Physica A 506 (2018) 767–772

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Newton's gravitation-force's classical average proof of a Verlinde's conjecture

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HIGHLIGHTS

- Verlinde made two gravitation conjectures.
- The first is that gravity is an entropic, emergent force.
- This has been recently proved, classically.
- Verlinde's second conjecture refers to gravitation's asymptotic behaviour.
- It asserts that it differs from Newton's one.
- We prove such conjecture here.

ARTICLE INFO

Article history:
Received 21 March 2018
Available online 4 May 2018

Keywords:
Verlinde's second conjecture
Boltzmann–Gibbs distribution
Divergences
Dimensional regularization

ABSTRACT

A surprising, gravity related Verlinde-conjecture, that generated immense interest, asserts that gravity is an emergent entropic force. We provided a classical proof of the assertion in [doi.org/j.physa.2018.03.019]. Here, we classically prove a related, second Verlinde-conjecture. This states that, at very large distances (r_0), gravity departs from its classical nature and begins to decay linearly with r_0.

1. Introduction

Entropic gravity, or emergent one, is a Verlinde's conjecture that has attracted immense interest [1]. He describes gravity as an entropic force (EF). This EF is endowed with macro-scale homogeneity, being at the same time subjected to quantum-disorder. Accordingly, this two-body EF would not be a fundamental interaction. Verlinde based his ideas on string theory, black hole physics, and quantum information theory. We provided a classical proof of the assertion in [2].

Such EF theory involves what we call here a second conjecture. It says that when gravity becomes vanishingly weak, at very large distances, it differs from its Newtonian quadratic nature because its strength starts to decay linearly with the inverse distance from a given mass. We intend to prove below this second conjecture.

The main technical ingredient of our proceedings is dimensional regularization (DR). DR [3,4] constitutes one of the greatest advances in the theoretical physics of the last 45 years, with applications in several branches of physics (see, for instance, [5–58].

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https://doi.org/10.1016/j.physa.2018.04.103
0378-4371/

It was believed that the classical Boltzmann–Gibbs (BG) probability distribution cannot yield finite results because the associated partition function $\mathcal{Z}$ diverges [59,60]. This belief did not take into account the possibility of analytical extensions, that could overcome divergences, e.g., at the origin. However, it was shown in Refs. [61,62], that $\mathcal{Z}$ can be calculated and yields finite results for Boltzmann–Gibbs and Tsallis entropies, using the 45-years old DR technique.

It is well known that, *at a quantum field theory level*, DR cannot cope with the gravitational field, since it is non-renormalizable. Our present challenge is quite different, though, because we deal with Newton's gravity at a *classical* level and we are not attempting renormalization.

We prove the second conjecture in three dimensions in Section 2 and in two dimensions in Section 3. The ensuing conclusions are drawn in Section 4.

2. The three-dimensional case

In [2], we classically verified Verlinde's emergent gravitation conjecture by starting with the ideal gas Hamiltonian, constructing the associated partition function, and from it the entropy. Then, following Verlinde's prescription for an entropic force, we showed that it had Newton's appearance. Now instead, we start from the gravitation Hamiltonian and compute the concomitant partition function.

The Boltzmann–Gibbs (BG) partition function $\mathcal{Z}_\nu$ for a Newton potential $\frac{GmM}{r}$ is [62]

$$\mathcal{Z}_\nu = \int_{\mathcal{W}} e^{-\beta\left(\frac{p^2}{2m}-\frac{GmM}{r}\right)} d^\nu x d^\nu p, \tag{2.1}$$

where the masses involved are M and m. We call $\mathcal{W} = R^\nu \oplus S^\nu(r_0)$, $S^\nu(r_0)$ being the spherical volume of radius r_0. For effecting the integration process one uses hyper-spherical coordinates and two integrals, each in ν dimensions. One is then left with just two radial coordinates (one in r-space and the other in p-space) and $2(\nu-1)$ angles. Accordingly [62],

$$\mathcal{Z}_\nu = \left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}\right]^2 \int_0^\infty e^{-\beta\frac{p^2}{2m}} p^{\nu-1} dp \int_0^{r_0} e^{\beta\frac{GmM}{r}} r^{\nu-1} dr. \tag{2.2}$$

We appeal here to a Table if Integrals [63] the integral

$$\int_0^\infty e^{-\beta\frac{p^2}{2m}} p^{\nu-1} dp = \left(\frac{2m}{\beta}\right)^{\frac{\nu}{2}} \frac{\Gamma\left(\frac{\nu}{2}\right)}{2}. \tag{2.3}$$

The remaining integral is cast as

$$\int_0^{r_0} e^{\beta\frac{GmM}{r}} r^{\nu-1} dr = \int_0^\infty e^{\beta\frac{GmM}{r}} r^{\nu-1} dr - \int_{r_0}^\infty e^{\beta\frac{GmM}{r}} r^{\nu-1} dr =$$
$$\Gamma(-\nu)\cos\pi\nu(\beta GmM)^\nu + \frac{r_0^\nu}{\nu}\phi\left(-\nu, 1-\nu; \frac{\beta GmM}{r_0}\right). \tag{2.4}$$

The first integral on the r.h.s has been evaluated in [62] while the second can be read off [63]. We call ϕ the confluent hypergeometric function. We arrive in this way to the following expression for $\mathcal{Z}_\nu$

$$\mathcal{Z}_\nu = \frac{2}{\Gamma\left(\frac{\nu}{2}\right)}\left(\frac{2\pi^2 m}{\beta}\right)^{\frac{\nu}{2}}\left[\Gamma(-\nu)\cos\pi\nu(\beta GmM)^\nu + \frac{r_0^\nu}{\nu}\phi\left(-\nu, 1-\nu; \frac{\beta GmM}{r_0}\right)\right]. \tag{2.5}$$

We need further appeal to [63] to find

$$\phi(-\nu, 1-\nu; z) = e^z\phi(1, 1-\nu; -z) = e^z\left[1 + \frac{z}{\nu-1} + \frac{z^2}{(\nu-1)(\nu-2)} + \right.$$
$$\left. \frac{z^3}{(\nu-1)(\nu-2)(\nu-3)}\phi(1, 4-\nu; -z)\right], \tag{2.6}$$

refining further $\mathcal{Z}$ as

$$\mathcal{Z}_\nu = \frac{2}{\Gamma\left(\frac{\nu}{2}\right)}\cos\pi\nu\left(2\pi^2\beta G^2 m^3 M^2\right)^{\frac{\nu}{2}}\Gamma(-\nu) +$$
$$\frac{2}{\Gamma\left(\frac{\nu}{2}\right)}\left(\frac{2\pi^2 m}{\beta}\right)^{\frac{\nu}{2}}\frac{r_0^\nu}{\nu}e^{\frac{\beta GmM}{r_0}}\left[1 + \frac{\beta GmM}{(\nu-1)r_0} + \frac{(\beta GmM)^2}{(\nu-1)(\nu-2)r_0^2}\right] +$$
$$\frac{2}{\Gamma\left(\frac{\nu}{2}\right)}\left(\frac{2\pi^2 m}{\beta}\right)^{\frac{\nu}{2}}\frac{r_0^{\nu-3}}{\nu}e^{\frac{\beta GmM}{r_0}}\frac{(\beta GmM)^3}{(\nu-1)(\nu-2)(\nu-3)}\phi\left(1, 4-\nu; -\frac{\beta GmM}{r_0}\right). \tag{2.7}$$

From (2.7) one gathers that poles emerge for any dimension ν, $\nu = 3$ included. Thus, appeal to dimensional regularization (DR) is mandatory. To this effect we use the DR-Bollini @ Giambiagi's technique's generalization given in [4]. In a nut-shell the (DR) process consists in this procedure: if we have, for instance, an expression $F(\nu)$ that diverges, say, for $\nu = 3$, our Bollini–Giambiagi's DR generalization consists in performing the Laurent-expansion of F around $\nu = 3$ and select afterwards, as the physical result for F, the $\nu = 3$-independent term in the expansion. The justification for such a procedure is clearly explained in [4].

The $\mathcal{Z}_\nu$'s Laurent expansion reads

$$\mathcal{Z}_\nu = \frac{a_{-1}}{\nu-3} + a_0 + \sum_{n=0}^{\infty} a_n(\nu-3)^n. \tag{2.8}$$

Physically then, from $\mathcal{Z}$, DR selects the a_0-term, i.e.,

$$\begin{aligned}\mathcal{Z} = a_0 = &-\frac{1}{3\sqrt{\pi}}(2\pi^2\beta G^2m^3M^2)^{\frac{3}{2}}\left[\ln(2\pi^2\beta G^2m^3M^2) - \mathbf{C} - \frac{17}{3}\right] + \\ &\frac{4}{3\sqrt{\pi}}\left(\frac{2\pi^2 m}{\beta}\right)^{\frac{3}{2}} r_0^3 e^{\frac{\beta GmM}{r_0}}\left[1 + \frac{\beta GmM}{2r_0} + \frac{(\beta GmM)^2}{2r_0^2}\right] + \\ &\frac{(2\pi^2\beta G^2m^3M^2)^{\frac{3}{2}}}{3\sqrt{\pi}}\left[\ln\left(\frac{2\pi^2 m r_0^2}{\beta}\right) + \mathbf{C} + 2\ln 2 - \frac{17}{3}\right] - \\ &\frac{2}{3\sqrt{\pi}} e^{\frac{\beta GmM}{r_0}}\left[(2\pi^2\beta G^2m^3M^2)^{\frac{3}{2}}\phi^{(1)}\left(1, 4-\nu; -\frac{\beta GmM}{r_0}\right)\right]_{\nu=3}\end{aligned} \tag{2.9}$$

where $\phi^{(1)}$ denotes derivative of ϕ with respect to $4-\nu$ [64].

We now analyse the 4 lines that make up Eq. (2.9) for very large r_0. In such an instance, the first line is constant, the second line grows as r_0^3, the third line grows logarithmically, and the fourth one is constant. Thus, when we pass to the limit of very large r_0 we find

$$\mathcal{Z} = \frac{4}{3\sqrt{\pi}}\left(\frac{2\pi^2 m}{\beta}\right)^{\frac{3}{2}} r_0^3. \tag{2.10}$$

We consider next the mean energy $\langle U\rangle$ and face

$$\langle \mathcal{U}\rangle_\nu = \frac{1}{\mathcal{Z}}\int_{\mathcal{M}} e^{-\beta\left(\frac{p^2}{2m} - \frac{GmM}{r}\right)}\left(\frac{p^2}{2m} - \frac{GmM}{r}\right) d^\nu x d^\nu p. \tag{2.11}$$

We have to treat $\langle U\rangle$ now in identical manner as we did above with $\mathcal{Z}$. We do not give the pertinent details to save space. The ensuing result reads

$$\begin{aligned}\langle \mathcal{U}\rangle_\nu = &-\frac{\nu}{\beta\Gamma\left(\frac{\nu}{2}\right)}\cos\pi\nu(2\pi^2\beta G^2m^3M^2)^{\frac{\nu}{2}}\Gamma(-\nu) + \\ &\frac{1}{\beta\Gamma\left(\frac{\nu}{2}\right)}\left(\frac{2\pi^2 m}{\beta}\right)^{\frac{\nu}{2}} r_0^\nu e^{\frac{\beta GmM}{r_0}}\left[1 - \frac{\beta GmM}{(\nu-1)r_0} - \frac{(\beta GmM)^2}{(\nu-1)(\nu-2)r_0^2}\right] - \\ &\frac{1}{\beta\Gamma\left(\frac{\nu}{2}\right)}\left(\frac{2\pi^2 m}{\beta}\right)^{\frac{\nu}{2}} r_0^{\nu-3} e^{\frac{\beta GmM}{r_0}}\frac{(\beta GmM)^3}{(\nu-1)(\nu-2)(\nu-3)}\phi\left(1, 4-\nu; -\frac{\beta GmM}{r_0}\right).\end{aligned} \tag{2.12}$$

Again, poles in ν ensue and we need DR once again, that is, the Laurent series for $\langle\mathcal{U}\rangle_\nu$ around $\nu = 3$. We arrive at

$$\langle \mathcal{U}\rangle_\nu = \frac{1}{\mathcal{Z}}\left[\frac{b_{-1}}{\nu-3} + b_0 + \sum_{n=0}^{\infty} b_n(\nu-3)^n\right] \tag{2.13}$$

and the physical term for $\mathcal{U}$, the one independent of $\nu - 3$, is now

$$\begin{aligned}\langle \mathcal{U}\rangle = \frac{b_0}{\mathcal{Z}} = \frac{1}{\mathcal{Z}}\Bigg\{ &\frac{1}{2\beta\sqrt{\pi}}(2\pi^2\beta G^2m^3M^2)^{\frac{3}{2}}\left[\ln(2\pi^2\beta G^2m^3M^2) - \mathbf{C} - 5\right] + \\ &\frac{2}{\beta\sqrt{\pi}}\left(\frac{2\pi^2 m}{\beta}\right)^{\frac{3}{2}} r_0^3 e^{\frac{\beta GmM}{r_0}}\left[1 - \frac{\beta GmM}{2r_0} - \frac{(\beta GmM)^2}{2r_0^2}\right] + \\ &\frac{(2\pi^2\beta G^2m^3M^2)^3}{2\beta\sqrt{\pi}}\left[\ln\left(\frac{8\pi^2 m r_0^2}{\beta}\right) + \mathbf{C} - 5\right] - \\ &\frac{1}{\beta\sqrt{\pi}} e^{\frac{\beta GmM}{r_0}}\left[(2\pi^2\beta G^2m^3M^2)^{\frac{3}{2}}\phi^{(1)}\left(1, 4-\nu; -\frac{\beta GmM}{r_0}\right)\right]_{\nu=3}\Bigg\}.\end{aligned} \tag{2.14}$$

Proceeding similarly to what was done with $\mathcal{Z}$ we have for very large r_0:

$$\langle \mathcal{U} \rangle = \frac{3}{2\beta}. \tag{2.15}$$

The following abbreviation is useful:

$$\mathcal{Z} = \alpha r_0^3 \quad ; \quad \alpha = \frac{4}{3\sqrt{\pi}} \left(\frac{2\pi^2 m}{\beta} \right)^{\frac{3}{2}}. \tag{2.16}$$

The entropy in the canonical ensemble reads now [62]

$$\mathcal{S} = \ln \mathcal{Z} + \beta \langle \mathcal{U} \rangle = \ln \mathcal{Z} + \frac{3}{2} \tag{2.17}$$

for very large r_0. Verlinde's entropic force is defined as

$$F_e = -\frac{\lambda}{\beta} \frac{\partial \mathcal{S}}{\partial r_0} = -\frac{\lambda}{\beta \mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial r_0}, \tag{2.18}$$

i.e.,

$$F_e = -\frac{3\lambda}{\beta r_0} = -\frac{GmM}{r_0}, \tag{2.19}$$

with λ

$$\lambda = \frac{\beta GmM}{3}. \tag{2.20}$$

We realize that (2.19) does have for F_e the form, at large distance, conjectured by Verlinde, QED. This is our main conclusion in the present Communication.

3. The planar case

In two dimensions one faces

$$\mathcal{Z}_2 = \int_{\mathcal{M}} e^{-\beta\left(\frac{p^2}{2m} + 2GmM \ln r\right)} d^2x d^2p, \tag{3.1}$$

where $\mathcal{M} = R^2 \oplus S^2(r_0)$ and $S^2(r_0)$ is the spherical volume of radius r_0. Using polar coordinates one has

$$\mathcal{Z}_\nu = 4\pi^2 \int_0^\infty e^{-\beta \frac{p^2}{2m}} pdp \int_0^{r_0} e^{-2\beta GmM \ln r} rdr \tag{3.2}$$

or

$$\mathcal{Z} = 4\pi^2 \int_0^\infty e^{-\beta \frac{p^2}{2m}} pdp \int_0^{r_0} r^{1-2\beta GmM} dr. \tag{3.3}$$

The first integral on the r.h.s. of (3.3) is straightforward. The second needs appeal to the integral regularization technique of I. M. Guelfand in Vol. 1 of his treatise [65]. This leads to

$$\int_0^{r_0} r^{1-2\beta GmM} dr = \frac{r_0^{2-2\beta GmM}}{2 - 2\beta GmM} \quad ; \quad 2 - 2\beta GmM \neq 0. \tag{3.4}$$

The planar partition function becomes

$$\mathcal{Z} = \frac{4\pi^2 m}{\beta} \frac{r_0^{2-2\beta GmM}}{2 - 2\beta GmM}, \tag{3.5}$$

while $\langle U \rangle$ is

$$\langle \mathcal{U} \rangle = \frac{1}{\mathcal{Z}} \int_M e^{-\beta\left(\frac{p^2}{2m} + 2Gmm \ln r\right)} \left(\frac{p^2}{2m} + 2GmM \ln r \right) d^\nu x d^\nu p. \tag{3.6}$$

We follow the steps of the preceding Section and obtain for the partition function times $\langle U \rangle$

$$\mathcal{Z}\langle \mathcal{U} \rangle = \frac{4\pi^2 m}{\beta^2} \frac{r_0^{2-2\beta GmM}}{2 - 2\beta GmM} \left[1 + 2\beta GmM \left(\ln r_0 - \frac{1}{2 - 2\beta GmM} \right) \right], \tag{3.7}$$

or

$$\langle \mathcal{U} \rangle = \frac{1}{\beta} \left[1 + 2\beta GmM \left(\ln r_0 - \frac{1}{2 - 2\beta GmM} \right) \right]. \tag{3.8}$$

From large r_0 one finds, from (3.8):

$$\langle \mathcal{U} \rangle = \frac{1}{\beta}\left(1 + 2\beta GmM \ln r_0\right). \tag{3.9}$$

Passing to the entropic force we face

$$F_e = -\frac{2\lambda}{\beta r_0} = -\frac{GmM}{r_0}, \tag{3.10}$$

with

$$\lambda = \frac{\beta GmM}{2}. \tag{3.11}$$

The statistically averaged planar entropic force's behaviour coincides with that of the three-dimensional one at very large r. This fact might tempt one to conjecture that, at large distances, the mass-distribution should be planar.

4. Conclusions

Two inspiring Verlinde's conjectures regarding the gravitational interaction have been proved at a classical statistical level.

First, that it is an emergent force derived from entropy, proved in [2]. Second, proved here, that at very large distances the interaction decays as $1/r$ and not as $1/r^2$.

Verlinde has revolutionized our conception of gravity. Here we have contributed our grain of sand to such revolution.

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