumption. If $\mathfrak{n} = \mathfrak{h} + \mathfrak{n}^{(r)}$, then

$$\begin{aligned} \mathfrak{n} &= \mathfrak{h} + \mu(\mathfrak{n}, \mathfrak{n}) = \mathfrak{h} + \mu(\mathfrak{h} + \mathfrak{n}^{(r)}, \mathfrak{h} + \mathfrak{n}^{(r)}) \\ &\subset \mathfrak{h} + \mu(\mathfrak{h}, \mathfrak{h}) + \mu(\mathfrak{h}, \mathfrak{n}^{(r)}) + \mu(\mathfrak{n}^{(r)}, \mathfrak{n}^{(r)}) \subset \mathfrak{h} + \mathfrak{n}^{(r+1)}, \end{aligned}$$

and the claim follows. □

*Proof of Lemma 4.2* Consider a sequence $(h^{(k)})_{k\geq 1} \subset \mathsf{G}_{(\mathfrak{v}_i)}$ such that $\lim_{k\to\infty} h^{(k)}\mu =: \bar{\mu}$ exists. We first claim that for each $v \in \mathfrak{n}$, the sequence $(h^{(k)}v)_{k\in\mathbb{N}}$ is bounded. To see that, set

$$\mathfrak{h} := \left\{ v \in \mathfrak{n} : (h^{(k)}v)_{k\in\mathbb{N}} \text{ is bounded} \right\} \subset \mathfrak{n}.$$

It is clearly a vector subspace of $\mathfrak{n}$. Moreover, given $v, w \in \mathfrak{h}$, we have

$$h^{(k)}\left(\mu(v, w)\right) = (h^{(k)} \cdot \mu)\left(h^{(k)}v, h^{(k)}w\right), \tag{8}$$

which is bounded uniformly in $k$ since $\left(h^{(k)} \cdot \mu\right)_{k\in\mathbb{N}}$ is bounded in $\Lambda^2(\mathfrak{n}^*) \otimes \mathfrak{n}$. Thus, $\mathfrak{h}$ is a Lie subalgebra of $(\mathfrak{n}, \mu)$. Also, $\mathfrak{v}_1 \oplus \mathfrak{v}_2 \subset \mathfrak{h}$ by definition of $\mathsf{G}_{(\mathfrak{v}_i)}$, thus by assumption we must have that $\mathfrak{h} + \mu(\mathfrak{n}, \mathfrak{n}) = \mathfrak{n}$. Lemma 4.3 now yields $\mathfrak{h} = \mathfrak{n}$.

The above claim implies that, after passing to a subsequence (which by simplicity we denote with the same indices), $h^{(k)}$ converges to some linear map $\bar{h}$. It is enough to show that $\bar{h}$ is invertible. Indeed, this would imply that $\bar{\mu} = \lim h^{(k)} \cdot \mu = \bar{h} \cdot \mu$, as desired.

Assume on the contrary this is not the case and let

$$\mathfrak{i} := \{v \in \mathfrak{n} : h^{(k)}v \to_{k\to\infty} 0\} \neq 0.$$

By Eq. 8, $\mathfrak{i}$ is an ideal in $(\mathfrak{n}, \mu)$, and $\mathfrak{i} \cap \mathfrak{z}(\mathfrak{n}, \mu) = 0$ since by assumption $\mathfrak{z}(\mathfrak{n}, \mu) \subset \mathfrak{v}_1 \oplus \mathfrak{v}_2$. This contradicts the fact that any nonzero ideal of a nilpotent Lie algebra must intersect its center [13, p. 13]. □

## 5 Closed Orbits and Zeroes of the Ricci Curvature

We now recall one of the most remarkable and useful facts about the Ricci curvature of nilmanifolds: formula (9) below, relating it to the 'GIT moment map' of the $\mathsf{Gl}(\mathfrak{n})$-representation (3). Its origins may be traced back to [11, Sect. 6.4]. Due to our needs in the present article, and in order to simplify the presentation, we have decided to avoid discussing real GIT, referring instead the interested reader to [3, 10, 12, 26] and the references therein.

It was observed in [20, Prop. 3.5] that in terms of the Lie bracket $\mu \in \Lambda^2(\mathfrak{n}^*) \otimes \mathfrak{n}$ of $\mathfrak{n}$, the Ricci curvature of a nilmanifold satisfies

$$\langle \mathrm{Ric}_{\mu}, A\rangle = \frac{1}{4}\langle \pi(A)\mu, \mu\rangle = \tfrac{1}{8}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \|\exp(tA) \cdot \mu\|^2, \qquad A \in \mathrm{End}(\mathfrak{n}), \tag{9}$$

where $\pi : \mathrm{End}(\mathfrak{n}) \to \mathrm{End}(\Lambda^2(\mathfrak{n}^*) \otimes \mathfrak{n})$ is the Lie algebra representation determined by differentiating the Lie group action (3) at the identity, and explicitly given by $(\pi(A)\mu)(\cdot,\cdot) := A\mu(\cdot,\cdot) - \mu(A\cdot,\cdot) - \mu(\cdot, A\cdot)$.

The next result is a simple generalisation of one of the directions in the Kempf-Ness theorem (see [15], and [26] for the $\mathbb{R}$-version). We point out that neither the representation space $\Lambda^2(\mathfrak{n}^*) \otimes \mathfrak{n}$ nor the fact that $\mathfrak{n}$ is nilpotent is essential here: one simply replaces the Ricci curvature by the real GIT moment map, and obtains a similar result for closed orbits in appropriate representations spaces of real reductive Lie groups (cf. [3]).

**Proposition 5.1** *Let $\mu \in \Lambda^2(\mathfrak{n}^*)\otimes\mathfrak{n}$ be the Lie bracket of a nilpotent Lie algebra and $\mathsf{G} \subset \mathsf{GL}(\mathfrak{n})$ a Lie subgroup with Lie algebra $\mathfrak{g}$. Assume that the orbit $\mathsf{G}\cdot\mu$ is closed. Then, there exists a bracket $\bar{\mu} \in \mathsf{G}\cdot\mu$ whose Ricci curvature satisfies $\mathrm{Ric}_{\bar{\mu}} \perp A$ for all $A \in \mathfrak{g}$.*

*Proof* For the closed set $\mathsf{G}\cdot\mu$ there exists $\bar{\mu} \in \mathsf{G}\cdot\mu$ minimising the distance to the origin. In particular, $\bar{\mu}$ is a critical point for $\|\cdot\|^2\big|_{\mathsf{G}\cdot\mu}$, thus (9) gives $\langle \mathrm{Ric}_{\bar{\mu}}, A\rangle = 0$ for all $A \in \mathfrak{g}$. □

The following consequence of Lemma 4.2 and Proposition 5.1 yields the existence of a left-invariant metric whose Ricci endomorphism has a 'large' kernel:

**Corollary 5.2** *Let $\mathfrak{n} = \mathfrak{v}_1 \oplus \mathfrak{v}_2 \oplus \mathfrak{v}_3$ be an orthogonal decomposition with*

$$\mathfrak{v}_1 = \mathfrak{z}(\mathfrak{n},\mu) \cap \mu(\mathfrak{n},\mathfrak{n})^{\perp}, \qquad (\mathfrak{v}_1 \oplus \mathfrak{v}_2) + \mu(\mathfrak{n},\mathfrak{n}) = \mathfrak{n}, \qquad \mathfrak{z}(\mathfrak{n},\mu) \subset \mathfrak{v}_1 \oplus \mathfrak{v}_2.$$

*Then, there exists $\bar{\mu} \in \mathsf{G}_{(\mathfrak{v}_i)}\cdot\mu$ such that $\mathfrak{v}_1 \oplus \mathfrak{v}_3 \subset \ker \mathrm{Ric}_{\bar{\mu}}$.*

*Proof* Lemma 4.2 implies that the orbit $\mathsf{G}_{(\mathfrak{v}_i)}\cdot\mu$ is closed, and from Proposition 5.1 applied to the Lie subgroup $\mathsf{G}_{(\mathfrak{v}_i)}$ we deduce the existence of $\bar{\mu} = h\cdot\mu \in \mathsf{G}_{(\mathfrak{v}_i)}\cdot\mu$, $h \in \mathsf{G}_{(\mathfrak{v}_i)}$, such that

$$\mathrm{Ric}_{\bar{\mu}} \perp \mathfrak{g}_{(\mathfrak{v}_i)}. \tag{10}$$

Let us show that this $\bar{\mu}$ satisfies the above stated property.

Firstly, since $h|_{\mathfrak{v}_1\oplus\mathfrak{v}_2} = \mathrm{Id}_{\mathfrak{v}_1\oplus\mathfrak{v}_2}$ and $\mathfrak{z}(\mathfrak{n},\mu) \subset \mathfrak{v}_1 \oplus \mathfrak{v}_2$, we have that

$$\mathfrak{z}(\mathfrak{n},\bar{\mu}) = h\,(\mathfrak{z}(\mathfrak{n},\mu)) = \mathfrak{z}(\mathfrak{n},\mu).$$

On the other hand, $h$ preserves $\mathfrak{v}_2 \oplus \mathfrak{v}_3$ thus $h^T(\mathfrak{v}_1) \subset \mathfrak{v}_1$. Hence, for $X \in \mathfrak{v}_1$, $Y, Z \in \mathfrak{n}$ we have

$$\langle X, \bar{\mu}(Y,Z)\rangle = \langle X, h\,\mu(h^{-1}Y, h^{-1}Z)\rangle = \langle h^T X, \mu(h^{-1}Y, h^{-1}Z)\rangle = 0,$$

from which $\mathfrak{v}_1 \subset \mathfrak{z}(\mathfrak{n},\bar{\mu}) \cap \bar{\mu}(\mathfrak{n},\mathfrak{n})^{\perp}$. Therefore, Lemma 2.1 implies that $\mathfrak{v}_1 \subset \ker\big(\mathrm{Ric}_{\bar{\mu}}\big)$.

To conclude the proof, let us see that $\mathfrak{v}_3 \subset \ker \mathrm{Ric}_{\bar{\mu}}$. Let $X \in \mathfrak{v}_3$, $\|X\| = 1$, and consider $A \in \mathrm{End}(\mathfrak{n})$ with $AX = \mathrm{Ric}_{\bar{\mu}} X$, $AY = 0$ for all $Y \perp X$. Since

$\mathfrak{v}_1 \subset \ker \mathrm{Ric}_{\bar{\mu}}$ and $\mathrm{Ric}_{\bar{\mu}}$ is self-adjoint, we must have $\mathrm{Ric}_{\bar{\mu}} X \in \mathfrak{v}_2 \oplus \mathfrak{v}_3$, from which $A \in \mathfrak{g}_{(\mathfrak{v}_i)}$. By Eq. 10 this implies that

$$0 = \mathrm{tr}(\mathrm{Ric}_{\bar{\mu}} A) = \langle \mathrm{Ric}_{\bar{\mu}} AX, X\rangle = \|\mathrm{Ric}_{\bar{\mu}} X\|^2,$$

as we wanted to show. □

## 6 The Linearisation of the Ricci Curvature

Let $\mathfrak{p} := \{A \in \mathfrak{gl}(\mathfrak{n}) : A = A^T\}$. For a subspace $\mathfrak{s} \subset \mathfrak{p}$, we denote by $\mathrm{pr}_{\mathfrak{s}} : \mathfrak{p} \to \mathfrak{s}$ the corresponding orthogonal projection.

We view the Ricci endomorphism as a map

$$\mathrm{Ric} \ : \mathsf{GL}(\mathfrak{n}) \cdot \bar{\mu} \to \mathfrak{p}, \qquad h \cdot \bar{\mu} \mapsto \mathrm{Ric}_{h\cdot\bar{\mu}}.$$

Let $L_{\bar{\mu}} : \mathfrak{p} \to \mathfrak{p}$ be the linear map given by

$$L_{\bar{\mu}}(E) = \mathrm{dRic}\,|_{\bar{\mu}}(\pi(E)\bar{\mu}). \tag{11}$$

Recall that $T_{\bar{\mu}}(\mathsf{GL}(\mathfrak{n}) \cdot \bar{\mu}) = \{\pi(E)\bar{\mu} : E \in \mathfrak{gl}(\mathfrak{n})\}$.

Notice that $L_{\bar{\mu}}$ is self-adjoint. Indeed, if $\bar{\mu}(t) := \exp(tE) \cdot \bar{\mu}$, then by Eq. 9 we have

$$\begin{aligned}\langle L_{\bar{\mu}} E, F\rangle \ &= \ \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\langle \mathrm{Ric}_{\bar{\mu}(t)}, F\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\frac{1}{4}\,\langle \pi(F)\bar{\mu}(t), \bar{\mu}(t)\rangle \\ &= \ \frac{1}{4}\,\langle \pi(F)\pi(E)\bar{\mu}, \bar{\mu}\rangle + \frac{1}{4}\,\langle \pi(F)\bar{\mu}, \pi(E)\bar{\mu}\rangle = \frac{1}{2}\,\langle \pi(E)\bar{\mu}, \pi(F)\bar{\mu}\rangle,\end{aligned}$$

for any $E, F \in \mathfrak{p}$, where in the last equality we have used the fact that $\pi(F)^T = \pi(F^T) = \pi(F)$. This computation also shows that $L_{\bar{\mu}}$ is positive semi-definite, with

$$\ker L_{\bar{\mu}} = \mathrm{Der}(\bar{\mu}) \cap \mathfrak{p}.$$

Indeed, $\mathrm{Der}(\bar{\mu}) = \{E \in \mathfrak{gl}(\mathfrak{n}) : \pi(E)\bar{\mu} = 0\}$. In particular, we have

**Lemma 6.1** *The projection $pr_{\mathfrak{s}} \circ L_{\bar{\mu}} : \mathfrak{p} \to \mathfrak{s}$ is surjective if and only if $\mathfrak{s} \cap Der(\bar{\mu}) = 0$.*

*Proof* We will show that the orthogonal complement of $\mathrm{pr}_{\mathfrak{s}} \circ L_{\bar{\mu}}(\mathfrak{p})$ in $\mathfrak{s}$ equals $\mathfrak{s} \cap \mathrm{Der}(\bar{\mu})$. If $S \in \mathfrak{s}$ belongs to the former, then also $S \perp L_{\bar{\mu}}(\mathfrak{p})$. Since $L_{\bar{\mu}}$ is self-adjoint, this implies that $S \in \ker L_{\bar{\mu}} = \mathrm{Der}(\bar{\mu}) \cap \mathfrak{p}$, so $S \in \mathfrak{s} \cap \mathrm{Der}(\bar{\mu})$. Conversely, let $S \in \mathfrak{s} \cap \mathrm{Der}(\bar{\mu})$. Then $S \in \ker L_{\bar{\mu}}$, thus for any $E \in \mathfrak{p}$ we have that

$$\langle \mathrm{pr}_{\mathfrak{s}} \circ L_{\bar{\mu}}(E), S\rangle = \langle L_{\bar{\mu}}(E), S\rangle = \langle E, L_{\bar{\mu}}(S)\rangle = 0.$$ □

## 7 Proof of Theorem A

By Lemma 2.3, we know that one of the inclusions holds. Let us now show that any triple as in the theorem's statement can be realised as the signature of the Ricci curvature of some left-invariant metric.

Any nilpotent Lie algebra may written as

$$\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{n}_1, \qquad \mathfrak{z}(\mathfrak{n}, \mu) = \mathfrak{a} \oplus \mathfrak{z}_1, \qquad \mathfrak{z}_1 := \mathfrak{z}(\mathfrak{n}, \mu) \cap \mu(\mathfrak{n}, \mathfrak{n}).$$

The subspaces $\mathfrak{a}$, $\mathfrak{n}_1$ are nilpotent ideals, $\mathfrak{a}$ is central, and we have a Lie algebra direct sum $\mathfrak{n} \simeq \mathbb{R}^a \oplus \mathfrak{n}_1$. Clearly, $\mu(\mathfrak{n}, \mathfrak{n}) \subset \mathfrak{n}_1$. Choose direct complements $\mathfrak{u}$ of $\mu(\mathfrak{n}, \mathfrak{n})$ in $\mathfrak{n}_1$, and $\mathfrak{m}$ of $\mathfrak{z}_1$ in $\mu(\mathfrak{n}, \mathfrak{n})$, so that we have the decompositions

$$\mathfrak{n} = \mathfrak{a} \oplus \overbrace{\mathfrak{u} \oplus \underbrace{\mathfrak{m} \oplus \mathfrak{z}_1}_{\mu(\mathfrak{n},\mathfrak{n})}}^{\mathfrak{n}_1}, \qquad \mathfrak{z}(\mathfrak{n}, \mu) = \mathfrak{a} \oplus \mathfrak{z}_1.$$

Let us fix an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ making the above decompositions orthogonal.

We begin by making the following reduction. Choose an integer $r \in [0, \min(a, m)]$ and consider any orthogonal decomposition $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1$ into subspaces, where $\dim \mathfrak{a}_0 = a - r$, $\dim \mathfrak{a}_1 = r$. Then $\mathfrak{n} = \mathfrak{a}_0 \oplus (\mathfrak{a}_1 \oplus \mathfrak{n}_1)$, with $\tilde{\mathfrak{n}} := \mathfrak{a}_1 \oplus \mathfrak{n}_1$ a nilpotent ideal. The simply connected Lie group $\mathsf{N}$ with left-invariant metric $g$ corresponding to $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ decomposes as a Riemannian product $\mathsf{N} = \mathbb{R}^{a-r} \times \tilde{\mathsf{N}}$, where the first factor is Euclidean (flat) and the second one is the simply connected Lie group with Lie algebra $\tilde{\mathfrak{n}}$, endowed with the corresponding left-invariant metric $\tilde{g}$. We clearly have

$$\sigma(\mathrm{Ric}\,(g)) = (0, a - r, 0) + \sigma(\mathrm{Ric}\,(\tilde{g})).$$

The theorem will follow if we show that for any triple of non-negative integers $(m^-, m^0, m^+)$ with $m^- + m^0 + m^+ = m - r$ we have

$$(u + r, 0, z + r) + (m^-, m^0, m^+) \in \sigma \mathrm{Ric}\,(\tilde{\mathfrak{n}}). \tag{12}$$

From now on and for the rest of the proof, we will focus on proving this assertion. To ease notation, we will simply ignore the subspace $\mathfrak{a}_0$ and assume that $\mathfrak{n} = \tilde{\mathfrak{n}}$. That is, we have

$$r = a := \dim \mathfrak{a} \leq \dim \mathfrak{m}$$

and aim to prove that for all $(m^-, m^0, m^+) \in \mathbb{Z}^3_{\geq 0}$ with $m^- + m^0 + m^+ = m - a$ we have

$$(u + a, 0, z + a) + (m^-, m^0, m^+) \in \sigma \mathrm{Ric}\,(\mathfrak{n}). \tag{13}$$

In terms of our fixed background scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$, we may parametrise all scalar products on $\mathfrak{n}$ via $\langle h \cdot, h \cdot \rangle$, $h \in \mathsf{GL}(\mathfrak{n})$. Recall that by Eq. 2, the endomorphism $\mathrm{Ric}_{h \cdot \mu}$ represents the bilinear form $\mathrm{Ric}_{\langle h \cdot, h \cdot \rangle}$ in a certain basis. Thus, Sylvester's law of inertia allows us to compute the signature of $\mathrm{Ric}_{\langle h \cdot, h \cdot \rangle}$ by looking at the signs of the eigenvalues of $\mathrm{Ric}_{h \cdot \mu}$. Throughout the proof, when referring to the signature of $\mathrm{Ric}_{h \cdot \mu}$, which we will write simply as $\sigma\left(\mathrm{Ric}_{h \cdot \mu}\right)$, we will always mean the signature of the bilinear form $\langle \mathrm{Ric}_{h \cdot \mu} \cdot, \cdot \rangle$.

The first step towards proving (13) is to establish that $(u, a + m, z) \in \sigma \mathrm{Ric}\,(\mathfrak{n})$. To that end, we apply Corollary 5.2 to the subspaces $\mathfrak{v}_1 = \mathfrak{a}$, $\mathfrak{v}_2 = \mathfrak{u} \oplus \mathfrak{z}_1$, $\mathfrak{v}_3 = \mathfrak{m}$. This yields a bracket $\bar{\mu} \in \mathsf{G}_{(\mathfrak{v}_i)} \cdot \mu$ with $\mathfrak{a} \oplus \mathfrak{m} \subset \ker \mathrm{Ric}_{\bar{\mu}}$. In other words, $s^0 \geq a + m$, where $(s^-, s^0, s^+) = \sigma(\mathrm{Ric}_{\bar{\mu}})$. On the other hand, $s^- \geq u$, $s^+ \geq z$ by Lemma 2.3. Since $a + m + u + z = \dim \mathfrak{n}$ we conclude that in fact we have $\sigma(\mathrm{Ric}_{\bar{\mu}}) = (u, a + m, z)$.

The strategy is now to prove that local variations of $\bar{\mu}$ within the orbit $\mathsf{GL}(\mathfrak{n}) \cdot \bar{\mu}$ yield all claimed Ricci signatures. Thus, we study $\sigma(\mathrm{Ric}_{h \cdot \bar{\mu}})$ for $h \in \mathsf{G}l(\mathfrak{n})$ in a

neighbourhood of the identity. More precisely, we only consider $h = \exp(E)$, for $E \in \mathfrak{p}$ in a neighbourhood of 0. According to the decomposition $\mathfrak{n} = \mathfrak{v}_1 \oplus \mathfrak{v}_2 \oplus \mathfrak{v}_3$ we have

$$\mathrm{Ric}_{\bar{\mu}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

with $R$ non-singular and $\sigma(R) = (u, 0, z)$. Let us write

$$\mathrm{Ric}_{\exp(E)\cdot\bar{\mu}} = \mathrm{Ric}_{\bar{\mu}} + A(E) = \begin{bmatrix} A_{11}(E) & A_{12}(E) & A_{13}(E) \\ A_{12}(E)^T & R + A_{22}(E) & A_{23}(E) \\ A_{13}(E)^T & A_{23}(E)^T & A_{33}(E) \end{bmatrix},$$

with $A(0) = 0$, $A_{i,j}(E) : \mathfrak{v}_j \to \mathfrak{v}_i$, $A_{ii}(E)^T = A_{ii}(E)$, and $E \mapsto A(E)$ a smooth map whose differential is given by

$$\mathrm{d}A|_0(E) = L_{\bar{\mu}}(E) = \mathrm{dRic}\,|_{\bar{\mu}}(\pi(E)\bar{\mu}), \qquad E \in \mathfrak{p}. \tag{14}$$

For small enough $E$ the operator $R + A_{22}(E)$ is invertible, allowing us to change basis with the endomorphism

$$Q := \begin{bmatrix} \mathrm{Id} & 0 & 0 \\ -(R + A_{22}(E))^{-1} A_{12}(E)^T & \mathrm{Id} & -(R + A_{22}(E))^{-1} A_{23}(E) \\ 0 & 0 & \mathrm{Id} \end{bmatrix}$$

to obtain an operator whose signature is easier to compute:

$$Q^T \mathrm{Ric}_{\exp(E)\cdot\bar{\mu}}\, Q = \begin{bmatrix} X_{11}(E) & 0 & X_{13}(E) \\ 0 & R + A_{22}(E) & 0 \\ X_{13}(E)^T & 0 & X_{33}(E) \end{bmatrix}.$$

Here, $X_{11}$, $X_{13}$, $X_{33}$ are defined by

$$\begin{aligned} X_{11} &:= A_{11} - A_{12}\,(R + A_{22})^{-1}\, A_{12}^T : \mathfrak{v}_1 \to \mathfrak{v}_1\,, \\ X_{13} &:= A_{13} - A_{12}\,(R + A_{22})^{-1}\, A_{23} : \mathfrak{v}_3 \to \mathfrak{v}_1\,, \\ X_{33} &:= A_{33} - A_{23}^T\,(R + A_{22})^{-1}\, A_{23} : \mathfrak{v}_3 \to \mathfrak{v}_3. \end{aligned}$$

A straightforward computation shows that their first variations are

$$(\mathrm{d}X_{i,j})\big|_0(E) = (\mathrm{d}A_{i,j})\big|_0(E), \qquad \forall\, i, j \in \{1, 3\}, \qquad E \in \mathfrak{p}. \tag{15}$$

In order to compute the signature of $Q^T \mathrm{Ric}_{\exp(E)\cdot\bar{\mu}}\, Q$, we need to understand $\sigma(X(E))$, where $X(E) : \mathfrak{v}_1 \oplus \mathfrak{v}_3 \to \mathfrak{v}_1 \oplus \mathfrak{v}_3$ is given by

$$X(E) = \begin{bmatrix} X_{11}(E) & X_{13}(E) \\ X_{13}(E)^T & X_{33}(E) \end{bmatrix}.$$

Let $\mathfrak{s} := \{A \in \mathfrak{p} : A|_{\mathfrak{v}_2} = 0, A(\mathfrak{v}_1) \subset \mathfrak{v}_3\}$, and notice that $X_{33}(E)$, $X_{13}(E) + X_{13}(E)^T \in \mathfrak{s}$, where we are abusing notation and viewing these as linear maps in $\mathrm{End}(\mathfrak{n})$ in the obvious way.

**Lemma 7.1** *We have that* $\mathfrak{s} \cap Der(\bar{\mu}) = 0$.

*Proof* Let $\tilde{D} \in \mathfrak{s} \cap \mathrm{Der}(\bar{\mu})$. Then $\tilde{D}|_{\mathfrak{v}_2} = 0$. Write $\bar{\mu} = h \cdot \mu$, $h \in \mathsf{G}_{(\mathfrak{v}_i)}$, so that $D := h^{-1}\tilde{D}h \in \mathrm{Der}(\mu)$. By definition of derivation, $\ker D \subset \mathfrak{n}$ is a Lie subalgebra. Therefore, $\mathfrak{h} := \ker D \cap \mathfrak{n}_1$ is a subalgebra of the ideal $\mathfrak{n}_1$. Since $\mathsf{G}_{(\mathfrak{v}_i)}$ acts trivially on $\mathfrak{v}_2$ we also have that $\mathfrak{v}_2 \subset \mathfrak{h}$. But $\mathfrak{u} \subset \mathfrak{v}_2$, hence it follows that $\mathfrak{h} + \mu(\mathfrak{n}_1, \mathfrak{n}_1) = \mathfrak{n}_1$, and by Lemma 4.3 we conclude that $\mathfrak{h} = \mathfrak{n}_1$. This implies that $\mathfrak{n}_1 = \mathfrak{v}_2 \oplus \mathfrak{v}_3 \subset \ker D$. Now $\mathsf{G}_{(\mathfrak{v}_i)}$ preserves $\mathfrak{n}_1$, from which we deduce that also $\mathfrak{v}_2 \oplus \mathfrak{v}_3 \subset \ker \tilde{D}$. Since $\tilde{D}$ is self-adjoint, the later gives $\tilde{D}(\mathfrak{v}_1) \subset \mathfrak{v}_1$. By definition of $\mathfrak{s}$, this yields $\tilde{D} = 0$ and concludes the proof. □

By Lemmas 6.1 and 7.1, the orthogonal projection of $L_{\bar{\mu}}$ onto $\mathfrak{s}$ is surjective. Using Eqs. 14 and 15 and the Implicit Function Theorem, this implies that for some neighbourhood $\mathcal{U}$ of 0 in $\mathfrak{p}$, the images $X_{13}(\mathcal{U})$ and $X_{33}(\mathcal{U})$ contain 0 as an interior point. In other words, they attain any given value whose norm is sufficiently small. As we will see, this is enough for constructing $E \in \mathfrak{p}$ such that $\mathrm{Ric}_{\exp(E)\cdot\bar{\mu}}$ has the desired signature.

Indeed, consider an arbitrary orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, with $\dim \mathfrak{m}_1 = a$. Let $(m^-, m^0, m^+) \in \mathbb{Z}^3_{\geq 0}$ with $m^- + m^0 + m^+ = m - a$, and choose a self-adjoint endomorphism $Y : \mathfrak{m}_2 \to \mathfrak{m}_2$ such that $\sigma(Y) = (m^-, m^0, m^+)$. Finally, choose a self-adjoint linear isomorphism $\mathfrak{a} \simeq \mathfrak{m}_1$. Then, by the above reasoning, there exists $E \in \mathfrak{p}$ such that

$$X(E) = \begin{bmatrix} X_{11}(E) & \mathrm{Id} & 0 \\ \mathrm{Id} & 0 & 0 \\ 0 & 0 & Y \end{bmatrix},$$

with blocks according to the decomposition $\mathfrak{a} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Clearly, we may assume $E$ is small enough so that $\sigma(R + A_{22}(E)) = \sigma(R)$. Finally, we have

$$\begin{aligned} \sigma(\mathrm{Ric}_{\exp(E)\cdot\bar{\mu}}) &= \sigma(Q^T \mathrm{Ric}_{\exp(E)\cdot\bar{\mu}} Q) = \sigma(R) + \sigma(X(E)) \\ &= \sigma(R) + \sigma(Y) + \sigma\left(\begin{bmatrix} X_{11}(E) & \mathrm{Id} \\ \mathrm{Id} & 0 \end{bmatrix}\right) \\ &= (u, 0, z) + (m^-, m^0, m^+) + (a, 0, a), \end{aligned}$$

and Eq. 13 follows. The last equality uses the fact that the signature remains invariant along a continuous path of invertible, self-adjoint operators:

$$\sigma\left(\begin{bmatrix} X_{11}(E) & \mathrm{Id} \\ \mathrm{Id} & 0 \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} t \cdot X_{11}(E) & \mathrm{Id} \\ \mathrm{Id} & 0 \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{bmatrix}\right) = (a, 0, a), \quad t \in [0, 1].$$

This concludes the proof of the theorem.

**Acknowledgements** The authors would like to thank Christoph Böhm for useful comments on a first draft of this article. Part of this research was carried out while the first named author was a Postdoctoral Research Fellow at The University of Queensland. She is very grateful to the staff and students of the School of Mathematics & Physics for their kindness and hospitality.

**Funding** The first named author was partially supported by grants from the Australian Government through the Australian Research Council's Discovery Projects funding scheme (project DP180102185), CONICET, FONCYT and SeCyT (Universidad Nacional Córdoba). The second named author is an ARC DECRA fellow.

## Declarations

**Conflict of Interest** The authors declare no competing interests.

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