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On the homotopy perturbation method for Boussinesq-like equations

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ABSTRACT

We comment on some analytical solutions to a class of Boussinesq-like equations derived recently by means of the homotopy perturbation method (HPM). We show that one may obtain exactly the same result by means of the Taylor series in the time variable. We derive more general results by means of traveling waves and argue that a curious superposition principle proposed by the author may not be of any mathematical or physical significance.
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1. Introduction

In a recent paper Yıldırım [1] solved a kind of generalized Boussinesq-like equations by means of the homotopy perturbation method (HPM) and derived new exact solitary solutions. His results revealed a kind of superposition principle for nonlinear equations. According to the author "This paper reveals first time that in some special cases addition of two solitary solutions satisfies exactly the nonlinear equation. Such phenomenon requires further mathematical study and physical explanation".

The purpose of this paper is to carry out a "further mathematical study" of the results derived by that author and the conclusions drawn from them.

2. On the HPM

Yıldırım [1] investigated $B(m,n)$ equations of the form

$$u_{tt} = (u^m)_{xx} - (u^n)_{xxxx}, \quad m,n > 1, \tag{1}$$

where the subscripts denote derivatives of the solution $u(x,t)$ with respect to its arguments. However, in the examples Yıldırım changed his mind and considered particular cases of

$$u_{tt} = -(u^m)_{xx} + (u^n)_{xxxx}. \tag{2}$$

Both equations exhibit completely different solutions and we mainly restrict our discussion to Eq. (2) in what follows.

Yıldırım applied HPM which looks like an ordinary perturbation theory with a fancy name. After introducing a perturbation parameter p, Yıldırım obtained a perturbation series of the form

$$u(x,t) = \sum_{j=0}^{\infty} v_j(x,t) p^j \tag{3}$$

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and stated that its convergence was proved by He [2]. However, that reference does not contain any convergence proof. As far as we know, the proof of convergence was suggested to He by an anonymous referee of an earlier paper [3] but to our knowledge no HPM user has ever verified if his particular implementation of the approach already met those convergence criteria.

After substituting $p = 1$ in the resulting perturbation series (3) Yıldırım realized that it was merely the Taylor series about $t = 0$:

$$u(x,t) = \sum_{j=0}^{\infty} f_j(x) t^j, \tag{4}$$

where $f_0(x) = u(x,0)$ and $f_1(x) = u_t(x,0)$. By inspection of the Taylor series Yıldırım [1] identified some exact analytical solutions to the nonlinear partial differential equations. It is worth noting that Yıldırım chose the particular initial conditions that he knew would lead to such analytical solutions. It is unlikely that one may be able to sum the time series exactly for two arbitrary functions $u(x,0)$ and $u_t(x,0)$; therefore, in an actual physical problem one has to rely on the accuracy of truncated time series that most probably will provide a quite poor description of the dynamics of the problem beyond a relatively small neighborhood of $t = 0$.

Obviously, we may substitute Eq. (4) into Eq. (2) obtain the coefficients $f_j(x)$ in a straightforward way. However, Yıldırım preferred the application of the lengthier HPM. To make this point clearer let us write Eq. (2) as $u_{tt} = F(u,u')$, where u' collects all the derivatives of $u(x,t)$ with respect to x, and F is a nonlinear function. Taking into account that $F(u,u')$ can be expanded in a Taylor series $F = F_0 + F_1 t + \ldots$ we easily derive a recurrence relation for the coefficients f_j:

$$f_{j+2} = \frac{F_j}{(j+1)(j+2)}, \quad j = 0, 1, \ldots, \tag{5}$$

where, obviously, F_j depends on $f_i, i = 0, 1, \ldots, j$. Since we know f_0 and f_1 as already indicated above then we obtain all the other coefficients $f_j(x)$ with $j > 1$ in a straightforward way. Considering that this procedure is a textbook example of calculus, is there any necessity for the application of HPM or any other more or less elaborate method to obtain the same series?. May the reader answer this question himself/herself.

Yıldırım obtained a kind of traveling-wave solutions to the $B(m,n)$ equations and identified a kind of superposition of waves that is unexpected in nonlinear problems. In what follows we analyze those results as well as such phenomenon.

3. On the solutions to the $B(m,n)$ equations

Following the standard procedure for obtaining traveling waves we write

$$u(x,t) = U(\xi), \quad \xi = x \pm vt \tag{6}$$

so that Eq. (2) becomes

$$\left[v^2 U + U^m - (U^n)''\right]'' = 0, \tag{7}$$

where the prime indicates differentiation with respect to ξ. For present purposes it is sufficient to consider polynomial solutions of the form

$$U(\xi) = \sum_{j=-M}^{N} a_j e^{\alpha j \xi} \tag{8}$$

keeping in mind that other expressions (like, for example, rational functions) may also be suitable.

We first discuss the $B(2,2)$ equation

$$u_{tt} + (u^2)_{xx} - (u^2)_{xxx} = 0 \tag{9}$$

and purposely leave the boundary conditions unspecified. In this particular case we have

$$U(\xi) = a_2 e^{\xi/2} + a_{-2} e^{-\xi/2} - \frac{2}{3} v^2, \tag{10}$$

where $\alpha = 1/4$ and a_2 and a_{-2} are arbitrary coefficients.

By means of the HPM Yıldırım [1] derived two solutions for Eq. (9) with different boundary conditions and then combined them into a more general one of the form

$$\begin{aligned} u^Y(x,t) &= a\cosh^2\left(\frac{\xi}{4}\right) + b\sinh^2\left(\frac{\xi}{4}\right), \\ a &= -\frac{4}{3} L v^2, \\ b &= \frac{4}{3}(1-L) v^2. \end{aligned} \tag{11}$$

Yıldırım [1] obtained other solutions by addition of a constant to the arguments of the hyperbolic functions. This strategy for generating new solutions is successful because if $U(\xi)$ is a solution to Eq. (7), then $U(\xi+\xi_0)$ is also a solution (with, of course, different boundary conditions). Eq. (11) is a particular case of our Eq. (10) with $a_2 = a_{-2}$ as follows from the fact that

$$u^Y(x,t) = \frac{a+b}{4}(e^{\xi/2}+e^{-\xi/2}) - \frac{2}{3}v^2. \tag{12}$$

At this point it is worth noting that the solutions that we are discussing are unbounded and, therefore, of little or no physical utility, whatsoever. However, there seems to be some interest in them anyway [1].

As promised above we now address the Yıldırım's superposition principle that is embodied in the solution (11) to Eq. (9). Note that $u^Y(x,0)$ and $u_t^Y(x,0)$ depend on the coefficients of the linear combination (through L). Therefore, the solutions with, say, $L=0$ ($a=0$) and $L=1$ ($b=0$) exhibit different boundary conditions and thereby correspond to different physical problems. In other words, Yıldırım's superposition principle is a linear combination of solutions to different problems. We do not think that such a property is of any physical relevance but, perhaps, we may be wrong.

In the case of the $B(3,3)$ equation

$$u_{tt} + (u^3)_{xx} - (u^3)_{xxx} = 0, \tag{13}$$

we have

$$U(\xi) = a_3 e^{\xi/3} + a_{-3} e^{-\xi/3},$$

$$\alpha = \frac{1}{9},$$

$$a_3 a_{-3} = -\frac{3}{8}v^2 \tag{14}$$

that leads to the two solutions obtained by Yıldırım when $a_3 = -a_{-3} = \pm\sqrt{6}v/4$.

Finally, we mention that Eqs. (1) and (2) are related by the transformation $t \to it$ (or $v \to iv$ for the traveling waves). Therefore, we easily derive the solutions to Eq. (1) from the ones just obtained.

We think that it is unnecessary to consider other cases because equations (1) and (2) do not appear to have any physical application whatsoever. They are merely tailor-made toy problems for the application of approximate methods of doubtful utility. Besides, the results and discussion above are more than enough for making our point.

4. Conclusions

- The HPM proposed by Yıldırım [1] to treat Boussinesq-like equations simply leads to the Taylor series of the traveling-wave solutions about $t=0$. We may apply the Taylor-series approach with initial conditions different to those that lead to traveling waves, but in such more general cases we are not certain to obtain analytical solutions.
- Instead of resorting to the HPM, one can obtain more general solutions by means of the textbook method of traveling-waves reduction.
- Yıldırım's superposition principle is just the linear combination of solutions to models with different boundary conditions and for this reason it is of questionable physical significance.

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