

Weakly (and not so weakly) bound states of a relativistic particle in one dimension



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ABSTRACT

We present the first exact calculation of the energy of the bound state of a one dimensional Dirac massive particle in weak short-range arbitrary potentials, using perturbation theory to fourth order (the analogous result for two dimensional systems with confinement along one direction and arbitrary mass is also calculated to second order). We show that the non-perturbative extension obtained using Padé approximants can provide remarkably good approximations even for deep wells, in certain range of physical parameters. As an example, we discuss the case of two gaussian wells, comparing numerical and analytical results, predicted by our formulas.

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1. Introduction

Almost 90 years have passed since Dirac established his famous equation, successfully combining Quantum Mechanics and Special Relativity, the two physical theories that completely changed our understanding of Nature at the beginning of the previous century. The importance of the Dirac equation can hardly be overstated: it predicts the existence of antimatter (discovered by Anderson in 1932), it explains the spin of the electron, recovering Pauli's theory in the low energy limit, and it also describes correctly the observed spectrum of the hydrogen atom, all at once. Another consequence of the Dirac equation, the *Zitterbewegung* (*trembling motion*) of the electron, has not been experimentally observed, although recently it has been simulated on physical systems composed of atoms which mimic the behavior of a free relativistic particle [1,2]. In recent years, the Dirac equation has also been used to describe the low energy spectrum of graphene, with either massless [3] or massive [4] excitations.

It is interesting to observe that even from the point of view of the theory, there are consequences of the Dirac equation that still need to be explored; our attention in the present paper is devoted to the study of the behavior of weakly bound relativistic states in

one and two dimensional systems. The non-relativistic counterpart of this problem, has been settled long time ago in a seminal paper by Simon [5], where the conditions for the existence of this bound state have been given and the analyticity (non-analyticity) of the energy in one (two) dimension has been established.

For the relativistic case, the conditions under which a Dirac particle is trapped in a one-dimensional potential have been identified in ref. [6]; more recently Cuenin and Siegel [7] have studied the weakly coupling eigenvalue asymptotics for the bound state of the one dimensional Dirac operator, perturbed by a matrix-valued and non-symmetric potential.

For the case of a non-relativistic particle in a one dimensional short-range potential, a formula for the energy of the bound state has been derived up to sixth order: Simon [5] reports an unpublished result obtained by Abarbanel, Callan and Goldberger [8], which is exact to third order in the parameter controlling the strength of the potential, whereas higher order corrections (up to order six) have been derived later [11,10,9] using different techniques. Interestingly, a similar analysis for the relativistic case is still lacking and this constitutes the main goal of the present paper.

The approach that we will follow in this paper has been originally proposed by Gat and Rosenstein [10], and applied to the non-relativistic version of the present problem (to third order in the perturbation parameter) and to a (1 + 1) dimensional QFT; in a recent work by two of the present authors, ref. [9], the method has been applied to calculate the energy of the bound state of an arbitrary shallow short range potential to sixth order.

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2. The method

We will first briefly describe how the method works for the non-relativistic problem and then discuss how it can be extended to its relativistic counterpart.

Let \hat{H} be the hamiltonian of the problem

$$\hat{H}(\lambda) = -\frac{d^2}{dx^2} + \lambda V(x) \quad (1)$$

where $V(x) < 0$ for $x \in (-\infty, \infty)$ and $\lim_{|x| \rightarrow \infty} V(x) = 0$. Here $\lambda > 0$ is a parameter that controls the strength of the potential well. As noticed in [10], one cannot use $\hat{H}(0)$ as the unperturbed hamiltonian, since, for $\lambda > 0$ the spectrum of \hat{H} contains (at least) one bound state, whereas the spectrum of $\hat{H}(0)$ is continuous.

Instead we use as unperturbed Hamiltonian the operator

$$\hat{H}_0 \equiv -\frac{d^2}{dx^2} - 2\beta\delta(x) \quad , \quad \beta > 0 \quad . \quad (2)$$

\hat{H}_0 has just one bound state with energy $\epsilon_0 = -\beta^2$ and a continuum of states, for $\epsilon > 0$ (the reader may refer to ref. [12] for a discussion of the one center δ interaction in one dimension). As a result, the Schrödinger equation

$$\left[\hat{H}_0 + \lambda V(x) \right] \psi(x) = E\psi(x) \quad (3)$$

can now be studied perturbatively in λ , working with a finite β and assuming $E = \sum_{n=0}^{\infty} \lambda^n \epsilon_n$ and $\psi(x) = \sum_{n=0}^{\infty} \lambda^n \phi_n(x)$. The infrared divergencies, which would spoil the perturbative expansion when $H(0)$ is used, manifest, at a given order, as inverse powers of β , and cancel out exactly, rendering each order perfectly finite.

Contrary to the approach followed in [10,9], where the standard Rayleigh–Schrödinger approach involving matrix elements was applied, here we obtain a perturbative solution of the Schrödinger equation in terms of the appropriate Green's functions.

To lowest order in λ one has the eigenvalue equation

$$\left(-\frac{d^2}{dx^2} - 2\beta\delta(x) \right) \phi_0(x) = \epsilon_0 \phi_0(x) \quad (4)$$

In this case the eigenvalue and eigenfunction are $\epsilon_0 = -\beta^2$ and $\phi_0(x) = \sqrt{\beta} e^{-\beta|x|}$ respectively.

To higher orders one obtains the equations

$$\mathcal{D}\phi_n(x) = -V(x)\phi_{n-1}(x) + \sum_{k=1}^n \epsilon_k \phi_{n-k}(x) \equiv \mathcal{S}_n(x) \quad (5)$$

where

$$\mathcal{D} \equiv \left(-\frac{d^2}{dx^2} - 2\beta\delta(x) + \beta^2 \right) \quad (6)$$

To deal with them one needs to consider the Green's function $G(x, y)$ defined by

$$\mathcal{D}G(x, y) = \delta(x - y) \quad (7)$$

and write the solution of order n as $\phi_n(x) = \int G(x, y) \mathcal{S}_n(y) dy$. The exact form of this and higher orders Green's functions can be found in ref. [9]. This equation needs to be complemented by the condition

$$\int \mathcal{S}_n(x) \phi_0(x) dx = 0 \quad ; \quad n \geq 1 \quad , \quad (8)$$

which removes the “secular terms” in the expansion. Equation (8) only gives the energy and the wave function at a given order.

This approach has the advantage of avoiding the appearance of infinite series and it allows one to consider more general eigenvalue equations, as in the case of a relativistic particle.

Let us now discuss the case of a relativistic particle in one or two dimensions, obeying the Dirac equation $\hat{H}\psi = E(\lambda)\psi$, where

$$\hat{H} = -i\sigma \cdot \nabla + \sigma_3 m + \lambda W(x) \quad (9)$$

and $\psi = (\psi_1 \ \psi_2)$ is a spinor (σ_i are the usual Pauli matrices).

Here $\sigma \cdot \nabla = \sigma_1 \partial_x$ for the one dimensional case and $\sigma \cdot \nabla = \sigma_1 \partial_x + \sigma_2 \partial_y$ for the two-dimensional one.

The potential, which depends only on x , is given by

$$W(x) = \frac{1}{2} [\sigma_3 (V(x) + U(x)) + \mathbf{1} (V(x) - U(x))] \quad ,$$

where $(V(x) + U(x))/2$ and $(V(x) - U(x))/2$ are a vector and a scalar potential respectively.

Equations of the form of (9) have been studied previously, in particular for the case of point-like interactions in one dimension [13] and for graphene and graphite systems, subject to piecewise-constant potentials [14,15].

We can work in one or two dimensions in an unified framework by using the ansatz $\exp[iqy]\psi(x)$ (the one dimensional case is recovered for $q = 0$) and write explicitly the Dirac equation in terms of its components

$$\begin{aligned} (-E + m + \lambda V)\psi_1 - i(q + \partial_x)\psi_2 &= 0 \\ -(E + m + \lambda U)\psi_2 + i(q - \partial_x)\psi_1 &= 0 \end{aligned} \quad (10)$$

Using the second equation we can express ψ_2 in terms of ψ_1 and then use it inside the first equation to obtain a second order differential equation for ψ_1 alone:

$$\begin{aligned} -\psi_1''(x) + \frac{\lambda U'(x)\psi_1'(x)}{E + m + \lambda U(x)} + \mathcal{V}(x)\psi_1(x) \\ = (E^2 - k^2(q))\psi_1(x) \end{aligned} \quad (11)$$

with

$$\mathcal{V}(x) \equiv \left(\lambda(m - E)U(x) + \lambda(E + m)V(x) + \lambda^2 U(x)V(x) \right) \quad (12)$$

and $k(q) \equiv \sqrt{q^2 + m^2}$.

When $U(x) = 0$ this equation takes a simpler form of a Schrödinger-like equation, with an energy dependent potential, as already pointed out by Coutinho and Nogami [6]. For the special case $E + m + \lambda U(x_0) = 0$ for some $x_0 \in \mathbb{R}$, in which the denominator in the second term of eq. (11) vanishes, the eigenfunction needs to obey the additional boundary condition $\psi_1'(x_0) = 0$ (see the discussion in Fig. 3).

Eq. (11) is now in the appropriate form to be attacked using the approach that we have previously described for the non-relativistic case, introducing an attractive delta potential of strength β , that allows to separate a single bound state from the continuum. This amounts to substituting $-\frac{d^2}{dx^2} \rightarrow -\frac{d^2}{dx^2} - 2\beta\delta(x) = \mathcal{D} - \beta^2$, previously defined, and then casting equation (11), in a compact form, formally similar to the nonrelativistic case, as

$$\left(\mathcal{D} - \beta^2 \right) \psi_1 = \tilde{\mathcal{W}} \psi_1 \quad , \quad (13)$$

where

$$\tilde{\mathcal{W}}(x) \equiv -\frac{\lambda U'(x)}{E + m + \lambda U(x)} \frac{d}{dx} + \left(E^2 - k^2(q) - \mathcal{V}(x) \right) \quad (14)$$

is an operator defined on the real line (with the possible exception of x_0 for which the denominator $E + m + \lambda U(x)$ vanishes).

After expressing both the energy and wave function as power series in λ ,

$$E = \sqrt{k^2 - \Delta}, \quad \Delta = \sum_{n=0}^{\infty} \delta_n \lambda^n, \quad \psi_1 = \sum_{n=0}^{\infty} \phi_n \lambda^n \quad (15)$$

and substituting into eq. (13), we obtain an infinite tower of second order differential equations, corresponding to different orders in λ , which can be solved starting from the lowest order. This situation is completely analogous to the non-relativistic case, although now \tilde{W} in eq. (13) is an operator and it is non-linear in λ . The main consequence of this fact is the rapid proliferation of terms contributing at a given perturbative order, as the order is increased.

Applying the method of Gat and Rosenstein to this equation, we have obtained the perturbative expression for the energy of the fundamental mode to fourth order in λ for the one-dimensional problem and to second order in λ for the two-dimensional model.

For instance to order λ^0 we obtain again eq. (4)

$$-\phi_0''(x) - 2\beta\delta(x)\phi_0(x) + \delta_0\phi_0(x) = 0 \quad (16)$$

where $\delta_0 = \beta^2$ and $\phi_0(x) = \sqrt{\beta}e^{-\beta|x|}$.

To order λ we obtain the equation

$$\begin{aligned} D\phi_1(x) &= -\delta_1\phi_0(x) + \left(-m + \sqrt{m^2 - \beta^2}\right)U(x)\phi_0(x) \\ &\quad - \left(m + \sqrt{m^2 - \beta^2}\right)V(x)\phi_0(x) - \frac{U'(x)}{m + \sqrt{m^2 - \beta^2}}\phi_0'(x) \\ &\equiv S_1(x) \end{aligned} \quad (17)$$

To eliminate the secular term we need to enforce the condition

$$\int S_1(x)\phi_0(x)dx = 0 \quad (18)$$

as done in the non-relativistic problem.

Given the normalization of the $\phi_0(x)$, this equation can be cast in the form

$$\begin{aligned} \delta_1 &= \int \left[\left(-m + \sqrt{m^2 - \beta^2}\right)U(x)\phi_0(x) \right. \\ &\quad \left. - \left(m + \sqrt{m^2 - \beta^2}\right)V(x)\phi_0(x) \right. \\ &\quad \left. - \frac{U'(x)}{m + \sqrt{m^2 - \beta^2}}\phi_0'(x) \right] \phi_0(x)dx \end{aligned} \quad (19)$$

In order to comply with equation (19), we need to express δ_1 as

$$\delta_1 = \sum_{\ell=0}^{\infty} \delta_1^{(\ell)} \beta^\ell \quad (20)$$

and determine the coefficients $\delta_1^{(\ell)}$ by expanding the rhs of (19) about $\beta = 0$:

$$\begin{aligned} \delta_1^{(0)} + \beta\delta_1^{(1)} + \beta^2\delta_1^{(2)} + \dots \\ = -2\beta m \int V(x)dx + \beta^2 \int |x| \left(4mV(x) + \frac{U'(x)}{2mx}\right) dx + \dots \end{aligned}$$

In this way we easily conclude that

$$\begin{aligned} \delta_1^{(0)} &= 0 \\ \delta_1^{(1)} &= -2m \int V(x)dx \\ \delta_1^{(2)} &= \int |x| \left(4mV(x) + \frac{U'(x)}{2mx}\right) dx \\ \dots \end{aligned}$$

At first sight, it may appear that the calculation of $\delta_1^{(\ell)}$, for $\ell > 0$, is not needed in order to obtain the exact perturbative expression of the relativistic energy since at the end of the calculation we are taking the limit $\beta \rightarrow 0^+$. This observation however is incorrect since the Green's functions contain infrared divergent terms which can provide finite contributions for $\beta \rightarrow 0^+$ when combined with the appropriate terms stemming from the expansion of δ_n .

To clarify this point it is convenient to consider the equation to order λ^2 :

$$\begin{aligned} D\phi_2(x) &= -\delta_1\phi_1(x) - \delta_2\phi_0(x) - V(x)U(x)\phi_0(x) \\ &\quad - \frac{\delta_1(U(x) - V(x))}{2\sqrt{m^2 - \beta^2}}\phi_0(x) \\ &\quad + \frac{U(x)U'(x)}{(m + \sqrt{m^2 - \beta^2})^2}\phi_0'(x) \\ &\quad - \frac{\delta_1 U'(x)}{2\sqrt{m^2 - \beta^2}(m + \sqrt{m^2 - \beta^2})^2}\phi_0'(x) \\ &\quad - \left((m - \sqrt{m^2 - \beta^2})U(x) \right. \\ &\quad \left. + (m + \sqrt{m^2 - \beta^2})V(x) \right) \phi_1(x) \\ &\quad - \frac{U'(x)}{m + \sqrt{m^2 - \beta^2}}\phi_1'(x) \equiv S_2(x) \end{aligned} \quad (21)$$

where

$$\phi_1(x) = \int \mathcal{G}^{(0)}(x, y)S_1(y)dy \quad (22)$$

and

$$\mathcal{G}^{(0)}(x, y) \equiv \frac{1}{4\beta} - \frac{1}{4}(|x| + 2|x - y| + |y|) + O(\beta) \quad (23)$$

is the appropriate Green's function, given in ref. [9] (observe how the source terms become considerably more complicated as the order of the calculation grows).

In this case we obtain the equation

$$\begin{aligned} \delta_2 &= - \int V(x)U(x)\phi_0^2(x)dx - \delta_1 \int \frac{U(x) - V(x)}{2\sqrt{m^2 - \beta^2}}\phi_0^2(x)dx \\ &\quad + \int \frac{U(x)U'(x)}{(m + \sqrt{m^2 - \beta^2})^2}\phi_0(x)\phi_0'(x)dx \\ &\quad - \delta_1 \int \frac{U'(x)}{2\sqrt{m^2 - \beta^2}(m + \sqrt{m^2 - \beta^2})^2}\phi_0(x)\phi_0'(x)dx \\ &\quad - \int \left((m - \sqrt{m^2 - \beta^2})U(x) \right. \\ &\quad \left. + (m + \sqrt{m^2 - \beta^2})V(x) \right) \phi_1(x)\phi_0(x)dx \\ &\quad - \int \frac{U'(x)}{m + \sqrt{m^2 - \beta^2}}\phi_0(x)\phi_1'(x)dx, \end{aligned} \quad (24)$$

where the contributions containing $\phi_1(x)$ and $\phi_1'(x)$ are potentially infrared divergent because of the behavior of eq. (23) for $\beta \rightarrow 0^+$. However, it is easy to see that this is not the case since the normalization factors $\sqrt{\beta}$ in the wave functions take care of the singular behavior of eq. (23) for $\beta \rightarrow 0^+$; as a matter of fact one has

$$\int \phi_0(x)f(x)\phi_1(x)dx = -\frac{m}{2} \int V(x)f(x)dx + O(\beta) \quad (25)$$

Finally, upon expanding eq. (24) in powers of β , one then obtains the expressions for the coefficients $\delta_2^{(\ell)}$, $\ell = 0, 1, \dots$.

Keeping in mind these technical details, the calculation can be carried out systematically at higher orders, with a rapidly increasing complexity, obtaining expressions for δ_n that are always free of infrared divergencies due to exact cancellations between divergent terms, occurring at each perturbation order (the same behavior was already observed for the non-relativistic problem in ref. [9]).

In what follows we present the leading behavior of δ_n ($n = 0, \dots, 4$) for $\beta \rightarrow 0^+$. Since the solutions for the one-dimensional case can be recovered from the corresponding two-dimensional expressions setting $q = 0$, we first assume $|q| \geq 0$, and report the coefficients of Δ up to second order

$$\begin{aligned} \delta_0 &= \beta^2, \\ \delta_1 &= -2\beta \mathcal{F}(k) + \mathcal{O}(\beta^2), \\ \delta_2 &= \mathcal{F}(k)^2 + \mathcal{O}(\beta), \end{aligned} \tag{26}$$

where

$$\mathcal{F}(k) = \frac{1}{2} \int dx ((m+k)V + (m-k)U). \tag{27}$$

For the one-dimensional case ($q = 0$), we have computed the energy corrections up to order four. Terms up to second order are obtained by making $\kappa = m$ in formula (26), while δ_3 and δ_4 are given by

$$\begin{aligned} \delta_3 &= 2m^3 \mathcal{F}_1 \mathcal{F}_{2,1} + \mathcal{O}(\beta), \\ \delta_4 &= m^4 \eta_4 - m^2 \kappa_4 + \mathcal{O}(\beta), \end{aligned} \tag{28}$$

where

$$\begin{aligned} \eta_4 &= (\mathcal{F}_1)^2 \mathcal{F}_{2,2} + 2\mathcal{F}_1 \mathcal{F}_{3,1} + (\mathcal{F}_{2,1})^2, \\ \kappa_4 &= (1/2) (\mathcal{F}_1 \mathcal{F}_{3,2} + (\mathcal{F}_1)^4). \end{aligned} \tag{29}$$

$\mathcal{F}_1, \dots, \mathcal{F}_{3,2}$ are functionals of V and U given by

$$\begin{aligned} \mathcal{F}_1 &= \mathcal{F}(m)/m = \int dx V(x), \\ \mathcal{F}_{2,1} &= \iint dx dy V(y) |x-y| V(x), \\ \mathcal{F}_{2,2} &= \iint dx dy V(y) (x-y)^2 V(x), \\ \mathcal{F}_{3,1} &= \iiint dx dy dz |x-y| |y-z| V(x) V(y) V(z), \\ \mathcal{F}_{3,2} &= \iiint dx dy dz \frac{|x-y| |x-z|}{(x-y)(x-z)} U(x) V(z) V(y). \end{aligned}$$

The energy of the bound state in one dimension reads

$$E^{(1)}(\lambda) = m + \tilde{E}(\lambda) + \lambda^4 \delta E + \mathcal{O}(\lambda^5), \tag{30}$$

where

$$\tilde{E} = -\frac{m\lambda^2}{2} \mathcal{F}_1^2 - m^2 \lambda^3 \mathcal{F}_1 \mathcal{F}_{2,1} - \frac{m^3 \lambda^4}{2} \eta_4 \tag{31}$$

is the non-relativistic formula previously obtained to fourth order working with the Schrödinger equation (see refs. [8,11,9]) and δE is the leading relativistic correction which appears to fourth order

$$\delta E = \frac{m}{2} \left(\kappa_4 - \frac{1}{4} \mathcal{F}_1^4 \right) \tag{32}$$

Note that, while $\tilde{E}(\lambda)$ is a functional of V only, δE is a functional both of V and U .

In two dimensions, for quasi-bound states of the form $\psi(x, y) = \exp(iq)\psi(x)$, the energy is given by

$$E^{(2)}(\lambda) = k - (\lambda^2/2k) \mathcal{F}(k)^2 + \mathcal{O}(\lambda^3). \tag{33}$$

For the case of a relativistic one-dimensional square well, discussed by Greiner [16] in detail, Eq. (30) reproduces the exact results up to fourth order. As a further test of our perturbation expressions we also consider the simple case in which $V(x) = -(1+\gamma)\delta(x)$, $U(x) = -(1-\gamma)\delta(x)$ and $q = 0$. In order to avoid the possible discontinuity of both functions $\psi_{1,2}(x)$ at $x = 0$ we set $\gamma = 1$. In this case $\psi_1(x)$ is continuous at $x = 0$ and a straightforward calculation shows that

$$E^{(1)} = m \frac{1-\lambda^2}{1+\lambda^2} = m - 2m\lambda^2 (1-\lambda^2 + \lambda^4 + \dots), \tag{34}$$

and $\psi(x) = \sqrt{\beta} e^{-\beta|x|}$, with $\beta = \sqrt{m^2 - E^2} = \frac{2m\lambda}{1+\lambda^2}$.

Note that $0 < \beta \leq \beta(\lambda = 1) = m$. Present perturbation theory yields the first three terms of the series (34) exactly.

In the perturbative region, $0 < \lambda \ll 1$, the relativistic correction δE , provides in general a tiny correction to the corresponding non-relativistic expression, \tilde{E} , implying that the weakly bound electron is essentially non-relativistic. This hierarchy can however be modified already at moderate values of λ . In this case, the energy of the bound state cannot lower indefinitely as the well becomes deeper and deeper, as in the non-relativistic case, since it is trapped between two continua, the continuum of positive energy states, for $E \geq m$, and the continuum of negative energy states, for $E \leq -m$. This behavior can be captured using a diagonal Padé approximant, which tends to a constant for $\lambda \rightarrow \infty$:

$$E_{\text{Padé}}^{(1)} = m + \frac{m^2 \lambda^2 \mathcal{F}_1^4}{-2m \mathcal{F}_1^2 + 4m^2 \lambda \mathcal{F}_1 \mathcal{F}_{2,1} + 2\lambda^2 (-2\delta E + m^3 (\eta_4 - 4\mathcal{F}_{2,1}^2))} \tag{35}$$

This formula provides a completely analytical expression for the energy of the relativistic bound state which can be used for larger values of λ ; the non relativistic case can either be obtained setting $\delta E \rightarrow 0$ in this expression, or using the simpler [2, 1] Padé approximant, which is linear as $\lambda \rightarrow \infty$:

$$E_{\text{Padé-nr}}^{(1)} = -\frac{m\lambda^2 \mathcal{F}_1^3}{4m\lambda \mathcal{F}_{2,1} - 2\mathcal{F}_1} \tag{36}$$

One way to assess the region of applicability of Eq. (35) is by identifying the region in parameter space where

$$\delta E > \frac{m^3}{2} (\eta_4 - 3\mathcal{F}_{2,1}^2) \tag{37}$$

is fulfilled. When this condition is met, the denominator of $E_{\text{Padé}}^{(1)}$ has no real pole and consequently the resummation is more accurate.

3. An example

As an example, we consider the gaussian wells $V(x) = -(1+\gamma)e^{-\alpha x^2}$ and $U(x) = -(1-\gamma)e^{-\alpha x^2}$, where $-1 \leq \gamma \leq 1$ is a parameter which controls the depths of $V(x)$ and $U(x)$.¹

In this case the inequality (37) reads

$$\pi\alpha(\gamma+5) > 8(-6+3\sqrt{3}+2\pi)(\gamma+1)m^2 \tag{38}$$

The region in parameter space where the inequality is fulfilled is displayed in Fig. 1, for three values of α . Notably the Padé has always real poles when δE is set to zero.

¹ Note that the case $|\gamma| > 1$ can be reduced to the present case by means of a redefinition of λ .

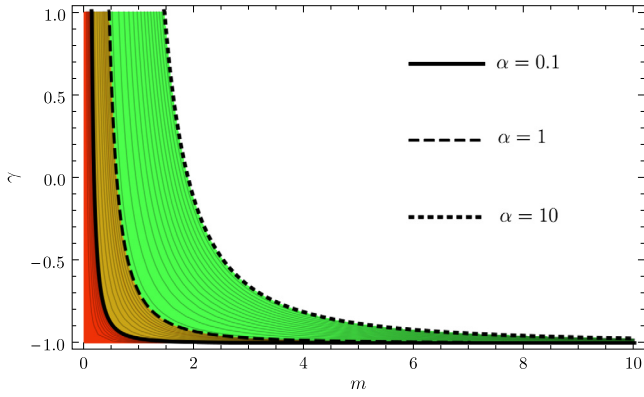


Fig. 1. Parameter space for the inequality (38) for three different values of α . (Color online.)

In Fig. 2 we plot the quantity $m - E$ for the case $m = 0.1$ and $\alpha = 1$, as a function of λ , for $\gamma = 1$ (the plot for $\gamma = 0$, not reported here is quite similar). The blue points represent the numerical values obtained applying the shooting method directly to Eq. (11); the red points are obtained solving the corresponding non-relativistic Schrödinger equation. These values are compared with the relativistic Padé of Eq. (35) (solid line), the non-relativistic [2, 2] Padé obtained setting $\delta E = 0$ (dashed line) and the non-relativistic Padé of Eq. (36) (dot-dashed line). The horizontal lines correspond to the limit values $m(1 + \gamma)$. While δE provides a tiny contribution at small λ , it plays an essential role at larger values of λ .

The normalized upper and lower components of the Dirac spinor, $\psi_{1,2}(x)$, are plotted in Fig. 3, for the case $m = 0.1$, $\alpha = 1$, $\gamma = 0$ and $\lambda = 1$. The corresponding probability density $\rho(x)$ is also displayed. $\psi_1(x)$ is obtained numerically using the shooting method. By inspection of the Eq. (11) we see that the coefficient of $\psi_1'(x)$ is singular when $E + m + \lambda U(x) = 0$: this forces the first derivative of the wavefunction to vanish at the singularity, represented by a vertical line in the plot. Out of this region the wave function decays exponentially as $\psi_1(x) \propto e^{-\sqrt{m^2 - E^2}x}$. The dashed line is a fit of the numerical results, within the interval $5 \leq x \leq 50$ and it corresponds to $\psi_1^{(fit)}(x) = 0.5929 \cdot e^{-0.0931 x}$. Note that $\Gamma^{(fit)} = 0.0931$ is in perfect agreement with the expected expression $\Gamma = \sqrt{m^2 - E^2}$.

This remarkable agreement can be appreciated from Fig. 4, where the constant Γ is extracted from the fit of the numerical results of the wave function $\psi_1(x)$, at different values of λ (the dots in the plot), and contrasted with the explicit expressions obtained using the Padé approximant of Eq. (35). While, in the non-relativistic case, the wave function decays more and more strongly as $\lambda \rightarrow \infty$, in the relativistic case the energy of the bound state obeys the inequality $-m < E < m$, and therefore $0 < \Gamma \leq m$. The particular behavior of the analytic formula for Γ when $\gamma = 0$, which breaks down at $\lambda \approx 10$, is easily explained by the fact that the Padé slightly underestimates the limiting energy for $\lambda > 10$, and as a result $\sqrt{m^2 - E^2}$ becomes imaginary.

4. Conclusions

We have calculated for the first time the energy of a relativistic bound state in a shallow short range potential in one dimension to fourth order in perturbation theory, proving that the first genuinely relativistic correction appears only at order four. We have confirmed this generally tiny contribution in a number of cases where it was possible to contrast our results with exact results available in the literature and with precise numerical calculations, carried out for the case of a pair of gaussian potentials.

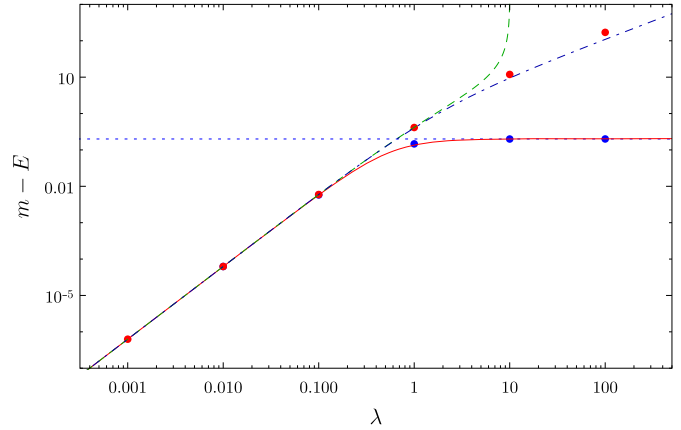


Fig. 2. Energy of the relativistic and non-relativistic bound states (blue and red dots respectively) compared with the relativistic and non-relativistic Padé approximants (solid, dashed and dot-dashed lines respectively). Here $m = 0.1$, $\alpha = 1$ and $\gamma = 1$. The horizontal lines are the limit values $m - E = m(1 + \gamma)$. (Color online.)

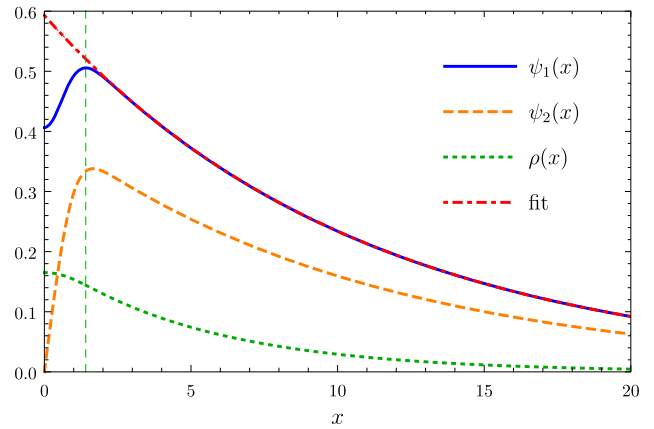


Fig. 3. Normalized upper and lower components $\psi_{1,2}(x)$ and probability density of the Dirac spinor for $m = 0.1$, $\alpha = 1$, $\gamma = 0$ and $\lambda = 1$, obtained using the shooting method. The vertical line is the location of the singularity $E + m + \lambda U(x)$, where $\psi_1'(x) = 0$. The dashed line is the fit of the numerical results between $x = 5$ and $x = 50$, $\psi_1^{(fit)}(x) = 0.5929 \cdot e^{-0.0931 x}$. (Color online.)

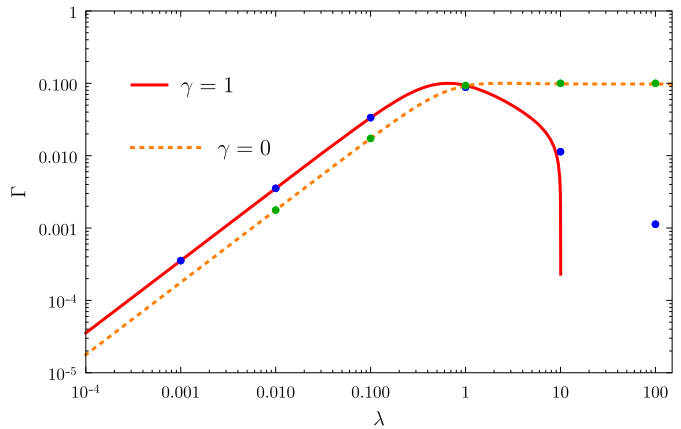


Fig. 4. Constant Γ of the exponential decay of the wave function. The dots (blue and green) are the values of Γ obtained from the fit of the numerical wave function at different values of λ at $\gamma = 1$ and $\gamma = 0$ respectively. The solid and dashed lines are the explicit expressions obtained using the Padé of Eq. (35). (Color online.)

We have also shown that it is possible to extend the perturbative analysis to the study of deep wells, by using a Padé approximant which captures the asymptotic behavior of the energy for $\lambda \rightarrow \infty$. The simple analytical formula that we have obtained has been tested for the (not exactly solvable) case of gaussian well, finding that the analytical approximation is in excellent agreement with the numerical results.

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