European Journal of Physics

PAPER

On the straightforward perturbation theory in classical mechanics

To cite this article: Paolo Amore and Francisco M Fernández 2018 *Eur. J. Phys.* **39** 055001

IOP Publishing European Journal of Physics
Eur. J. Phys. 39 (2018) 055001 (9pp) https://doi.org/10.1088/1361-6404/aac358

On the straightforward perturbation theory in classical mechanics

Paolo Amore[1] and **Francisco M Fernández**[2,3]

[1] Facultad de Ciencias, Universidad de Colima, Bernal Díaz del Castillo 340, Colima, Colima, Mexico
[2] INIFTA (CONICET), División Química Teórica, Blvd. 113 S/N, Sucursal 4, Casilla de Correo 16, 1900 La Plata, Argentina

E-mail: paolo.amore@gmail.com and framfer@gmail.com

Received 21 March 2018, revised 19 April 2018
Accepted for publication 9 May 2018
Published 7 June 2018

Abstract
We show that it is possible to extract useful information from the straightforward perturbation theory in classical mechanics. Although the secular terms make the perturbation series useless for large times, these expansions yield the perturbation corrections for the period exactly and may even be useful for sufficiently short times. We think that the present analysis may be suitable for an advanced undergraduate course on classical mechanics.

Keywords: anharmonic oscillator, perturbation theory, secular terms, Lindstedt–Poincaré

(Some figures may appear in colour only in the online journal)

1. Introduction

Straightforward application of perturbation theory to classical periodic systems gives rise to the so-called secular terms, unbounded perturbation corrections that spoil the periodic behaviour at sufficiently large times. Such unwanted terms come from resonant contributions to the perturbation equations. Most textbooks on classical mechanics discuss this problem in detail and describe alternative perturbation approaches free from secular terms [1, 2]. One such approach is the Lindstedt–Poincaré method, which consists of explicitly introducing the oscillation frequency into the equations of motion and setting its perturbation coefficients to remove the resonant terms. In this way, one obtains the perturbation corrections to the trajectory and to the frequency at the same time [1, 2].

[3] Author to whom any correspondence should be addressed.

The purpose of this paper is to investigate the possibility of extracting useful information from the perturbation series with secular terms. We believe that the analysis of this old problem from another perspective may be beneficial for undergraduate students taking an introductory course in classical mechanics.

In section 2 we introduce a simple model and convert the equation of motion into a dimensionless differential equation. In this way, we reduce the number of independent model parameters. We show the exact expression for the dimensionless period. In section 3 we derive the straightforward perturbation series with secular terms from the corresponding equations in matrix form. In section 4 we outline the Lindstedt–Poincaré method and obtain a perturbation series free from the unwanted secular terms (again using the matrix approach). In section 5 we compare the results of both series and show that the perturbation expansion with secular terms provides some useful information about the periodic system. Finally, in section 6 we comment on the main results of the paper and draw conclusions.

2. The model

For simplicity and concreteness we consider a particle of mass m under the effect of the anharmonic potential

$$V(x) = \frac{k}{2}x^2 + \frac{\kappa}{4}x^4. \tag{1}$$

The equation of motion is

$$m\ddot{x}(t) = -kx(t) - \kappa x(t)^3, \tag{2}$$

where an overdot indicates differentiation with respect to time, and we assume the initial conditions $x(0) = x_0$, $\dot{x}(0) = 0$.

If we define

$$\tau = \Omega_0 t, \;\; \Omega_0 = \sqrt{\frac{k}{m}}, \; q = \frac{x}{x_0}, \tag{3}$$

the equation of motion (2) can be rewritten as

$$q''(\tau) = -q(\tau) - \lambda q(\tau)^3, \;\; \lambda = \frac{\kappa}{k}x_0^2, \tag{4}$$

where a prime indicates differentiation with respect to the dimensionless time τ and λ is a dimensionless perturbation parameter. The initial conditions become $q(0) = 1$ and $q'(0) = 0$.

It is well known that the Duffing equations (2) and (4) can be solved exactly in terms of elliptic functions [2]. For example, we easily obtain the following expression for the dimensionless period:

$$T = \frac{4\sqrt{2}}{\sqrt{\lambda + 2}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 + \frac{\lambda}{2+\lambda}\sin^2\theta}}, \tag{5}$$

which will be useful for testing the performance of the approximate methods based on perturbation theory.

3. Straightforward perturbation theory

In order to apply perturbation theory we rewrite the dimensionless equation of motion (4) as a system of two first-order differential equations:

$$\begin{aligned} q' &= v, \\ v' &= -q - \lambda q^3. \end{aligned} \tag{6}$$

If we expand

$$\begin{aligned} q(\tau) &= \sum_{j=0}^{\infty} q_j(\tau)\lambda^j, \\ v(\tau) &= \sum_{j=0}^{\infty} v_j(\tau)\lambda^j, \end{aligned} \tag{7}$$

we obtain the perturbation equations

$$\begin{aligned} q_n' &= v_n, \\ v_n' &= -q_n - (q^3)_{n-1}, \ \ (q^3)_{n-1} = \sum_{j=0}^{n-1} q_{n-1-j} \sum_{k=0}^{j} q_k q_{j-k}. \end{aligned} \tag{8}$$

It is convenient to rewrite these equations in matrix form:

$$\begin{aligned} &\mathbf{X}_n' = \mathbf{K}\mathbf{X}_n + \mathbf{R}_n, \\ &\mathbf{X}_n = \begin{pmatrix} q_n \\ v_n \end{pmatrix}, \ \mathbf{K} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \mathbf{R}_n = \begin{pmatrix} 0 \\ -(q^3)_{n-1} \end{pmatrix}, \end{aligned} \tag{9}$$

because its solution is straightforward:

$$\mathbf{X}_n(\tau) = \exp(\mathbf{K}\tau)\left[\mathbf{W}_n + \int_0^{\tau} \exp(-\mathbf{K}s)\mathbf{R}_n\right], \ \ \mathbf{W}_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}, \tag{10}$$

where the column vector $\mathbf{W}_n$ provides the initial conditions $\mathbf{X}_n(0)$ of the system of first-order differential equations. The analytical calculation of

$$\exp(\mathbf{K}\tau) = \begin{pmatrix} \cos(\tau) & \sin(\tau) \\ -\sin(\tau) & \cos(\tau) \end{pmatrix} \tag{11}$$

offers no difficulty [3, 4]. At order zero we have

$$q_0(\tau) = \cos(\tau), \ \ v_0 = -\sin(\tau), \tag{12}$$

which is consistent with the initial conditions $q(0) = 1$ and $v(0) = 0$ provided that $a_n = b_n = 0$ for all $n > 0$.

The first two perturbation corrections for the dimensionless coordinate $q(\tau)$ and velocity $v(\tau)$,

$$q_1(\tau) = \frac{\cos(3\tau)}{32} - \frac{\cos(\tau)}{32} - \frac{3\tau\sin(\tau)}{8},$$
$$q_2(\tau) = \frac{\cos(5\tau)}{1024} - \frac{3\cos(3\tau)}{128} + \frac{23\cos(\tau)}{1024} + \tau\left[\frac{3\sin(\tau)}{32} - \frac{9\sin(3\tau)}{256}\right] - \frac{9\tau^2\cos(\tau)}{128},$$
$$v_1(\tau) = -\frac{11\sin(\tau)}{32} - \frac{3\sin(3\tau)}{32} - \frac{3\tau\cos(\tau)}{8},$$
$$v_2(\tau) = \frac{73\sin(\tau)}{1024} + \frac{9\sin(3\tau)}{256} - \frac{5\sin(5\tau)}{1024} - \frac{3\tau\cos(\tau)}{64} - \frac{27\tau\cos(3\tau)}{256} + \frac{9\tau^2\sin(\tau)}{128}, \quad (13)$$

exhibit secular terms proportional to powers of τ that are unbounded and in principle inconsistent with a periodic motion [1, 2].

4. The Lindstedt–Poincaré method

One of the standard strategies for removing the secular terms from the straightforward perturbation theory is the Lindstedt–Poincaré method [1, 2], which consists of defining a new time variable $s = \omega\tau$, where ω is the dimensionless frequency of the periodic motion (if Ω is the actual frequency then $\omega = \Omega/\Omega_0$). In this way the equations of motion (6) become

$$\omega\xi'(s) = \eta(s),$$
$$\omega\eta'(s) = -\xi(s) - \lambda\xi(s)^3, \quad (14)$$

where $\xi(s) = q(s/\omega)$ and $\eta(s) = v(s/\omega)$.

If we expand

$$\xi(s) = \sum_{j=0}^{\infty} \xi_j(s)\lambda^j,$$
$$\eta(s) = \sum_{j=0}^{\infty} \eta_j(s)\lambda^j,$$
$$\omega(s) = \sum_{j=0}^{\infty} \omega_j(s)\lambda^j, \quad \omega_0 = 1, \quad (15)$$

the perturbation equations become

$$\xi_n' = \eta_n - \sum_{j=1}^{n} \omega_j(s)\xi_{n-j}',$$
$$\eta_n' = -\xi_n - (\xi^3)_{n-1} - \sum_{j=1}^{n} \omega_j(s)\eta_{n-j}'. \quad (16)$$

We can solve these equations very easily in matrix form by just modifying the column vector $\mathbf{R}_n$ in equation (10) because the matrix $\mathbf{K}$ is exactly the same.

The resulting equations depend on the expansion coefficients ω_j that we choose so that the secular terms vanish [1, 2]. In this way we obtain

$$\xi_1(s) = -\frac{1}{32}[\cos(s) + \cos(3s)],$$
$$\eta_1(s) = -\frac{1}{32}[11\sin(s) + 3\sin(3s)],$$
$$\xi_2(s) = \frac{1}{1024}[23\cos(s) - 24\cos(3s) + \cos(5s)],$$
$$\eta_2(s) = \frac{1}{1024}[73\sin(s) + 36\sin(3s) - 5\sin(5s)], \quad (17)$$

which are periodic because the perturbation coefficients for the frequency

$$\omega_1 = \frac{3}{8},$$
$$\omega_2 = -\frac{21}{256}, \quad (18)$$

were chosen in such a way that the coefficients of the secular terms vanish. We do not expand upon this particular point because it is discussed in detail in most textbooks [1, 2].

Equations (17) suggest that

$$\xi_j(s) = \sum_{k=0}^{j} a_{kj}\cos[(2k+1)s],$$
$$\eta_j(s) = \sum_{k=0}^{j} b_{kj}\sin[(2k+1)s]. \quad (19)$$

For example, the coefficients for the next two perturbation corrections are

$$a_{03} = -\frac{547}{32768},\ a_{13} = \frac{297}{16384},\ a_{23} = -\frac{3}{2048},\ a_{33} = \frac{1}{32768},$$
$$b_{03} = -\frac{1109}{32768},\ b_{13} = -\frac{333}{16384},\ b_{23} = \frac{45}{8192},\ b_{33} = -\frac{7}{32768},$$
$$a_{04} = \frac{6713}{524288},\ a_{14} = -\frac{15121}{1048576},\ a_{24} = \frac{883}{524288},\ a_{34} = -\frac{9}{131072},$$
$$a_{44} = \frac{1}{1048576},$$
$$b_{04} = \frac{11281}{524288},\ b_{14} = \frac{14043}{1048576},\ b_{24} = -\frac{2765}{524288},\ b_{34} = \frac{105}{262144},$$
$$b_{44} = -\frac{9}{1048576}, \quad (20)$$

provided that

$$\omega_3 = \frac{81}{2048},$$
$$\omega_4 = -\frac{6549}{262144}. \quad (21)$$

In order to compare equations (13) and (17) we should take into account that $s = \omega\tau$, where $\omega \approx 1 + \omega_1\lambda + \omega_2\lambda^2$ at second order. Clearly, the period of the perturbation corrections (17) in terms of the dimensionless time τ is $T = 2\pi/\omega$ and its λ-power series agrees with the one derived from the exact expression (5).

5. Comparison of the perturbation results

In order to compare the perturbation approaches with and without the secular terms we define the partial sums

$$q^{[N]}(\tau) = \sum_{j=0}^{N} q_j(\tau)\lambda^j, \tag{22}$$

and similar expressions for v, ξ, η, etc. The exact solution to the dimensionless Duffing equation (4) satisfies $q(\tau + T) = q(\tau)$, where T is the dimensionless period of the motion. In particular, $q(T) - q(0) = 0$ appears to be a suitable equation for the approximate calculation of the period from an approximate representation of $q(\tau)$ that is reasonably accurate in the neighbourhood of $\tau = 0$. Since our approximate expressions are λ-power series it seems reasonable to try to estimate the perturbation corrections to the dimensionless period T from the equation

$$q^{[N]}(T^{[N]}) - q^{[N]}(0) = \mathcal{O}(\lambda^{N+1}),$$
$$T^{[N]} = \sum_{j=0}^{N} T_j\lambda^j, \quad T_0 = 2\pi. \tag{23}$$

The calculation exhibits some surprising features; for example

$$q^{[3]}(T^{[1]}) - q^{[3]}(0) = \mathcal{O}(\lambda^4), \tag{24}$$

with just

$$T_1 = -\frac{3\pi}{4}. \tag{25}$$

On the other hand, from

$$v^{[4]}(T^{[4]}) - v^{[4]}(0) = \mathcal{O}(\lambda^5), \tag{26}$$

we obtain (25) as well as

$$T_2 = \frac{57\pi}{128},$$
$$T_3 = -\frac{315\pi}{1024},$$
$$T_4 = \frac{30345\pi}{131072}. \tag{27}$$

We conclude that it is more efficient to obtain the expansion for the period from $v^{[N]}(\tau)$ although the results discussed below clearly show that $q^{[N]}(\tau)$ is more accurate for this model. In any case, we appreciate that the unwanted secular terms in the perturbation series do not prevent us from obtaining the correct expansion for the period. Note that the perturbation corrections ω_j obtained above by means of the Lindstedt–Poincaré method are related to the perturbation corrections T_k obtained here by

$$\frac{2\pi}{\omega^{[N]}} = T^{[N]} + \mathcal{O}(\lambda^{N+1}), \tag{28}$$

and both agree with the Taylor expansion of the exact expression (5) about $\lambda = 0$. Since the perturbation series with secular terms yield the correct perturbation series for the period it appears to be reasonable to argue that they may be sufficiently accurate in the interval $0 \leqslant \tau \lesssim T$.

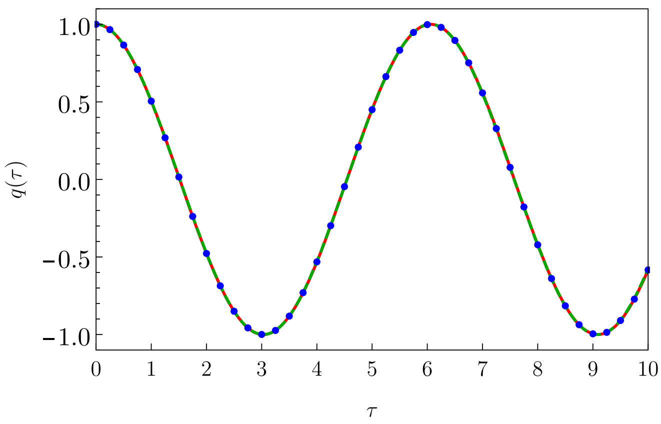

Figure 1. Exact $q(\tau)$ (solid circles, blue), perturbation theory with secular terms (dashed line, green) and Lindstedt–Poincaré series (solid line, red) for $\lambda = 0.1$.

In order to verify the conclusions drawn above we calculated $q^{[2]}(\tau)$ and $v^{[2]}(\tau)$ with both perturbation approaches and compared them with the exact result (obtained numerically with sufficient precision) for moderate times (not much larger than T). Figure 1 shows that the agreement of both perturbation series for $q(\tau)$ is remarkable for $\lambda = 0.1$. This fact is not surprising because the chosen strength λ of the nonlinear term is rather small. The upper panel of figure 2 shows that the Lindstedt–Poincaré series is clearly more accurate than the series with secular terms for $\lambda = 0.5$. The conclusion is that the former series deteriorates less noticeably than the latter as λ increases. The lower panel of figure 2 shows $v^{[2]}$ for $\lambda = 0.5$. Again we appreciate that the Lindstedt–Poincaré series is more accurate than the series with secular terms, but if we compare both panels we conclude that the series with secular terms yields more accurate results for $q^{[2]}(\tau)$ than for $v^{[2]}(\tau)$. However, as argued above, it is more efficient to use the secular series for $v^{[N]}(\tau)$ than the one for $q^{[N]}(\tau)$ with the purpose of calculating the coefficients of the perturbation series for the period.

6. Conclusions

It is clear that perturbation series with secular terms are unsuitable for the description of the periodic motion of an oscillator for sufficiently large τ, and for this reason one commonly resorts to the Lindstedt–Poincaré expansion. This well-known fact is discussed in most textbooks on classical mechanics [1, 2]. However, the former series provide considerable information about the dynamics of the oscillator. We have shown that they yield the exact perturbation expansion for the period and, if λ is not too large, they also give a reasonable estimate of the motion when $0 \leqslant \tau \leqslant T$. In such a case we can use these results for all τ because we know that the exact coordinate satisfies $q(\tau + T) = q(\tau)$.

We think that the present discussion of the perturbation series for a simple model may be of interest in an undergraduate course on classical mechanics. Since the calculation of perturbation corrections of orders larger than the second one by hand may be extremely tedious and error prone, the present pedagogical proposal may be suitable for stimulating the application of available computer algebra systems. Reasonable mastering of such software is certainly beneficial for physics students because it enables them to try somewhat complicated

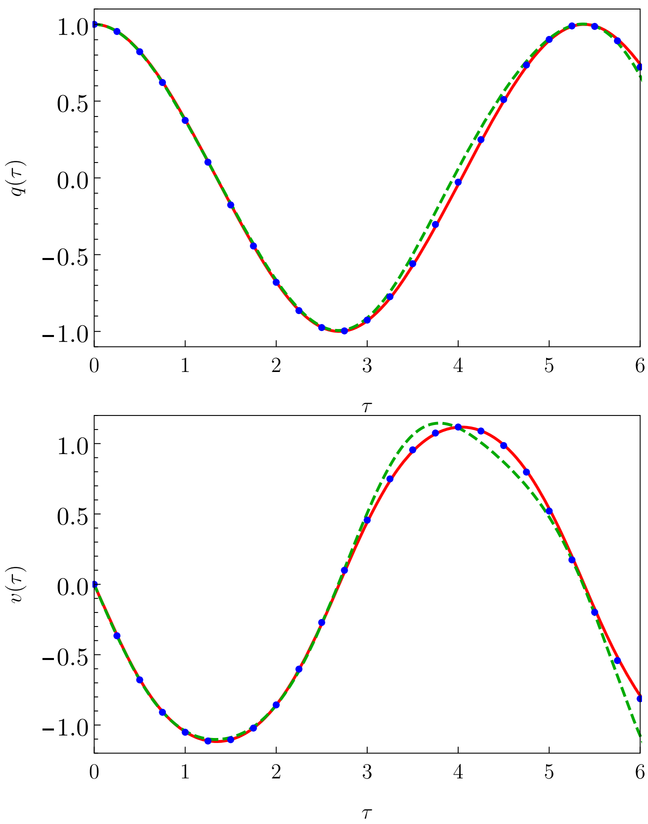

Figure 2. Exact $q(\tau)$ (solid circles, blue), perturbation theory with secular terms (dashed line, green) and Lindstedt–Poincaré series (solid line, red) for $\lambda = 0.5$ and $v(\tau)$ (lower figure) for $\lambda = 0.5$.

analytical calculations without spending too much time. The matrix method outlined in this paper is particularly useful for the application of computer algebra systems.

As already indicated above, the aim of this paper is just to compare the straightforward perturbation theory with an improved one. It is well known that there are other approaches for the treatment of dynamical systems that the students may also try. In the course one can also introduce the methods of renormalization, multiple scales, averaging or the variation of parameters that appear in well-known available textbooks [2]. One can also mention a pedagogical approach to the renormalization group [5].

ORCID iDs

Paolo Amore https://orcid.org/0000-0002-1321-9437
Francisco M Fernández https://orcid.org/0000-0003-0393-790X

References

[1] Marion J B 1965 *Classical Dynamics of Particles and Systems* (New York: Academic)
[2] Nayfeh A H 1981 *Introduction to Perturbation Techniques* (New York: Wiley)

[3] Apostol T M 1969 *Calculus* (Waltham: Blaisdell)
[4] Fernández F M and Castro E A 1996 *Algebraic Methods in Quantum Chemistry and Physics* (Boca Raton, FL: CRC Press, Inc)
[5] Kirkinis E 2012 *SIAM Rev.* **54** 374