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Dimensional regularization of Renyi's statistical mechanics

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HIGHLIGHTS

- We study the q-partition function of the Harmonic Oscillator.
- We study the poles of this q-partition function in phase space.
- We use Dimensional Regularization of Bollini and Giambiagi.
- We find interesting gravitational effects.

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ABSTRACT

We show that typical Renyi's statistical mechanics' quantifiers exhibit poles. We are referring to the partition function $\mathcal{Z}$ and the mean energy $\langle\mathcal{U}\rangle$. Renyi's entropy is characterized by a real parameter α. The poles emerge in a numerable set of rational numbers belonging to the α-line. Physical effects of these poles are studied by appeal to dimensional regularization, as usual. Interesting effects are found, as for instance, gravitational ones. In particular, negative specific heats.

1. Introduction

Renyi's information measure S_R is a generalization of both Hartley's and Shannon's entropic quantifiers of our ignorance regarding a system's structural characteristics. S_R is regarded as a quite important measure in several science's areas. We may cite, for example, ecology, quantum information, Heisenberg's XY spin chain model, theoretical computing, conformal field theory, quantum quenching, diffusion processes, etc. (See [1–12]).

The Renyi entropy is also relevant in statistics as signaling diversity and complexity measure [13,14]. S_R is defined as [1]:

$$S_R = \frac{1}{1-\alpha} \ln \left(\int_M P^\alpha d\mu \right). \tag{1.1}$$

where P is probability's density and M is the manifold where the integral is defined.

We will investigate here poles that emerge in computing the most important Renyi's statistical quantities for the harmonic oscillator (HO). We wish to ascertain the physical significance of these poles.

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To such an end we appeal to the dimensional regularization methodology developed by Bollini and Giambiagi [15–18], and by 't Hooft and Veltman [19]. plus its generalization, developed in [20].

Dimensional regularization is one of the most important theoretical achievements of contemporary physics, that has generated thousands of papers in variegated branches of physics. We mention that Ref. [21] purported to place upper bounds on Renyi entropy outside the poles.

Why the HO? We do **not** mean to unravel HO's peculiarities here. This is a very well known system already, of course, described by $\alpha = 1$ (when S_R becomes Shannon's entropy). We use the HO because of its simplicity, so that closed-form formulas become available. This enormously facilitates our pole-research and we thus obtain indications as of how to proceed in more complex situations. This work is an unavoidable preliminary step to be taken before tackling such situations.

We will separately treat the one, two, and three dimensions cases, as the poles are different for each dimension. We start below with some general considerations and will heavily rely on Ref. [22], which should be recommended as a useful prerequisite due to Tsallis and Renyi entropies have similar probabilities distributions.

2. Theoretical considerations for the α-region outside the poles

2.1. $\alpha > 1$

We showed in [22] that the classical Renyi-HO partition function is, for $\alpha > 1$,

$$\mathcal{Z} = \frac{\pi^\nu}{\Gamma(\nu)} \int_0^\infty \frac{u^{\nu-1}}{[1+\beta(1-\alpha)u]_+^{\frac{1}{1-\alpha}}} du. \tag{2.1}$$

where ν is the space dimension and $u = p^2 + q^2$. Here $\beta = \frac{1}{kT}$ and k is the Boltzmann's constant. The integral therein involved is

$$\mathcal{Z} = \frac{\pi^\nu}{[\beta(\alpha-1)]^\nu} \frac{\Gamma\left(\frac{\alpha}{\alpha-1}\right)}{\Gamma\left(\frac{\alpha}{\alpha-1}+\nu\right)} \tag{2.2}$$

Here we have used the result of Ref. [23]:

$$\int_0^u x^{\nu-1}(u-x)^{\mu-1}dx = u^{\mu+\nu-1}B(\mu,\nu)$$

Also, for the mean energy $\langle \mathcal{U} \rangle$, we have from [22]

$$\langle \mathcal{U} \rangle = \frac{\pi^\nu}{\Gamma(\nu)\mathcal{Z}} \int_0^\infty \frac{u^\nu}{[1+\beta(1-\alpha)u]_+^{\frac{1}{1-\alpha}}} du, \tag{2.3}$$

whose result is

$$\langle \mathcal{U} \rangle = \frac{\nu\pi^\nu}{\mathcal{Z}[\beta(\alpha-1)]^{\nu+1}} \frac{\Gamma\left(\frac{\alpha}{\alpha-1}\right)}{\Gamma\left(\frac{\alpha}{\alpha-1}+\nu+1\right)}, \tag{2.4}$$

or equivalently,

$$\langle \mathcal{U} \rangle = \frac{\nu}{\beta[\alpha+\nu(\alpha-1)]}. \tag{2.5}$$

The entropy can be expressed via $\mathcal{Z}$ and $\langle \mathcal{U} \rangle$ as (see [22])

$$\mathcal{S} = \ln \mathcal{Z} + \frac{1}{1-\alpha} \ln[1+(1-\alpha)\beta\langle \mathcal{U} \rangle]. \tag{2.6}$$

Using (2.2)–(2.5) we cast $\mathcal{S}$ as

$$\mathcal{S} = \ln\left\{\left[\frac{\pi}{\beta(\alpha-1)}\right]^\nu \frac{\Gamma\left(\frac{\alpha}{\alpha-1}\right)}{\Gamma\left(\frac{\alpha}{\alpha-1}+\nu\right)}\right\} + \frac{1}{1-\alpha} \ln\left[\frac{\alpha}{[\alpha+\nu(\alpha-1)]}\right]. \tag{2.7}$$

We gather from (2.2) that $\mathcal{Z}$ is positive and finite for $\alpha > 1$.

2.2. $0 < \alpha < 1$

Instead, for $0 < \alpha < 1$ one has

$$\mathcal{Z} = \frac{\pi^\nu}{\Gamma(\nu)} \int_0^\infty \frac{u^{\nu-1}}{[1+\beta(1-\alpha)u]^{\frac{1}{1-\alpha}}} du, \tag{2.8}$$

and

$$\mathcal{Z} = \frac{\pi^{\nu}}{[\beta(1-\alpha)]^{\nu}} \frac{\Gamma\left(\frac{1}{1-\alpha} - \nu\right)}{\Gamma\left(\frac{1}{1-\alpha}\right)}. \tag{2.9}$$

Let us pass now to the mean energy. For $\langle\mathcal{U}\rangle$ we have

$$\langle\mathcal{U}\rangle = \frac{\pi^{\nu}}{\Gamma(\nu)\mathcal{Z}} \int_0^{\infty} \frac{u^{\nu}}{[1+\beta(1-\alpha)u]^{\frac{1}{1-\alpha}}} du, \tag{2.10}$$

or

$$\langle\mathcal{U}\rangle = \frac{\nu\pi^{\nu}}{\mathcal{Z}[\beta(1-\alpha)]^{\nu+1}} \frac{\Gamma\left(\frac{1}{1-\alpha} - \nu - 1\right)}{\Gamma\left(\frac{1}{1-\alpha}\right)}, \tag{2.11}$$

that can be recast as

$$\langle\mathcal{U}\rangle = \frac{\nu}{\beta[\alpha - \nu(1-\alpha)]}. \tag{2.12}$$

Here

$$\mathcal{S} = \ln\left\{\left[\frac{\pi}{\beta(1-\alpha)}\right]^{\nu} \frac{\Gamma\left(\frac{1}{1-\alpha} - \nu\right)}{\Gamma\left(\frac{1}{1-\alpha}\right)}\right\} + \frac{1}{1-\alpha} \ln\left[\frac{\alpha}{[\alpha - \nu(1-\alpha)]}\right]. \tag{2.13}$$

Here we find poles in the partition function.

Outside these poles we should have

$$\frac{1}{1-\alpha} - \nu > 0$$

or

$$\frac{1}{1-\alpha} - \nu < 0 \quad ; \quad \Gamma\left(\frac{1}{1-\alpha} - \nu\right) > 0. \tag{2.14}$$

This is due to the fact that the partition function should be both positive and finite. For the second case we use the equality [23]:

$$\Gamma\left(\frac{1}{1-\alpha} - \nu\right) = -\frac{\pi}{\sin\pi\left(\nu - \frac{1}{1-\alpha}\right)\Gamma\left(\nu + 1 - \frac{1}{1-\alpha}\right)} \tag{2.15}$$

and ascertain that

$$\sin\pi\left(\nu - \frac{1}{1-\alpha}\right) < 0. \tag{2.16}$$

Thus,

$$2p+1 < \nu - \frac{1}{1-\alpha} < 2p+2 \quad ; \quad p = 0, 1, 2, 3, \ldots \tag{2.17}$$

This chain of inequalities shows that α and ν are related to each other.

3. The divergences of the theory

Remember beforehand the well known fact that a classical entropy is defined only up to an arbitrary constant. From (2.9), $\mathcal{Z}$' poles arise when the Gamma arguments become [24]

$$\frac{1}{1-\alpha} - \nu = -p \quad \text{for} \quad p = 0, 1, 2, 3, \ldots, \tag{3.1}$$

or, equivalently, for

$$\alpha = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{\nu-2}{\nu-1}, \frac{\nu-1}{\nu}. \tag{3.2}$$

For $\langle\mathcal{U}\rangle$'s poles we have

$$\frac{1}{1-\alpha} - \nu - 1 = -p \quad \text{for } p = 0, 1, 2, 3, \ldots, \tag{3.3}$$

or, equivalently,

$$\alpha = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{\nu-1}{\nu}, \frac{\nu}{\nu+1}. \tag{3.4}$$

4. The one-dimensional scenario

In one dimension $\mathcal{Z}$ is regular and $\langle\mathcal{U}\rangle$ has a singularity at $\alpha = \frac{1}{2}$. For $\alpha \neq \frac{1}{2}$, $\mathcal{Z}$ and $\langle\mathcal{U}\rangle$ can be easily evaluated. The result is straightforward

$$\mathcal{Z} = \frac{\pi}{\beta\alpha}, \tag{4.1}$$

$$\langle\mathcal{U}\rangle = \frac{1}{\beta(2\alpha-1)}. \tag{4.2}$$

As a consequence, we have for $\mathcal{S}$

$$\mathcal{S} = \ln\left(\frac{\pi}{\beta\alpha}\right) + \frac{1}{1-\alpha}\ln\left(\frac{\alpha}{2\alpha-1}\right). \tag{4.3}$$

When $\alpha = \frac{1}{2}$, we have for $\mathcal{Z}$

$$\mathcal{Z} = \frac{2\pi}{\beta}, \tag{4.4}$$

a regular value. Regularization is needed then only for $\langle\mathcal{U}\rangle$.

4.1. Dealing with the divergences

In order to proceed with such regularizing procedure, the main idea is to write $\langle\mathcal{U}\rangle$ as a function of the dimension ν in the fashion (from (2.11))

$$\langle\mathcal{U}\rangle_\nu = \frac{2^{\nu+1}\nu\pi^\nu}{\mathcal{Z}\beta^{\nu+1}}\Gamma(1-\nu), \tag{4.5}$$

and carefully dissect this expression by appealing to Laurent's expansion. We note first that [23]

$$\Gamma(1-\nu) = -\frac{1}{\nu-1} + \boldsymbol{C} + \sum_{k=1}^{\infty} b_k(\nu-1)^k, \tag{4.6}$$

where $\boldsymbol{C}$ is Euler's constant. Let

$$f(\nu) = \frac{2^{\nu+1}\nu\pi^\nu}{\beta^{\nu+1}}. \tag{4.7}$$

The Laurent expansion of $f(\nu)$ in $\nu = 1$ is

$$f(\nu) = \frac{4\pi}{\beta^2} + \frac{4\pi}{\beta^2}\left[1 + \ln\left(\frac{2\pi}{\beta}\right)\right](\nu-1) + \sum_{k=2}^{\infty} c_k(\nu-1)^k. \tag{4.8}$$

Using (4.6)–(4.8) we obtain

$$\langle\mathcal{U}\rangle_\nu = \frac{1}{\mathcal{Z}}\left\{\frac{4\pi}{\beta^2(1-\nu)} + \frac{4\pi}{\beta^2}\left[\boldsymbol{C} - 1 - \ln\left(\frac{2\pi}{\beta}\right)\right] + \sum_{k=2}^{\infty} d_k(\nu-1)^k\right\}. \tag{4.9}$$

Use now the $\mathcal{Z}$ value of (4.4) and find

$$\langle\mathcal{U}\rangle_\nu = \frac{2}{\beta(1-\nu)} + \frac{2}{\beta}\left[\boldsymbol{C} - 1 - \ln\left(\frac{2\pi}{\beta}\right)\right] + \sum_{k=1}^{\infty} a_k(\nu-1)^k. \tag{4.10}$$

Dimensional regularization's prescriptions assert that the $\langle\mathcal{U}\rangle$-physical value is given by the $\nu-1$-independent term in

$$\langle\mathcal{U}\rangle = \frac{2}{\beta}\left[\boldsymbol{C} - 1 - \ln\left(\frac{2\pi}{\beta}\right)\right]. \tag{4.11}$$

Using then (4.4)–(4.11) we find

$$\mathcal{S} = \ln\left\{\frac{2\pi}{\beta}\left[\boldsymbol{C} - \ln\left(\frac{2\pi}{\beta}\right)\right]^2\right\}. \tag{4.12}$$

Note that, classically, $\mathcal{S}$ is defined up to an additive constant.

5. The two-dimensional case

For two dimensions, $\mathcal{Z}$ has a singularity at $\alpha = \frac{1}{2}$ and $\langle \mathcal{U} \rangle$ has singularities at $\alpha = \frac{1}{2}$ and $\alpha = \frac{2}{3}$. Save for the case of these singularities, we can evaluate their values of the main statistical quantities without the use of dimensional regularization. Thus, we obtain

$$\mathcal{Z} = \frac{\pi^2}{\beta^2 \alpha (2\alpha - 1)}, \tag{5.1}$$

$$\langle \mathcal{U} \rangle = \frac{2}{\beta(3\alpha - 2)}, \tag{5.2}$$

$$\mathcal{S} = \ln\left[\frac{\pi^2}{\beta^2\alpha(2\alpha - 1)}\right] + \frac{1}{1-\alpha}\ln\left(\frac{\alpha}{3\alpha - 2}\right). \tag{5.3}$$

5.1. The $\alpha = 1/2$ pole

For $\alpha = \frac{1}{2}$ we must employ the treatment of the preceding Section, i.e., regularize, both $\mathcal{Z}$ and $\mathcal{U}$. We start with $\mathcal{Z}$

$$\mathcal{Z} = \left(\frac{2\pi}{\beta}\right)^{\nu} \Gamma(2 - \nu). \tag{5.4}$$

We recall that

$$\Gamma\,(2-\nu) = -\frac{1}{\nu - 2} + \boldsymbol{C} + \sum_{k=1}^{\infty} b_k (\nu - 2)^k, \tag{5.5}$$

and define

$$f(\nu) = \frac{2^{\nu}\pi^{\nu}}{\beta^{\nu}}. \tag{5.6}$$

The associated Laurent expansion is

$$f(\nu) = \frac{4\pi^2}{\beta^2} + \frac{4\pi^2}{\beta^2}\ln\left(\frac{2\pi}{\beta}\right)(\nu - 2) + \sum_{k=2}^{\infty} c_k(\nu - 2)^k. \tag{5.7}$$

Using (5.5)–(5.7) we find

$$\mathcal{Z}_{\nu} = -\frac{4\pi^2}{\beta^2(\nu - 2)} + \frac{4\pi^2}{\beta^2}\left[\boldsymbol{C} - \ln\left(\frac{2\pi}{\beta}\right)\right] + \sum_{k=1}^{\infty} a_k(\nu - 2)^k, \tag{5.8}$$

and the physical value for he partition function becomes

$$\mathcal{Z} = \frac{4\pi^2}{\beta^2}\left[\boldsymbol{C} - \ln\left(\frac{2\pi}{\beta}\right)\right]. \tag{5.9}$$

Since $\mathcal{Z}$ must be positive, we find the following upper bound for T

$$T < \frac{e^{\boldsymbol{C}}}{2\pi k} \tag{5.10}$$

For $\mathcal{U}$ the situation is similar. From (3.4) we have

$$\langle \mathcal{U} \rangle = \frac{\nu}{\mathcal{Z}\pi}\left(\frac{2\pi}{\beta}\right)^{\nu+1}\Gamma(1-\nu). \tag{5.11}$$

Define

$$f(\nu) = \frac{\nu 2^{\nu+1}\pi^{\nu}}{\beta^{\nu+1}(1-\nu)}. \tag{5.12}$$

The pertinent Laurent expansion is

$$f(\nu) = -\frac{16\pi^2}{\beta^2} + \frac{16\pi^2}{\beta^2}\left[\frac{1}{2} - \ln\left(\frac{2\pi}{\beta}\right)\right](\nu - 2) + \sum_{k=2}^{\infty} c_k(\nu - 2)^k. \tag{5.13}$$

From (5.5)–(5.13) we find

$$\langle \mathcal{U} \rangle_{\nu} = \frac{1}{\mathcal{Z}}\left\{\frac{16\pi^2}{\beta^3(\nu - 2)} + \frac{16\pi^2}{\beta^3}\left[\ln\left(\frac{2\pi}{\beta}\right) - \boldsymbol{C} - \frac{1}{2}\right] + \sum_{k=1}^{\infty} a_k(\nu - 2)^k\right\}. \tag{5.14}$$

Thus, $\mathcal{U}$'s physical value is (remember (5.10))

$$\langle\mathcal{U}\rangle = \frac{4}{\beta}\,\frac{\ln\left(\frac{2\pi}{\beta}\right) - \boldsymbol{C} - \frac{1}{2}}{\boldsymbol{C} - \ln\left(\frac{2\pi}{\beta}\right)}. \tag{5.15}$$

Using (5.9)–(5.15) we find

$$\mathcal{S} = \ln\left\{\frac{4\pi^2}{\beta^2}\,\frac{\left[\ln\left(\frac{2\pi}{\beta}\right) - \boldsymbol{C} - 1\right]^2}{\boldsymbol{C} - \ln\left(\frac{2\pi}{\beta}\right)}\right\}. \tag{5.16}$$

5.2. The $\alpha = 2/3$ pole

For $\alpha = \frac{2}{3}$, $\mathcal{Z}$ is finite and $\langle\mathcal{U}\rangle$ has a pole. The procedure for finding their physical values is similar to that for the case $\alpha = \frac{1}{2}$. For this reason, we merely indicate the results obtained for $\mathcal{Z}$, $\langle\mathcal{U}\rangle$, and $\mathcal{S}$. One finds

$$\mathcal{Z} = \frac{9\pi^2}{2\beta^2}, \tag{5.17}$$

$$\langle\mathcal{U}\rangle = \frac{6}{\beta}\left[\boldsymbol{C} - \frac{1}{2} - \ln\left(\frac{3\pi}{\beta}\right)\right], \tag{5.18}$$

$$\mathcal{S} = \ln\left\{\frac{36\pi^2}{\beta^2}\left[\boldsymbol{C} - \ln\left(\frac{2\pi}{\beta}\right)\right]^3\right\}. \tag{5.19}$$

6. The three-dimensional instance

In three dimensions, $\mathcal{Z}$ has poles at $\alpha = \frac{1}{2}$ and $\alpha = \frac{2}{3}$ while $\langle\mathcal{U}\rangle$ exhibits them at $\alpha = \frac{1}{2}$, $\alpha = \frac{2}{3}$, and $\alpha = \frac{3}{4}$. Outside the poles one has for $\mathcal{Z}$, $\langle\mathcal{U}\rangle$, and $\mathcal{S}$, respectively,

$$\mathcal{Z} = \frac{\pi^3}{\beta^3\alpha(2\alpha-1)(3\alpha-2)}, \tag{6.1}$$

$$\langle\mathcal{U}\rangle = \frac{3}{\beta(4\alpha-3)}. \tag{6.2}$$

$$\mathcal{S} = \ln\left[\frac{\pi^3}{\beta^3\alpha(2\alpha-1)(3\alpha-2)}\right] + \frac{1}{1-\alpha}\ln\left(\frac{\alpha}{4\alpha-3}\right). \tag{6.3}$$

In this case α should satisfy the condition $\alpha < \frac{5}{4}$ for the mean energy to be a positive quantity.

6.1. The $\alpha = 1/2$ pole

For $\alpha = \frac{1}{2}$ we have

$$\mathcal{Z}_\nu = \left(\frac{2\pi}{\beta}\right)^\nu \Gamma(2-\nu). \tag{6.4}$$

The Laurent expansion is tackled as above. One finds

$$\mathcal{Z}_\nu = -\frac{8\pi^3}{\beta^3(\nu-3)} + \frac{8\pi^3}{\beta^3}\left[\ln\left(\frac{2\pi}{\beta} - 1 - \boldsymbol{C}\right)\right] + \sum_{k=1}^{\infty} a_k(\nu-3)^k. \tag{6.5}$$

From (6.5) it is easy to obtain the physical value of $\mathcal{Z}$ as

$$\mathcal{Z} = \frac{8\pi^3}{\beta^3}\left[\ln\left(\frac{2\pi}{\beta}\right) - 1 - \boldsymbol{C}\right]. \tag{6.6}$$

Since $\mathcal{Z}$ is positive, one is led to the bound

$$T > \frac{e^{\boldsymbol{C}+1}}{2\pi k}. \tag{6.7}$$

In a similar vein, we have for $\langle \mathcal{U} \rangle$

$$\langle \mathcal{U} \rangle = \frac{1}{\beta} \frac{\frac{7}{2} + 3\boldsymbol{C} - 3\ln\left(\frac{2\pi}{\beta}\right)}{\ln\left(\frac{2\pi}{\beta}\right) - \boldsymbol{C} - 1} \tag{6.8}$$

and from (6.6) and (6.8)

$$\mathcal{S} = \ln\left\{\frac{2\pi^3}{\beta^3} \frac{\left[2\boldsymbol{C} + 3 - 2\ln\left(\frac{2\pi}{\beta}\right)\right]^2}{\ln\left(\frac{2\pi}{\beta}\right) - 1 - \boldsymbol{C}}\right\}. \tag{6.9}$$

6.2. *The $\alpha = 2/3$ and $\alpha = 3/4$ poles*

For $\alpha = \frac{2}{3}$ and $\alpha = \frac{3}{4}$ we give only the corresponding results, since the calculations are entirely similar to those for the case $\alpha = \frac{1}{2}$. Thus, for , $\alpha = \frac{2}{3}$ we have

$$\mathcal{Z} = \frac{27\pi^3}{2\beta^3}\left[\boldsymbol{C} - \ln\left(\frac{3\pi}{\beta}\right)\right]. \tag{6.10}$$

Here one requires

$$T < \frac{e^{\boldsymbol{C}}}{3\pi k} \tag{6.11}$$

$$\langle \mathcal{U} \rangle = \frac{1}{\beta} \frac{9\ln\left(\frac{3\pi}{\beta}\right) - 6 - 9\boldsymbol{C}}{\boldsymbol{C} - \ln\left(\frac{3\pi}{\beta}\right)} \tag{6.12}$$

$$\mathcal{S} = \ln\left\{\frac{27\pi^3}{2\beta^3} \frac{\left[\ln\left(\frac{9\pi^2}{\beta^2}\right) - 2 - 2\boldsymbol{C}\right]^3}{\left[c - \ln\left(\frac{2\pi}{\beta}\right)\right]^2}\right\}. \tag{6.13}$$

For $\alpha = \frac{3}{4}$ we have

$$\mathcal{Z} = \frac{32\pi^3}{3\beta^3}, \tag{6.14}$$

$$\langle \mathcal{U} \rangle = \frac{4}{\beta}\left[3\boldsymbol{C} - 1 - 3\ln\left(\frac{4\pi}{\beta}\right)\right], \tag{6.15}$$

$$\mathcal{S} = \ln\left\{\frac{5\pi^3}{\beta^3}\left[\boldsymbol{C} - \ln\left(\frac{4\pi}{\beta}\right)\right]^4\right\}. \tag{6.16}$$

7. Specific heats

We set $k \equiv k_B$. For $\nu = 1$, in the regular case we have for the specific heat C:

$$\mathcal{C} = \frac{k}{2\alpha - 1}. \tag{7.1}$$

For $\nu = 2$ one has

$$\mathcal{C} = \frac{2k}{3\alpha - 2}. \tag{7.2}$$

Finally, for $\nu = 3$ one ascertains that

$$\mathcal{C} = \frac{3k}{4\alpha - 3}. \tag{7.3}$$

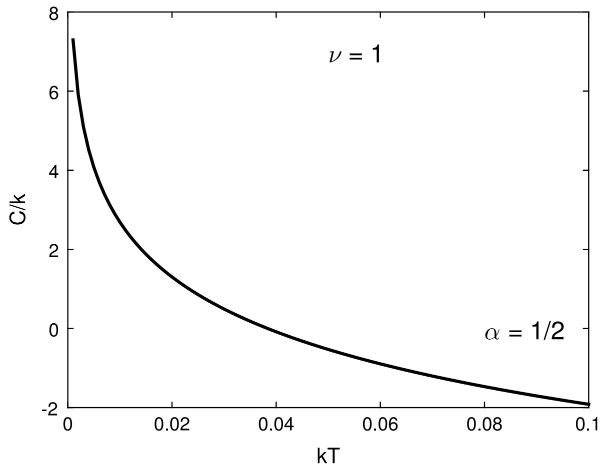

Fig. 1. One dimension: specific heats at the pole versus temperature T.

7.1. Specific heats at the poles

For $\nu = 1; \alpha = \frac{1}{2}$

$$\mathcal{C} = 2k(\boldsymbol{C} - 2 - \ln 2\pi kT). \tag{7.4}$$

For $\nu = 2; \alpha = \frac{1}{2}$

$$\mathcal{C} = \frac{2k(2\ln 2\pi kT - 1 - 2\boldsymbol{C})}{\boldsymbol{C} - \ln 2\pi kT} - \frac{2k}{(\boldsymbol{C} - \ln 2\pi kT)^2}. \tag{7.5}$$

For $\nu = 2$ and $\alpha = \frac{2}{3}$ one has

$$\mathcal{C} = 6k\left(\boldsymbol{C} - \frac{3}{2} - \ln 3\pi kT\right). \tag{7.6}$$

For $\nu = 3; \alpha = \frac{1}{2}$,

$$\mathcal{C} = k\frac{3\boldsymbol{C} + \frac{7}{2} - 3\ln 2\pi kT}{\ln 2\pi kT - \boldsymbol{C} - 1} - \frac{k}{2(\ln 2\pi kT - \boldsymbol{C} - 1)^2}. \tag{7.7}$$

For $\nu = 3$ and $\alpha = \frac{2}{3}$ one finds

$$\mathcal{C} = k\frac{9\ln 3\pi kT - 6 - 9\boldsymbol{C}}{\boldsymbol{C} - \ln 3\pi kT} - \frac{6k}{(\boldsymbol{C} - \ln 3\pi kT)^2}. \tag{7.8}$$

Finally, for $\nu = 3$ and $\alpha = \frac{3}{4}$ we obtain

$$\mathcal{C} = 4k\left(3\boldsymbol{C} - 4 - 3\ln 4\pi kT\right). \tag{7.9}$$

Figs. 1–4 plot the mean energy's pole-specific heats within their allowed temperature ranges, for one, two, and three dimensions, respectively. The most distinguished feature emerges in the cases in which we deal with $\langle U\rangle$-poles for which Z is regular. We see in such a case that negative specific heats arise. Such an occurrence has been associated to self-gravitational systems [25]. In turn, Verlinde has associated this type of systems to an entropic force [26]. We have derived an entropic force for the Harmonic oscillator in Refs. [27–29]. It is natural to conjecture then that such a force may appear at the energy associated poles.

That negative specific heats(NSH) do occur in gravitational systems does not logically entail that the presence of a negative specific heat in a generic system guarantees gravitational effects. However, given that, classically, gravitational systems are the only ones known to be endowed with NSH, it is a plausible inference that our Renyi systems may display them. Plausibility is not tantamount to certainty, of course.

Notice also that temperature ranges are restricted. There is an T-upper bound, and one may wish to remember, in this respect, the notion of Hagedorn temperature [30], as an example a temperature's upper bound. In two and three dimensions there is also a lower bound, so that the system (at the poles) would be stable only in a limited T-range.

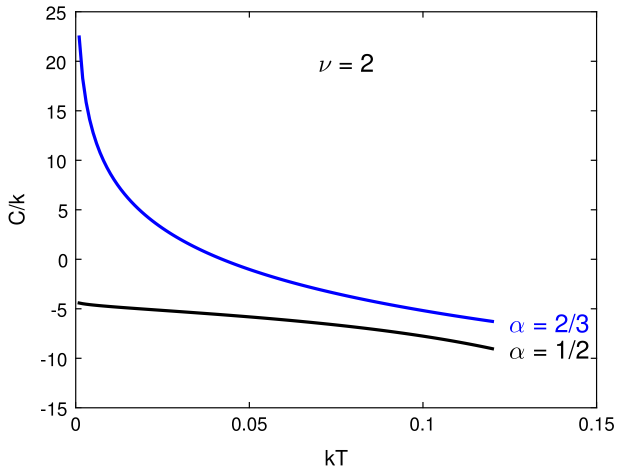

Fig. 2. Two dimensions: specific heats at the two poles versus temperature T.

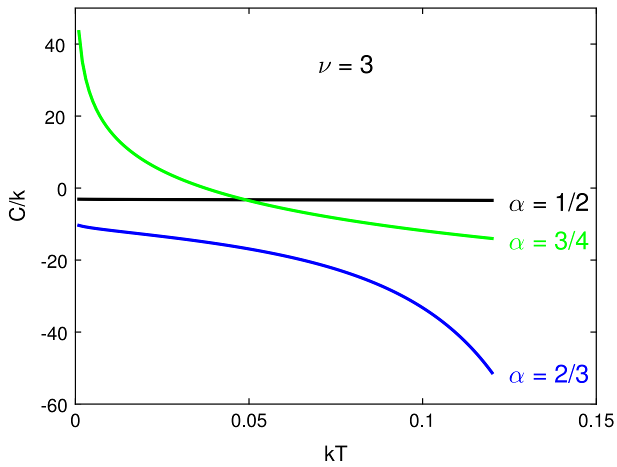

Fig. 3. Three dimensions: specific heats at the three poles versus temperature T.

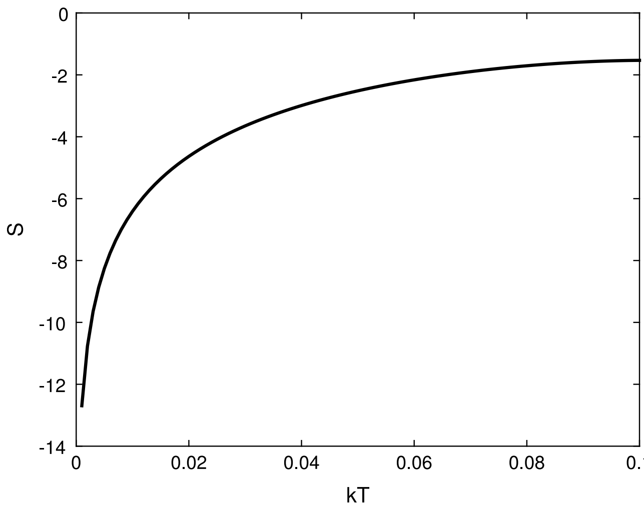

Fig. 4. Three dimensions: the entropy for $\alpha = 3/4$.

8. Discussion

In this work we have appealed to an elementary regularization procedure to study the poles in the partition function and the mean energy that appear, for specific, discrete q-values, in Renyi's statistics of the harmonic oscillator. We studied the thermodynamic behavior at the poles and found interesting peculiarities. The analysis was made in one, two, three, and 3 dimensions. Amongst the pole-traits we emphasize:

- The poles appear, both in the partition function and the mean energy, for $0 < \alpha < 1$.
- These poles are an artifact of having $\alpha \neq 1$.
- We have proved that there is an upper bound to the temperature at the poles, confirming the findings of Ref. [31], in the sense that, for $\alpha \neq 1$, the heath bath of the canonical ensemble must be finite.
- In some cases, Renyi's' entropies are positive only for a restricted temperature-range. Lower T bounds seem to be a new trait discovered here.
- Negative specific heats, characteristic trait of self-gravitating systems, are encountered. That negative specific heats(NSH) do occur in gravitational systems does not logically entail that the presence of a negative specific heat in a generic system guarantees gravitational effects. However, given that, classically, gravitational systems are the only ones known to be endowed with NSH, it is a plausible inference that our Renyi systems may display them. Plausibility is not tantamount to certainty, of course.
- It is true that some magnetic systems exhibit negative specific heats. But magnetism is a quantum-relativistic phenomenon and our treatment is strictly classical, so that our negative specific heats can only be associated to gravitation.

Our physical results derive only from statistics, not from mechanical effects. This fact reminds us of a similar occurrence in the case of the entropic force conjectured by Verlinde [26].

Indeed, the poles arise only because $\alpha \neq 1$. They are a property of the entropic quantifier, not of the Hamiltonian. Indeed, only for $\alpha \neq 1$ a Gamma function appears in the partition function. This Gamma function may display poles.

Future research should be concerned with cases where it is already known in advance that $\alpha \neq 1$. For these cases, the traits here discovered may acquire some degree of physical "reality".

The importance of the present communication resides in that fact of having disclosed Renyi's entropy traits that could not have been suspected before and we found the way of exhibiting the physics at the poles in Statistical Mechanics.

Appendix

In this appendix we will show that equation $\frac{\partial S}{\partial \beta} = \beta \frac{\partial \langle U \rangle}{\partial \beta}$, which is valid for Boltzmann–Gibbs entropy, it is not so for Renyi's one. The Boltzmann–Gibbs entropy is given by:

$$\mathcal{S} = \ln \mathcal{Z} + \beta \langle \mathcal{U} \rangle. \tag{A.1}$$

from which we obtain

$$\frac{\partial \mathcal{S}}{\partial \beta} = \frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial \beta} + \langle \mathcal{U} \rangle + \beta \frac{\partial \langle \mathcal{U} \rangle}{\partial \beta}. \tag{A.2}$$

As

$$\frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial \beta} + \beta \langle \mathcal{U} \rangle = 0, \tag{A.3}$$

then

$$\frac{\partial \mathcal{S}}{\partial \beta} = \beta \frac{\partial \langle \mathcal{U} \rangle}{\partial \beta}. \tag{A.4}$$

For Renyi's entropy we have, instead,

$$\mathcal{S} = \ln \mathcal{Z} + \frac{1}{1-\alpha} \ln[1 + (1-\alpha)\beta \langle \mathcal{U} \rangle]. \tag{A.5}$$

Thus,

$$\frac{\partial \mathcal{S}}{\partial \beta} = \frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial \beta} + \frac{\langle \mathcal{U} \rangle}{1 + (1-\alpha)\beta \langle \mathcal{U} \rangle} + \frac{\beta}{1 + (1-\alpha)\beta \langle \mathcal{U} \rangle} \frac{\partial \langle \mathcal{U} \rangle}{\partial \beta}. \tag{A.6}$$

From (A.6) we see that (A.4) is not valid for Renyi's entropy.

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