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A Gamma convergence approach to the critical Sobolev embedding in variable exponent spaces

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ABSTRACT

In this paper, we study the critical Sobolev embeddings $W^{1,p(\cdot)}(\Omega) \subset L^{p^*(\cdot)}(\Omega)$ for variable exponent Sobolev spaces from the point of view of the Γ-convergence. More precisely we determine the Γ-limit of subcritical approximation of the best constant associated with this embedding. As an application we provide a sufficient condition for the existence of extremals for the best constant.

1. Introduction

The purpose of this paper is to analyze the Sobolev immersion theorem for variable exponent spaces in the critical range from the point of view of the Γ-convergence. Our motivation comes from the existence problem for extremals of these immersions. By extremals we mean functions $u \in W_0^{1,p(\cdot)}(\Omega)$ where the infimum

$$S = S(p(\cdot), q(\cdot), \Omega) := \inf_{v \in W_0^{1,p(\cdot)}(\Omega)} \frac{\|\nabla v\|_{p(\cdot)}}{\|v\|_{q(\cdot)}} \tag{1.1}$$

is attained. Here Ω is a smooth bounded subset of $\mathbb{R}^n$, and $p, q : \overline{\Omega} \to \mathbb{R}$ are two functions satisfying the following assumptions:

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(H1) p is Log-Hölder continuous on Ω (see (2.2) below), $q \in C(\overline{\Omega})$,
(H2) $1 < p_- := \inf_{\overline{\Omega}} p \le p_+ := \sup_{\overline{\Omega}} p < n$,
(H3) $1 \le q(x) \le p^*(x) := np(x)/(n - p(x))$ for any $x \in \overline{\Omega}$.

We refer to the next section for the definition and basic properties of the variable exponent Sobolev spaces appearing in (1.1). Notice that the exponent p^* is critical from the Sobolev point of view. We shall also assume that

(H4) the set $\mathcal{A} := \{x \in \overline{\Omega}\colon q(x) = p^*(x)\}$ is non-empty.

Because of (H4) the embedding of $W^{1,p(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$ is not compact, making non-trivial the problem of existence of an extremal for S.

This problem was recently treated in [14] where the authors provide sufficient conditions to ensure the existence of such extremals. The approach in [14] was the so-called *direct method of the calculus of variations*. That is, they considered a minimizing sequence for S and find a sufficient condition that ensured the compactness of such sequence.

In this paper, we follow a different approach. Instead of looking for minimizing sequences for S, we approximate the critical problems by subcritical ones, where the existence of extremals is easily obtained, and then pass to the limit. In fact, following G. Palatucci in [29] and [30] where the constant exponent case is studied, we want to determine the asymptotic behavior in the sense of the Γ-convergence of the subcritical approximations

$$S_\varepsilon := S(p(\cdot), q(\cdot) - \varepsilon, \Omega) = \inf_{v \in W_0^{1,p(\cdot)}(\Omega)} \frac{\|\nabla v\|_{p(\cdot)}}{\|v\|_{q(\cdot)-\varepsilon}}, \quad \varepsilon > 0,$$

and then deduce the behavior of their associated extremals u_ε. We thus introduce the functional $F_\varepsilon : \mathcal{B}(\Omega) \to \mathbb{R}$, $\varepsilon > 0$, defined by

$$F_\varepsilon(u) := \int_\Omega |u|^{q(\cdot)-\varepsilon}\, dx,$$

where

$$\mathcal{B}(\Omega) := \Big\{u \in W_0^{1,p(\cdot)}(\Omega),\ \|\nabla u\|_{p(\cdot),\Omega} \le 1\Big\}, \tag{1.2}$$

with the purpose of finding its Γ-limit as $\varepsilon \to 0$.

This approach not only provides us with the existence of extremals for the critical embeddings but also gives us the asymptotic behavior of the subcritical extremals as the exponent q reaches a critical one. As in the constant exponent case, a concentration phenomenon occurs in the sense that the subcritical extremals concentrate at some point. In the constant exponent case the location of this point is related to the geometry of Ω via its Robin function (see e.g. [18]). The Gamma convergence turns out to be a useful tool in such analysis as was shown in [2] and in general in the study of the asymptotic behavior of variational problem (see e.g. [7]). On the other hand the study of such concentration phenomena in the variable exponent setting is a recent and rapidly growing area (see e.g. [1,13,14,16,15,19,24]). In particular the results in [13,16,15] let us think that the location of the concentration point may result of interplay between the exponents p and q on the one hand, and on the geometry of Ω on the other hand. The results of this paper are a first step toward a finer comprehension of the concentration phenomenon in the variable exponent setting.

In view of the concentration–compactness principle stated in (2.7)–(2.9) below, it turns out to be convenient to extend F_ε to the space

$$\mathcal{X} = \mathcal{X}(\Omega) = \Big\{(u,\mu) \in W_0^{1,p(\cdot)}(\Omega) \times \mathcal{M}(\bar\Omega)\colon \mu(\bar\Omega) \le 1,\ \mu = |\nabla u|^{p(\cdot)}\,dx + \tilde\mu + \sum_{i\in I} \mu_i \delta_{x_i}\Big\},$$

where $\mathcal{M}(\bar\Omega)$ is the space of bounded measures over $\bar\Omega$ and, in the decomposition of μ, $\tilde\mu$ is a nonnegative measure without atoms, the set I is at most countable, the μ_i are positive real numbers, and the atoms x_i belongs to the critical set $\mathcal{A}$ defined in hypothesis (H4).

We say that a sequence $\{(u_\varepsilon,\mu_\varepsilon)\}_{\varepsilon>0} \subset \mathcal{X}$ converges in $\mathcal{X}$ to (u,μ), which is denoted by $(u_\varepsilon,\mu_\varepsilon) \xrightarrow{\tau} (u,\mu)$, if $u_\varepsilon \rightharpoonup u$ weakly in $L^{q(\cdot)}(\Omega)$ and $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\bar\Omega)$. We recall that $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ means that $\int \phi\,d\mu_\varepsilon \to \int \phi\,d\mu$ for any $\phi \in C(\bar\Omega)$.

We then extend F_ε to the whole space $\mathcal{X}$ by

$$F_\varepsilon(u,\mu) = \begin{cases} \int_\Omega |u|^{q(\cdot)-\varepsilon}\,dx & \text{if } \mu = |\nabla u|^{p(\cdot)}\,dx + \tilde\mu \\ 0 & \text{otherwise in } \mathcal{X} \end{cases}$$

We also consider the limit functional $F^*\colon \mathcal{X} \to \mathbb{R}$ defined by

$$F^*(u,\mu) := \int_\Omega |u|^{q(\cdot)}\,dx + \sum_{i\in I} \mu_i^{\frac{p^*(x_i)}{p(x_i)}}\,\bar S_{x_i}^{-p^*(x_i)},$$

where $\bar S_{x_i}$, $i \in I$, is the localized best Sobolev constant at x_i defined in (2.4).

Our main result is the following

Theorem 1.1. *Assume that p and q satisfies assumptions (H1)–(H4). The functionals $\{F_\varepsilon\}_{\varepsilon>0}$ Γ-converge to F^* in the sense that for any $(u,\mu) \in \mathcal{X}$ there holds that:*

- *For every sequence $\{(u_\varepsilon,\mu_\varepsilon)\}_{\varepsilon>0} \subset \mathcal{X}$ converging to (u,μ) in $\mathcal{X}$, we have*

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,\mu_\varepsilon) \le F^*(u,\mu). \tag{1.3}$$

- *There exists a sequence $\{(u_k,\mu_k)\}_k \subset \mathcal{X}$ converging in $\mathcal{X}$ to (u,μ) with the property that for any sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ converging to 0 as $j \to \infty$, there exists subsequence $\{\varepsilon_{j_k}\}_{k\in\mathbb{N}} \subset \{\varepsilon_j\}_{j\in\mathbb{N}}$ such that*

$$\liminf_{k\to\infty} F_{\varepsilon_{j_k}}(u_k,\mu_k) \ge F^*(u,\mu). \tag{1.4}$$

Remark 1.2. Observe that this is not the usual definition of Γ-convergence. The most common definition is to replace the lim inf inequality (1.4) by

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,\mu_\varepsilon) \ge F^*(u,\mu), \tag{1.5}$$

for some pair $(u_\varepsilon,\mu_\varepsilon)$ converging to (u,μ). These conditions are not equivalent. In fact, (1.5) implies (1.4). However, the condition (1.4) is sufficient for our purposes. Indeed it is easily seen that the following property, one of the most useful consequence of the Γ-convergence, holds: if $(u_\varepsilon,\mu_\varepsilon)$ is a maximizer of F_ε, then any cluster point of the sequence $\{(u_\varepsilon,\mu_\varepsilon)\}_\varepsilon$ is a maximizer of F^* and $\lim_{\varepsilon\to 0} \max F_\varepsilon = \max F^*$. This property will be useful in the proof of Theorem 1.6 below.

Remark 1.3. What we called Γ-convergence in Theorem 1.1 is what other authors called Γ^+-convergence following De Giorgi's original notation – see [29].

Remark 1.4. We mention that Γ-convergence in the framework of variable exponent spaces has already been used to study homogenization problems in [3] and [4].

Define

$$\tilde{S}^{-1} := \tilde{S}(p(\cdot), q(\cdot), \Omega) = \sup_{u\in\mathcal{B}(\Omega)} \int_\Omega |u|^{q(\cdot)}\, dx, \tag{1.6}$$

where $\mathcal{B}(\Omega)$ is defined in (1.2). We also define a local best constant $\tilde{S}_{x_0}^{-1}$, $x_0 \in \mathcal{A}$, in a similar way as in (2.4) by

$$\widetilde{S}_{x_0}^{-1} := \lim_{\varepsilon\to 0} \left(\sup_{u\in\mathcal{B}(B_\varepsilon(x_0)\cap\Omega)} \int_{B_\varepsilon(x_0)} |u|^{q(\cdot)}\, dx \right), \qquad x_0 \in \mathcal{A}. \tag{1.7}$$

Noticing that $\mathcal{B}(B_{x_0}(\varepsilon)\cap\Omega) \subset \mathcal{B}(\Omega)$, we have that

$$\sup_{x_0\in\mathcal{A}} \widetilde{S}_{x_0}^{-1} \leq \widetilde{S}^{-1}. \tag{1.8}$$

We also prove in Lemma 3.3 below that $\widetilde{S}_{x_0}^{-1} = \bar{S}_{x_0}^{-q(x_0)}$ where $\bar{S}_{x_0}$ is defined in (2.4) and appear in the definition of F^*.

We now consider the subcritical approximations $\tilde{S}_\varepsilon^{-1}$ of $\tilde{S}^{-1}$ defined by

$$\tilde{S}_\varepsilon^{-1} := \tilde{S}(p(\cdot), q(\cdot)-\varepsilon, \Omega)^{-1} = \sup_{u\in\mathcal{B}(\Omega)} \int_\Omega |u|^{q(\cdot)-\varepsilon}\, dx.$$

We first prove that

Proposition 1.5. *There holds that*

$$\lim_{\varepsilon\to 0} \tilde{S}_\varepsilon^{-1} = \tilde{S}^{-1}.$$

In the same spirit as in [14, Theorem 4.2], we can deduce from the Γ-convergence of F_ε to F the asymptotic behavior of extremals for $\tilde{S}_\varepsilon$:

Theorem 1.6. *Assume that p and q satisfies assumptions* (H1)–(H4) *and also that $q_- > p_+$. Let $u_\varepsilon \in \mathcal{B}(\Omega)$ be an extremal for $\tilde{S}_\varepsilon^{-1}$, i.e.*

$$\int_\Omega |u_\varepsilon|^{q(\cdot)-\varepsilon}\, dx = \tilde{S}_\varepsilon^{-1}.$$

Then the following alternative holds:

(1) *either the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ has a strongly convergent subsequence in $L^{q(\cdot)}(\Omega)$ and the strong limit is an extremal for $\tilde{S}^{-1}$,*
(2) *or the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ concentrates around a single point $x_0 \in \mathcal{A}$ in the sense that*

$$|u_\varepsilon|^{q(\cdot)}\, dx \rightharpoonup \tilde{S}^{-1}\,\delta_{x_0} \quad \text{and} \quad |\nabla u_\varepsilon|^{p(\cdot)}\, dx \rightharpoonup \delta_{x_0}.$$

Moreover

$$\tilde{S}_{x_0}^{-1} = \tilde{S}^{-1}. \tag{1.9}$$

As an immediate consequence of (1.8) and (1.9), we obtain the following sufficient condition for the existence of an extremal for $\tilde{S}^{-1}$:

Corollary 1.7. *If* $\sup_{x\in\mathcal{A}} \tilde{S}_x^{-1} < \tilde{S}^{-1}$, *then any sequence of extremals for* $\tilde{S}_\varepsilon^{-1}$ *converges, up to a subsequence, to some* $u \in \mathcal{B}(\Omega)$ *which is an extremal for* $\tilde{S}^{-1}$. *In particular, there exists an extremal for* $\tilde{S}^{-1}$.

This kind of sufficient condition of existence is common in the study of problems with critical exponent. In the constant exponent case, it goes back to [6,9,23]. In the variable exponent case, it was recently established and used by the authors in [14,13,15,16] where precise condition on the exponents p and q were provided for this condition to hold.

2. Preliminary notations

2.1. Lebesgue and Sobolev spaces with variable exponents

Let Ω be smooth open bounded subset of $\mathbb{R}^n$. Given a measurable function $p\colon \Omega \to [1, +\infty)$, the Lebesgue variable exponent space $L^{p(\cdot)}(\Omega)$ is defined as

$$L^{p(\cdot)}(\Omega) := \left\{ u \in L^1_{\text{loc}}(\Omega)\colon \int_\Omega |u|^{p(\cdot)}\,dx < +\infty \right\}.$$

This space is endowed with the norm

$$\|u\|_{p(\cdot)} := \inf\left\{ \lambda > 0\colon \int_\Omega \left|\frac{u}{\lambda}\right|^{p(\cdot)} dx \le 1 \right\},$$

which turns $L^{p(\cdot)}(\Omega)$ into a Banach space. Assuming moreover that

$$1 < p_- := \inf_\Omega p \le p_+ := \sup_\Omega p < +\infty, \tag{2.1}$$

it can be proved that $L^{p(\cdot)}(\Omega)$ separable and reflexive.

These spaces where first considered in the seminal W. Orlicz' paper [28] in 1931 but then where left behind as the author pursued the study of the spaces that now bear his name. The first systematic study of these spaces appeared in H. Nakano's works at the beginning of the 1950s [26,27] where he developed a general theory in which the spaces $L^{p(\cdot)}(\Omega)$ were a particular example of the more general spaces he was considering. Even though some progress was made after Makano's work (see in particular the works of the Polish school H. Hudzik, A. Kamińska and J. Musielak in e.g. [20,21,25]), it was only in the last 20 years that major progress has been accomplished mainly due to the following facts:

- The discovery of a very weak condition ensuring the boundedness of the Hardy–Littlewood maximal operator in these spaces, i.e. the log-Hölder condition that implies, to begin with, that test functions are dense in $L^{p(\cdot)}(\Omega)$.
- The discovery of the connection of these spaces with the modeling of the so-called *electrorheological fluids* [32].
- The application that variable exponents have shown in image processing [10].

A complete presentation of variable Lebesgue and Sobolev spaces can be found in the book [11] and [5].

Of central importance in the above mentioned applications are the variable exponent Sobolev spaces $W^{1,p(\cdot)}(\Omega)$ defined as

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in W^{1,1}_{\mathrm{loc}}(\Omega) \colon u, \partial_i u \in L^{p(\cdot)}(\Omega)\ i = 1, \dots, n\right\},$$

and the subspace of functions with zero boundary values

$$W^{1,p(\cdot)}_0(\Omega) := \overline{\{u \in W^{1,p(\cdot)}(\Omega) \colon u \text{ has compact support}\}},$$

where the closure is taken in the $W^{1,p(\cdot)}(\Omega)$-norm $\|\cdot\|_{1,p(\cdot)}$ that is defined as

$$\|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

As first noticed in [33], when p is log-Hölder in the sense that

$$\sup_{x,y\in\Omega} |(p(x) - p(y)) \log(|x - y|)| < +\infty, \tag{2.2}$$

it can be proved that the space $C^\infty_c(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$ and in $W^{1,p(\cdot)}_0(\Omega)$, and also that the Poincaré inequality holds i.e. there exists a constant $C = C(\Omega, p) > 0$ such that

$$\|u\|_{p(\cdot)} \le C\|\nabla u\|_{p(\cdot)}$$

for any $u \in W^{1,p(\cdot)}_0(\Omega)$. It follows in particular that $\|\nabla u\|_{p(\cdot)}$ is an equivalent norm in $W^{1,p(\cdot)}_0(\Omega)$.

2.2. *Critical Sobolev embedding*

A major tool in order to study existence and regularity properties of solutions to partial differential equations is the Sobolev embedding theorem. For variable exponents spaces this theorem has been established in [22] (see also [12]). Given a measurable function $q\colon \Omega \to [1, +\infty)$, it basically says that, assuming $p^+ < n$, we have a continuous embedding

$$W^{1,p(\cdot)}_0(\Omega) \subset L^{q(\cdot)}(\Omega)$$

if and only if $q(x) \le p^*(x) := np(x)/(n - p(x))$. Moreover, when the exponent q is *strictly subcritical* in the sense that

$$\inf_{x\in\Omega}(p^*(x) - q(x)) > 0,$$

then this embedding is compact (see e.g. [11]). On the other hand when the *critical set*

$$\mathcal{A} := \{x \in \overline{\Omega} \colon q(x) = p^*(x)\} \tag{2.3}$$

is not empty, the immersion is no longer compact in general (see [24] for some very restricted cases where $\mathcal{A} \neq \emptyset$ but the immersion still remains compact). The existence of extremal for the best constant S defined in (1.1) is then not granted. Indeed the well-known Pohozaev identity implies that when p is constant and Ω is star-shaped then S is not attained.

Recall that in [14] the definition of $\mathcal{A}$ does not contain the points on the boundary of Ω. That is not correct, and the set $\mathcal{A}$ in [14] has to be the same as in (2.3). However the results in [14] still holds with minor modifications.

The problem of existence of extremals for S in the variable exponent setting was recently considered in [14] where the authors provided sufficient existence conditions. A fundamental tool used in their proof, as well as in almost every problem dealing with critical exponent in general, is the so called *Concentration Compactness Principle* (CCP) that was introduced by P. L. Lions in the 80's (see [23]) and was recently extended to the variable exponent setting in [17] (see also the refinement in [14]). This version of the CCP relies on a notion of localized Sobolev constant defined as follows. For $x \in \mathcal{A}$ we define the localized best Sobolev constant $\bar{S}_x$ as

$$\bar{S}_x \coloneqq \lim_{\varepsilon\to 0} S(p(\cdot), q(\cdot), B_\varepsilon(x)\cap\Omega) = \lim_{\varepsilon\to 0} \inf_{v\in W_0^{1,p(\cdot)}(\Omega)} \frac{\|\nabla v\|_{p(\cdot),B_\varepsilon(x)\cap\Omega}}{\|v\|_{q(\cdot),B_\varepsilon(x)\cap\Omega}}. \tag{2.4}$$

Notice that

$$0 < S(p(\cdot), q(\cdot), \Omega) \le \inf_{x\in\mathcal{A}} \bar{S}_x, \tag{2.5}$$

and that for any $x_0 \in \mathcal{A}$,

$$\bar{S}_{x_0} \le \inf_{u\in C_c^\infty(\mathbb{R}^n)} \frac{\|\nabla u\|_{p(x_0)}}{\|u\|_{p^*(x_0)}}. \tag{2.6}$$

Observe that the r.h.s. of this inequality is the best constant in the usual Sobolev embedding in $\mathbb{R}^n$ with constant exponent $p(x_0)$. We refer to [14] for a proof of this inequality.

The CCP proved in [17] and refined in [14] states that given a weakly convergent sequence $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{1,p(\cdot)}(\Omega)$ with weak limit u, there exists a countable set of indices I, positive real numbers $\{\mu_i\}_{i\in I}$, $\{\nu_i\}_{i\in I} \subset \mathbb{R}_+$, points $\{x_i\}_{i\in I} \in \mathcal{A}$ and nonnegative measures μ, ν such that

$$|u_k|^{q(\cdot)}\,dx \overset{*}{\rightharpoonup} d\nu = |u|^{q(\cdot)}\,dx + \sum_{i\in I} \nu_i\,\delta_{x_i}, \tag{2.7}$$

$$|\nabla u_k|^{p(\cdot)}\,dx \overset{*}{\rightharpoonup} d\mu \ge |\nabla u|^{p(\cdot)}\,dx + \sum_{i\in I} \mu_i\,\delta_{x_i}, \tag{2.8}$$

$$\bar{S}_{x_i}\nu_i^{\frac{1}{p^*(x_i)}} \le \mu_i^{\frac{1}{p(x_i)}} \qquad \text{for any } i\in I. \tag{2.9}$$

It is also easily checked that the nonnegative measure

$$\tilde{\mu} \coloneqq \mu - \left(|\nabla u|^{p(\cdot)}\,dx + \sum_{i\in I} \mu_i\,\delta_{x_i}\right)$$

has no atoms.

By analyzing the behavior of minimizing sequences using the CCP, it is proved in [14] that if the inequality in (2.5) is strict, $q_- < p_+$ and p, q are slightly more regular than merely Log-Hölder continuous, more precisely if

$$\lim_{\Omega\ni y\to x} (p(y)-p(x))\log(|x-y|) = \lim_{\Omega\ni y\to x} (q(y)-q(x))\log(|x-y|) = 0, \text{ uniformly in } x\in\Omega \tag{2.10}$$

is satisfied, then there exists an extremal for $S(p(\cdot), q(\cdot), \Omega)$, i.e. a function $u \in W_0^{1,p(\cdot)}(\Omega)$ where the infimum in (1.1) is attained.

3. Proof of Theorem 1.1

We divided the proof into two subsections: one for the lim sup inequality (1.3) and other for the lim inf inequality (1.4). The strategy of the proof is completely analogous to that of [2] and [29] where the constant exponent case is treated with difficulties specific to the variable exponent setting. We mention that an alternative approach to that used here in the proof of the lim inf-inequality has recently appeared in [31] when dealing with the fractional Laplacian.

3.1. *Proof of the* lim sup *property* (1.3)

Consider a sequence $\{(u_\varepsilon, \mu_\varepsilon)\}_{\varepsilon>0} \subset \mathcal{X}$ converging as $\varepsilon \to 0$ to some $(u_0, \mu_0) \in \mathcal{X}$. We can assume without loss of generality that $\mu_\varepsilon = |\nabla u_\varepsilon|^{p(\cdot)}\,dx + \tilde{\mu}_\varepsilon$ for all $\varepsilon > 0$ where $\tilde{\mu}_\varepsilon$ is a non-negative and non-atomic measure.

We first assume that $\tilde{\mu}_\varepsilon = 0$ i.e. $\mu_\varepsilon = |\nabla u_\varepsilon|^{p(\cdot)}\,dx$. Then by Hölder inequality (see [11, lemma 3.2.20]):

$$\begin{aligned} F_\varepsilon(u_\varepsilon, \mu_\varepsilon) &= \int_\Omega |u_\varepsilon|^{q(\cdot)-\varepsilon}\,dx \\ &\le \Big(\frac{1}{(\frac{q(\cdot)}{q(\cdot)-\varepsilon})^-} + \frac{1}{(\frac{q(\cdot)}{\varepsilon})^-} \Big) \| |u_\varepsilon|^{q(\cdot)-\varepsilon} \|_{\frac{q(\cdot)}{q(\cdot)-\varepsilon}} \|1\|_{\frac{q(\cdot)}{\varepsilon}}. \end{aligned}$$

Since $\left(\frac{q(\cdot)}{q(\cdot)-\varepsilon}\right)^- \to 1$, $\left(\frac{q(\cdot)}{\varepsilon}\right)^- \to \infty$, and $\|1\|_{\frac{q(\cdot)}{\varepsilon}} \to 1$ as $\varepsilon \to 0$, we obtain

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \mu_\varepsilon) \le \limsup_{\varepsilon\to 0} \| |u_\varepsilon|^{q(\cdot)-\varepsilon} \|_{\frac{q(\cdot)}{q(\cdot)-\varepsilon}}.$$

Up to some subsequence, by the CCP, there exists $u \in W_0^{1,p(\cdot)}(\Omega)$ and measures $\nu, \mu \in \mathcal{M}(\overline{\Omega})$ of the form $\nu = |u|^{q(\cdot)}dx + \sum_{i\in I} \nu_i \delta_{x_i}$, $\mu = |\nabla u|^{p(\cdot)}dx + \tilde{\mu} + \sum_{i\in I} \mu_i \delta_{x_i}$ such that

$$\begin{aligned} &u_\varepsilon \rightharpoonup u \qquad \text{weakly in } W_0^{1,p(\cdot)}(\Omega) \text{ and in } L^{q(\cdot)}(\Omega), \\ &|u_\varepsilon|^{q(\cdot)}\,dx \overset{*}{\rightharpoonup} \nu \quad \text{and} \quad |\nabla u_\varepsilon|^{p(\cdot)}\,dx \overset{*}{\rightharpoonup} \mu. \end{aligned}$$

Observe that $u = u_0$ and $\mu_0 = \mu$ since $(u_\varepsilon, \mu_\varepsilon = |\nabla u_\varepsilon|^{p(\cdot)}\,dx) \overset{\tau}{\to} (u_0, \mu_0)$ in $\mathcal{X}$. It follows that

$$\rho_{q(x)}(u_\varepsilon) := \int_\Omega |u_\varepsilon|^{q(\cdot)}\,dx \to \int_\Omega |u_0|^{q(\cdot)}\,dx + \sum_{i\in I} \nu_i \qquad \text{as } \varepsilon \to 0.$$

Since

$$\| |u_\varepsilon|^{q(\cdot)-\varepsilon} \|_{\frac{q(\cdot)}{q(\cdot)-\varepsilon}} \le \max\left\{ \rho_{q(\cdot)}(u_\varepsilon)^{(\frac{q(\cdot)}{q(\cdot)-\varepsilon})^+}, \rho_{q(\cdot)}(u_\varepsilon)^{(\frac{q(\cdot)}{q(\cdot)-\varepsilon})^-} \right\},$$

we obtain, in view of (2.9), that

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, \mu_\varepsilon) \le \int_\Omega |u_0|^{q(\cdot)}\,dx + \sum_{i\in I} \bar{S}_{x_i}^{-p^*(x_i)} \mu_i^{\frac{p^*(x_i)}{p(x_i)}} = F^*(u_0, \mu_0). \tag{3.1}$$

This is the limsup inequality.

We now assume that $\mu_\varepsilon = |\nabla u_\varepsilon|^{p(\cdot)}\,dx + \tilde{\mu}_\varepsilon$ for all $\varepsilon > 0$ where $\tilde{\mu}_\varepsilon$ is a non-negative and non-atomic measure. Since $\tilde{\mu}_\varepsilon(\overline{\Omega}) \le \mu_\varepsilon(\overline{\Omega}) \le 1$, we can assume that $\tilde{\mu}_\varepsilon$ converge weakly as $\varepsilon \to 0$ to some non-negative measure $\tilde{\mu}_0$. Notice that $\tilde{\mu}_0$ may have atoms. Passing to the limit in $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu_0$ gives $\mu_0 = \mu + \tilde{\mu}_0$ with μ as before. Notice that according to the definition of F_ε, we have $F_\varepsilon(u_\varepsilon, \mu_\varepsilon) = F_\varepsilon(u_\varepsilon, |\nabla u_\varepsilon|^{p(\cdot)}\, dx)$. Applying the first part of the proof to $\mu_\varepsilon - \tilde{\mu}_\varepsilon = |\nabla u_\varepsilon|^{p(\cdot)}\, dx$, we deduce from (3.1) that

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, \mu_\varepsilon) = \limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, |\nabla u_\varepsilon|^{p(\cdot)}\, dx) \le F^*(u_0, \mu_0 - \tilde{\mu}_0).$$

It follows from $\mu_0 = \mu + \tilde{\mu}_0 \ge \tilde{\mu}_0$ that any atom x_0 of $\tilde{\mu}_0$ is an atom of μ_0 with $\mu_0(\{x_0\}) \ge \tilde{\mu}_0(\{x_0\})$. Thus $F^*(u_0, \mu_0 - \tilde{\mu}_0) \le F^*(u_0, \mu_0)$ which concludes the proof. □

3.2. *Proof of the* lim inf *property* (1.4)

The proof of the lim inf property (1.4) follows the original scheme of [2] and mainly consists in proving it in two particular cases: when μ has no atoms (see Proposition 3.1) and when μ is purely atomic with a finite number of atoms (see Proposition 3.2).

We first prove the lim inf property when μ has no atoms.

Proposition 3.1. *Let $(u, \mu) \in \mathcal{X}$ such that μ has no atomic part i.e. $\mu = |\nabla u|^{p(\cdot)}\, dx + \widetilde{\mu}$. Then*

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \mu_\varepsilon) = F^*(u, \mu)$$

for every sequence $\{(u_\varepsilon, \mu_\varepsilon = |\nabla u_\varepsilon|^{p(\cdot)}\, dx + \tilde{\mu}_\varepsilon)\}_{\varepsilon>0} \subset \mathcal{X}$ converging to (u, μ) as $\varepsilon \to 0$.

Notice that we can take in particular the constant sequence $(u_\varepsilon, \mu_\varepsilon) = (u, \mu)$.

Proof. Consider a sequence $\{(u_\varepsilon, \mu_\varepsilon := |\nabla u_\varepsilon|^{p(\cdot)}\, dx + \tilde{\mu}_\varepsilon)\}_{\varepsilon>0} \subset \mathcal{X}$ converging to (u, μ). According to the CCP, the atoms of the measure $\nu := \lim |u_\varepsilon|^{q(\cdot)}\, dx$ (limit in $\mathcal{M}(\bar{\Omega})$ – which exists up to a subsequence) are also atoms of μ. Since by assumption μ has no atomic part, we deduce that ν also has no atoms so that the measure $|u_\varepsilon|^{q(\cdot)}\, dx$ weakly converges to $|u|^{q(\cdot)}\, dx$. In particular $\lim_{\varepsilon\to 0} \int_\Omega |u_\varepsilon|^{q(\cdot)}\, dx = \int_\Omega |u|^{q(\cdot)}\, dx$ i.e. $\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{q(\cdot)} = \|u_\varepsilon\|_{q(\cdot)}$. Since we also have the weak convergence of u_ε to u in $L^{q(\cdot)}(\Omega)$, which is a uniformly convex Banach space since $1 < q_- \le q_+ < \infty$ (see [11, thm 3.4.9]), we deduce that $u_\varepsilon \to u$ strongly in $L^{q(\cdot)}(\Omega)$. Up to a subsequence we can further assume that the convergence holds a.e.

As in the proof of the lim sup property we obtain that

$$\limsup_{\varepsilon\to 0} \int_\Omega |u_\varepsilon|^{q(\cdot)-\varepsilon}\, dx \le \int_\Omega |u|^{q(\cdot)}\, dx.$$

Moreover using Fatou lemma,

$$\liminf_{\varepsilon\to 0} \int_\Omega |u_\varepsilon|^{q(\cdot)-\varepsilon}\, dx \ge \int_\Omega |u|^{q(\cdot)}\, dx.$$

Hence

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \mu_\varepsilon) = \lim_{\varepsilon\to 0} \int_\Omega |u_\varepsilon|^{q(\cdot)-\varepsilon}\, dx = \int_\Omega |u|^{q(\cdot)}\, dx = F^*(u, \mu),$$

as we wanted to show. □

We now prove the lim inf property assuming that μ is purely atomic with a finite number of atoms and total mass strictly less that 1.

Proposition 3.2. *Consider $(u, \mu) \in \mathcal{X}$ of the form $(u, \mu) = (0, \sum_{i=0}^{k} \mu_i \delta_{x_i})$ with $x_i \in \mathcal{A}$ and $\mu_i > 0$ such that $\mu(\bar{\Omega}) = \sum_i \mu_i < 1$. Then there exists a sequence $(u_k, |\nabla u_k|^{p(\cdot)}) \in \mathcal{X}$ converging in $\mathcal{X}$ to (u, μ) with the property that for any sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ converging to 0 as $j \to \infty$, there exists subsequence $\{\varepsilon_{j_k}\}_{k\in\mathbb{N}} \subset \{\varepsilon_j\}_{j\in\mathbb{N}}$ such that*

$$\lim_{k\to\infty} F_{\varepsilon_{j_k}}(u_k, |\nabla u_k|^{p(\cdot)}) = F^*(u, \mu).$$

The proof relies on the following two lemmas. The first one gives the relation between the two localized Sobolev constants $\bar{S}_{x_0}$ and $\widetilde{S}_{x_0}$ defined in (2.4) and (1.7) for a point $x_0 \in \mathcal{A}$.

Lemma 3.3. *For any $x_0 \in \mathcal{A}$,*

$$\widetilde{S}_{x_0}^{-1} = \bar{S}_{x_0}^{-q(x_0)}.$$

Proof. First, suppose that $\widetilde{S}_{x_0}^{-1} > 1$. So there exists $\varepsilon_0 > 0$ such that

$$\sup_{u\in\mathcal{B}(B_\varepsilon(x_0)\cap\Omega)} \int_{B_\varepsilon(x_0)} |u|^{q(\cdot)}\, dx > 1 \qquad \text{for all } \varepsilon \le \varepsilon_0.$$

It follows that

$$\sup_{u\in\mathcal{B}(B_\varepsilon(x_0)\cap\Omega)} \|u\|_{q(\cdot),B_\varepsilon(x_0)}^{q_\varepsilon^-} \le \sup_{u\in\mathcal{B}(B_\varepsilon(x_0)\cap\Omega)} \int_{B_\varepsilon(x_0)} |u|^{q(\cdot)}\, dx \le \sup_{u\in\mathcal{B}(B_\varepsilon(x_0)\cap\Omega)} \|u\|_{q(\cdot),B_\varepsilon(x_0)}^{q_\varepsilon^+},$$

where $q_\varepsilon^- := \inf_{B_\varepsilon(x_0)} q(\cdot)$ and $q_\varepsilon^+ := \sup_{B_\varepsilon(x_0)} q(\cdot)$. Notice that

$$\sup_{u\in\mathcal{B}(B_\varepsilon(x_0)\cap\Omega)} \|u\|_{q(\cdot),B_\varepsilon(x_0)} = \left(\inf_{u\in\tilde{\mathcal{B}}(B_\varepsilon(x_0)\cap\Omega)} \|\nabla u\|_{p(\cdot),B_\varepsilon(x_0)} \right)^{-1},$$

where $\tilde{\mathcal{B}}(U) = \{u \in W_0^{1,p(\cdot)}(U) \colon \|u\|_{q(\cdot),U} \le 1\}$ for any open set $U \subset \mathbb{R}^n$. So, recalling that

$$\lim_{\varepsilon\to 0} \inf_{u\in\tilde{\mathcal{B}}(B_\varepsilon(x_0)\cap\Omega)} \|\nabla u\|_{p(\cdot),B_\varepsilon(x_0)} = \bar{S}_{x_0}$$

in view of (1.7), we get

$$\bar{S}_{x_0}^{-q(x_0)} = \tilde{S}_{x_0}^{-1}.$$

The case where $\widetilde{S}_{x_0}^{-1} \le 1$ is analogous. □

Lemma 3.4. *For any $x_0 \in \mathcal{A}$ there exists a sequence $(u_l)_l \subset W_0^{1,p(\cdot)}(\Omega)$ such that*

$$(u_l, |\nabla u_l|^{p(\cdot)}) \xrightarrow{\tau} (0, \delta_{x_0}) \qquad \text{and} \qquad \int_\Omega |u_l|^{q(\cdot)}\, dx \to \tilde{S}_{x_0}^{-1}.$$

As a consequence, for any sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ such that $\varepsilon_j \to 0$ as $j \to \infty$, there exists subsequence $\{\varepsilon_{j_l}\}_{l\in\mathbb{N}} \subset \{\varepsilon_j\}_{j\in\mathbb{N}}$ such that

$$\lim_{l\to\infty} \int |u_l|^{q(\cdot)-\varepsilon_{j_l}}\, dx = \bar{S}_{x_0}^{-q(x_0)} = \tilde{S}_{x_0}^{-1}.$$

Proof. The existence of a subsequence $\{\varepsilon_{j_l}\}_{l\in\mathbb{N}}$ follows from the fact that

$$\lim_{\varepsilon\to 0}\int_\Omega |u|^{q(\cdot)-\varepsilon}\,dx = \int_\Omega |u|^{q(\cdot)}\,dx \qquad \text{for any } u\in L^{q(\cdot)}(\Omega). \tag{3.2}$$

Indeed this is an application of the dominated convergence theorem noticing that

$$|u|^{q(\cdot)-\varepsilon} = |u|^{q(\cdot)-\varepsilon}1_{\{|u|\geq 1\}} + |u|^{q(\cdot)-\varepsilon}1_{\{|u|\leq 1\}} \leq |u|^{q(\cdot)}+1.$$

Concerning the existence of $(u_l)_l$, notice that for any $l>0$ there exists $\varepsilon_l>0$, $\varepsilon_l\to 0$, such that

$$\left|\widetilde{S}_{x_0}^{-1} - \widetilde{S}(p(\cdot),q(\cdot),B_{\varepsilon_l}(x_0)\cap\Omega)^{-1}\right| < \frac{1}{l},$$

and that there exists $u_l\in W_0^{1,p(\cdot)}(B_{\varepsilon_l}(x_0)\cap\Omega)$ such that $\|\nabla u_l\|_{p(\cdot)}\leq 1$ and

$$\widetilde{S}(p(\cdot),q(\cdot),B_{\varepsilon_l}(x_0)\cap\Omega)^{-1} - \frac{1}{l} < \int_{B_{\varepsilon_l}(x_0)} |u_l|^{q(\cdot)}\,dx \leq \widetilde{S}(p(\cdot),q(\cdot),B_{\varepsilon_l}(x_0)\cap\Omega)^{-1}.$$

In particular

$$\left|\widetilde{S}_{x_0}^{-1} - \int_\Omega |u_l|^{q(\cdot)}\,dx\right| \leq \frac{2}{l}.$$

Observe that $u_l\in W_0^{1,p(\cdot)}(\Omega)$ with $\|\nabla u_l\|_{p(\cdot)}\leq 1$ so that $(u_l)_l$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$. Moreover $\operatorname{supp} u_l\subset B_{\varepsilon_l}(x_0)\cap\Omega)$. It easily follows that for any $\phi\in C_c^\infty(\Omega)$, $\int u_l\phi\,dx\to 0$, so that $u_l\to 0$ weakly in $L^{q(\cdot)}(\Omega)$, and also that $\|u_l\|_{q(\cdot)}\to \tilde{S}_{x_0}^{-1/p^*(x_0)} = \bar{S}_{x_0}^{-1}$ in view of the previous lemma. Moreover notice that $\alpha := \liminf\|\nabla u_l\|_{p(\cdot)}>0$. Indeed otherwise $u_l\to 0$ strongly in $W^{1p(\cdot)}(\Omega)$, and in particular in $L^{q(\cdot)}(\Omega)$, so that $\tilde{S}_{x_0}^{-1}=0$ i.e. $\bar{S}_{x_0}=+\infty$. This contradicts (2.6). Then, in view of the definition (1.7) of $\bar{S}_{x_0}$ we obtain that

$$\lim_{l\to+\infty}\|u_l\|_{q(\cdot)} = \bar{S}_{x_0}^{-1} \leq \liminf_{l\to+\infty}\frac{\|u_l\|_{q(\cdot)}}{\|\nabla u_l\|_{q(\cdot)}} \leq \liminf_{l\to+\infty}\frac{\|u_l\|_{q(\cdot)}}{\alpha}.$$

It follows that $\lim\|\nabla u_l\|_{p(\cdot)}=1$. As a consequence $|\nabla u_k|^{p(\cdot)}\rightharpoonup\delta_{x_0}$ weakly in the sense of measures. □

We can now prove Proposition 3.2:

Proof of Proposition 3.2. We prove the claim in the case $k=2$ i.e. for μ of the form $\mu=\mu_0\delta_{x_0}+\mu_1\delta_{x_1}$ with $x_0,x_1\in\mathcal{A}$ and $\mu_0,\mu_1>0$, $\mu(\bar{\Omega})=\mu_0+\mu_1<1$. We denote by $u_{0,k}$ and $u_{1,k}$ the functions corresponding to the points x_0 and x_1 given by Lemma 3.4, and by $\{\varepsilon_{j_k}\}_{k\in\mathbb{N}}$ a subsequence such that:

$$\begin{aligned}
&(u_{0,k},|\nabla u_{0,k}|^{p(\cdot)}) \xrightarrow{\tau} (0,\delta_{x_0}), \qquad \lim_{k\to\infty}\int |u_{0,k}|^{q(\cdot)-\varepsilon_{j_k}}\,dx = \bar{S}_{x_0}^{-q(x_0)},\\
&(u_{1,k},|\nabla u_{1,k}|^{p(\cdot)}) \xrightarrow{\tau} (0,\delta_{x_1}), \qquad \lim_{k\to\infty}\int |u_{1,k}|^{q(\cdot)-\varepsilon_{j_k}}\,dx = \bar{S}_{x_1}^{-q(x_1)}.
\end{aligned} \tag{3.3}$$

Since $x_0\neq x_1$, the supports of $u_{0,k}$ and $u_{1,k}$ are disjoint for ε_{j_k} small. It follows that the functions

$$u_k := \mu_0^{\frac{1}{p(x_0)}}u_{0,k} + \mu_1^{\frac{1}{p(x_1)}}u_{1,k}$$

satisfy for any given $\psi \in C(\bar{\Omega})$ that

$$\int_\Omega \psi |\nabla u_k|^{p(\cdot)}\,dx = \int \mu_0^{\frac{p(x)}{p(x_0)}} \psi |\nabla u_{0,k}|^{p(\cdot)}\,dx + \int \mu_1^{\frac{p(x)}{p(x_1)}} \psi |\nabla u_{1,k}|^{p(\cdot)}\,dx$$

$$\to \mu_0 \psi(x_0) + \mu_1 \psi(x_1) = \int \psi\, d\mu$$

in view of (3.3). In particular $\lim_{k\to\infty} \int_\Omega |\nabla u_k|^{p(\cdot)}\,dx = \mu_0 + \mu_1 < 1$. Hence $(u_k, |\nabla u_k|^{p(\cdot)})$ belongs to $\mathcal{X}$ for ε_{j_k} small and converges to $(0,\mu)$ as $\varepsilon \to 0$.

Moreover, recalling the $u_{0,k}$ and $u_{1,k}$ have disjoint support, we have

$$\begin{aligned}
F_{\varepsilon_{j_k}}(u_k, |\nabla u_k|^{p(\cdot)}) &= \int_\Omega |u_k|^{q(\cdot)-\varepsilon_{j_k}}\,dx = \int_\Omega \mu_0^{\frac{q(\cdot)-\varepsilon_{j_k}}{p(x_0)}} |u_{0,k}|^{q(\cdot)-\varepsilon_{j_k}}\,dx + \int_\Omega \mu_1^{\frac{q(\cdot)-\varepsilon_{j_k}}{p(x_1)}} |u_{1,k}|^{q(\cdot)-\varepsilon_{j_k}}\,dx \\
&= (1+o(1)) \int_\Omega \mu_0^{\frac{q(\cdot)}{p(x_0)}} |u_{0,k}|^{q(\cdot)-\varepsilon_{j_k}}\,dx + (1+o(1)) \int_\Omega \mu_1^{\frac{q(\cdot)}{p(x_1)}} |u_{1,k}|^{q(\cdot)-\varepsilon_{j_k}}\,dx \\
&= \mu_0^{\frac{q(x_0)}{p(x_0)}} \bar{S}_{x_0}^{-q(x_0)} + \mu_1^{\frac{q(x_1)}{p(x_1)}} \bar{S}_{x_1}^{-q(x_1)} + o(1) \\
&= F^*(0, \mu_0\delta_{x_0} + \mu_1\delta_{x_1}) + o(1).
\end{aligned}$$

Now, we note that if for any sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ such that $\varepsilon_j \to 0$ as $j \to \infty$, there exists subsequence $\{\varepsilon_{j_k}\}_{k\in\mathbb{N}} \subset \{\varepsilon_j\}_{j\in\mathbb{N}}$ and a sequence $(u_k, |\nabla u_k|^{p(\cdot)}) \in \mathcal{X}$ converging in $\mathcal{X}$ to (u,μ) such that

$$\lim_{k\to\infty} F_{\varepsilon_{j_k}}(u_k, |\nabla u_k|^{p(\cdot)}) = F^*(0, \mu_0\delta_{x_0} + \mu_1\delta_{x_1}).$$

In particular, this finishes the proof of Proposition 3.2 in the case $k = 2$. The proof when μ has an arbitrary finite number of atoms is similar. □

The next lemma was first proved in [2] and [30] in the constant exponent case and allows to deduce the general case from the two particular cases stated in Propositions 3.1 and 3.2. Since its proof is identical to that of [2, lemma 4.1] and [29] we omit it.

Lemma 3.5. *If the* lim inf *property (1.4) holds for every $(u,\mu) \in \mathcal{X}$ such that*

(1) $\mu(\overline{\Omega}) < 1$,
(2) $\mu = |\nabla u|^{p(\cdot)} + \widetilde{\mu} + \sum_{i=0}^n \mu_i \delta_{x_i}$,
(3) $dist(\overline{supp(|u| + \tilde{\mu})}, \bigcup_{i=1}^n \{x_i\}) > 0$,

then it holds for any $(u,\mu) \in \mathcal{X}$.

Finally, we can prove the principal result.

Proof of the lim inf inequality. We only have to check the hypotheses of Lemma 3.5. Given some $(u,\mu) \in \mathcal{X}$ as in Lemma 3.5, we can descompose μ as $\mu = \mu^1 + \mu^2$ with $\mu^1 = \sum_{i=0}^n \mu_i \delta_{x_i}$ and $\mu^2 = |\nabla u|^{p(\cdot)} + \widetilde{\mu}$. Moreover there exist relatively open subsets $A, B \subset \overline{\Omega}$ such that $\overline{\text{supp}(\mu^1)} \subset \overline{A}$ and $\overline{\text{supp}(|u| + \tilde{\mu})} \subset \overline{B}$ and $\overline{A} \cap \overline{B} = \emptyset$.

By Propositions 3.1 and 3.2, there exist sequences $(u_k^1, \mu_k^1 = |\nabla u_k^1|^{p(\cdot)}) \in \mathcal{X}$ and $(u_k^2, \mu_k^2 = |\nabla u_k^2|^{p(\cdot)} + \tilde{\mu}_k^2) \in \mathcal{X}$ with $u_k^1 \in W_0^{1,p(\cdot)}(A)$, $u_k^2 \in W_0^{1,p(\cdot)}(B)$ converging in $\mathcal{X}$ to $(0,\mu^1)$ and (u,μ^2) respectively, and a subsequence $(\varepsilon_{j_k})_k$ of (ε_j) satisfying

$$F_{\varepsilon_{j_k}}(u_k^1, \mu_k^1) \to F^*(0, \mu^1) \text{ and } F_{\varepsilon_{j_k}}(u_k^2, \mu_k^2) \to F^*(u, \mu^2).$$

Consider $u_k = u_k^1 + u_k^2$ and $\mu_k = \mu_k^1 + \mu_k^2$. As u_k^1 and u_k^2 have disjoint support, it is easily seen as in the proof of Proposition 3.2, that (u_k, μ_k) belongs to $\mathcal{X}$ and converges to (u, μ). Moreover

$$\begin{aligned} F_{\varepsilon_{j_k}}(u_k, \mu_k) &= F_{\varepsilon_{j_k}}(u_k^1, \mu_k^1) + F_{\varepsilon_{j_k}}(u_k^2, \mu_k^2) \\ &= F^*(0, \mu^1) + F^*(u, \mu^2) + o(1) \\ &= \int_\Omega |u|^{q(\cdot)}\, dx + \sum_{i=0}^n \mu_i^{\frac{p^*(x_i)}{p(x_i)}} \overline{S}_{x_i}^{-p^*(x_i)} + o(1) \\ &= F^*(u, \mu) + o(1). \end{aligned}$$

This finishes the proof. $\square$

4. Proof of Proposition 1.5 and Theorem 1.6

4.1. Proof of Proposition 1.5

Using Hölder inequality (A.1), we have for any $u \in \mathcal{B}(\Omega)$ that

$$\begin{aligned} \int_\Omega |u|^{q(\cdot)-\varepsilon}\, dx &\le \left(\frac{1}{\left(\frac{q}{q-\varepsilon}\right)_-} + \frac{1}{\left(\frac{q}{\varepsilon}\right)_-} \right) \left(\int_\Omega |u|^{q(\cdot)}\, dx \right)^{\frac{1}{\left(\frac{q}{q-\varepsilon}\right)_-}} \|1\|_{\left(\frac{q(\cdot)}{\varepsilon}\right)'} \\ &= (1 + o(1)) \Big(\int_\Omega |u|^{q(\cdot)}\, dx \Big)^{1+o(1)} \end{aligned}$$

from which we deduce that $\limsup_{\varepsilon\to 0} \tilde{S}_\varepsilon^{-1} \le \tilde{S}^{-1}$.

We prove the opposite inequality. For a given $\delta > 0$, consider $u_\delta \in \mathcal{B}(\Omega)$ such that

$$\int_\Omega |u_\delta|^{q(\cdot)} \ge \tilde{S}^{-1} - \delta.$$

Recalling (3.2), we then have

$$\liminf_{\varepsilon\to 0} \tilde{S}_\varepsilon^{-1} \ge \liminf_{\varepsilon\to 0} \int_\Omega |u_\delta|^{q(\cdot)-\varepsilon} = \int_\Omega |u_\delta|^{q(\cdot)}\, dx \ge \tilde{S}^{-1} - \delta.$$

The proof is now complete.

4.2. Proof of Theorem 1.6

Before proving Theorem 1.6 we need the following Sobolev type inequality deduced from the definition of $\tilde{S}$:

Proposition 4.1. *For any $u \in W_0^{1,p(\cdot)}(\Omega)$,*

$$\int_\Omega |u|^{q(\cdot)}\, dx \le \tilde{S}^{-1} \max\left\{ \|\nabla u\|_{p(\cdot)}^{q^+}, \|\nabla u\|_{p(\cdot)}^{q^-} \right\}. \tag{4.1}$$

Proof. Let $u \in W_0^{1,p(\cdot)}(\Omega)$. By definition of the norm $\|\cdot\|_{p(\cdot)}$, there holds

$$\int_\Omega \Big(\frac{|\nabla u|}{\|\nabla u\|_{p(\cdot)}}\Big)^{p(\cdot)}\,dx = 1.$$

It follows that $v := \frac{u}{\|\nabla u\|_p}$ is admissible for $\tilde S^{-1}$ so that

$$\int_\Omega \frac{|u|^{q(\cdot)}}{\|\nabla u\|_{p(\cdot)}^{q(\cdot)}}\,dx \le \tilde S^{-1}.$$

The result follows noticing that $\|\nabla u\|_p^{q(x)} \le \max\left\{\|\nabla u\|_{p(\cdot)}^{q^+}, \|\nabla u\|_{p(\cdot)}^{q^-}\right\}$ for a.e. $x \in \Omega$. □

We can now prove Theorem 1.6.

Proof of Theorem 1.6. Observe that as an immediate consequence of the Γ-convergence of F_ε to F^* as stated in Theorem 1.1, we have

$$\limsup_{\varepsilon\to 0} \sup_{\mathcal{X}} F_\varepsilon = \sup_{\mathcal{X}} F^*. \tag{4.2}$$

See [7] and Remark 1.2. Noticing that $\sup_{\mathcal{X}} F_\varepsilon = \tilde S_\varepsilon^{-1} \to \tilde S^{-1}$ as $\varepsilon \to 0$ according to Proposition 1.5, we obtain

$$\sup_{\mathcal{X}} F^* = \tilde S^{-1}. \tag{4.3}$$

Being subcritical, the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)-\varepsilon}(\Omega)$ is compact for any $\varepsilon > 0$. It follows that there exists an extremal $u_\varepsilon \in \mathcal{B}(\Omega)$ for $\tilde S_\varepsilon^{-1}$ i.e.

$$\int_\Omega |u_\varepsilon|^{q(\cdot)-\varepsilon}\,dx = \tilde S_\varepsilon^{-1} = \tilde S^{-1} + o(1). \tag{4.4}$$

According to the CCP (2.7)–(2.9), we can assume that $(u_\varepsilon, |\nabla u_\varepsilon|^{p(\cdot)}\,dx) \to (u,\mu)$ in $\mathcal{X}$, where the measure μ can be written as $\mu = |\nabla u|^{p(\cdot)}\,dx + \tilde\mu + \sum_{i\in I}\mu_i\delta_{x_i}$. The lim sup property (1.3) and the definition of F^* then gives

$$\limsup_{\varepsilon\to 0}\int_\Omega |u_\varepsilon|^{q(\cdot)-\varepsilon}\,dx \le F^*(u,\mu) = \int_\Omega |u|^{q(\cdot)}\,dx + \sum_{i\in I}\mu_i^{\frac{p^*(x_i)}{p(x_i)}}\bar S_{x_i}^{-p^*(x_i)}.$$

We then obtain in view of (4.3), (4.4) that (u,μ) is an extremal for F^* i.e.

$$F^*(u,\mu) = \int_\Omega |u|^{q(\cdot)}\,dx + \sum_{i\in I}\mu_i^{\frac{p^*(x_i)}{p(x_i)}}\bar S_{x_i}^{-p^*(x_i)} = \tilde S^{-1}. \tag{4.5}$$

In view of Lemma 3.3 and inequality (1.8) we have for any $i \in I$ that

$$\bar S_{x_i}^{-p^*(x_i)} = \tilde S_{x_i}^{-1} \le \tilde S^{-1}.$$

These inequalities together with the Sobolev inequality (4.1) allow to deduce from the 2nd equality in (4.5) that

$$1 \le \max \left\{ \|\nabla u\|_{p(\cdot)}^{q_+}, \|\nabla u\|_{p(\cdot)}^{q_-} \right\} + \sum_{i \in I} \mu_i^{\frac{p^*(x_i)}{p(x_i)}}.$$

Moreover since

$$1 \ge \mu(\bar{\Omega}) \ge \int_\Omega |\nabla u|^{p(\cdot)}\, dx + \sum_i \mu_i, \tag{4.6}$$

we have

$$\max \left\{ \|\nabla u\|_{p(\cdot)}^{q_+}, \|\nabla u\|_{p(\cdot)}^{q_-} \right\} = \|\nabla u\|_{p(\cdot)}^{q_-} \le \Big(\int_\Omega |\nabla u|^{p(\cdot)}\, dx \Big)^{\frac{q_-}{p_+}}.$$

It follows that

$$1 \le \Big(\int_\Omega |\nabla u|^{p(\cdot)}\, dx \Big)^{\frac{q_-}{p_+}} + \sum_{i \in I} \mu_i^{\frac{p^*(x_i)}{p(x_i)}}.$$

Since $\frac{q_-}{p_+}, \frac{p^*(x_i)}{p(x_i)} > 1$ for any i, and noticing that each term of the above sum is less than 1 in view of (4.6), we obtain that

$$1 \le \Big(\int_\Omega |\nabla u|^{p(\cdot)}\, dx \Big)^{\frac{q_-}{p_+}} + \sum_{i \in I} \mu_i^{\frac{p^*(x_i)}{p(x_i)}} \le \int_\Omega |\nabla u|^{p(\cdot)}\, dx + \sum_i \mu_i \le 1,$$

where the 2nd inequality is strict, leading to a contradiction, if one of the terms in the sum belongs to $(0, 1)$. It follows that

(i) either $\int_\Omega |\nabla u|^{p(\cdot)}\, dx = 0$ and all the μ_i are 0 except one $\mu_{i_0} = 1$,
(ii) or $\mu_i = 0$ for any $i \in I$ and $\int_\Omega |\nabla u|^{p(\cdot)}\, dx = 1$.

In the first case (i), the CCP (2.7)–(2.9) reduces to

$$|u_\varepsilon|^{q(\cdot)}\, dx \overset{*}{\rightharpoonup} \nu_{i_0}\, \delta_{x_{i_0}}, \qquad |\nabla u_\varepsilon|^{p(\cdot)}\, dx \overset{*}{\rightharpoonup} \delta_{x_{i_0}}, \qquad \nu_{i_0} \le \tilde{S}_{x_{i_0}}^{-1}.$$

Then using Hölder inequality as at the beginning of the proof of Proposition 1.5,

$$\tilde{S}^{-1} = \lim_{\varepsilon \to 0} \int_\Omega |u_\varepsilon|^{q(\cdot)-\varepsilon}\, dx \le \limsup_{\varepsilon \to 0} \int_\Omega |u_\varepsilon|^{q(\cdot)}\, dx = \nu_{i_0} \le \tilde{S}_{x_{i_0}}^{-1} \le \tilde{S}^{-1}.$$

It follows that $\nu_{i_0} = \tilde{S}^{-1}$ and we obtain the second alternative in Theorem 1.6.

In the second case (ii), it follows from (4.5) that u is an extremal for $\tilde{S}^{-1}$. Since $u_\varepsilon \to u$ a.e. and $\int_\Omega |u_\varepsilon|^{q(\cdot)}\, dx \to \int_\Omega |u|^{q(\cdot)}\, dx$, we obtain using the Brezis–Lieb Lemma (see [8] and also [17, Lemma 3.4]) that

$$\int_\Omega |u_\varepsilon - u|^{q(\cdot)}\, dx = \int_\Omega |u_\varepsilon|^{q(\cdot)}\, dx - \int_\Omega |u|^{q(\cdot)}\, dx + o(1) = o(1)$$

i.e. $u_\varepsilon \to u$ strongly in $L^{q(\cdot)}(\Omega)$. This ends the proof of Theorem 1.6. $\square$

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Appendix A

We state and prove an easy version of the Hölder inequality, that, although being well known (see e.g. [11]), is not the most common version. So we provide here with a proof for the sake of completeness.

Lemma A.1. *Let $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$ where $1 < p_- \leq p(\cdot) \leq p_+ < \infty$ and $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$ is the conjugate exponent. Then*

$$\int_\Omega f(x)g(x)\,dx \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}.$$

In particular

$$\int_\Omega f(x)g(x)\,dx \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \max\left\{ \left(\int_\Omega |f(x)|^{p(x)}\,dx\right)^{1/p_-} ; \left(\int_\Omega |f(x)|^{p(x)}\,dx\right)^{1/p_+} \right\} \|g\|_{p'(\cdot)}. \tag{A.1}$$

Proof. Let $\lambda = \|f\|_{p(\cdot)}$ and $\mu = \|g\|_{p'(\cdot)}$. By Young's inequality, we get

$$\begin{aligned}
\int_\Omega \frac{f(x)}{\lambda}\frac{g(x)}{\mu}\,dx &\leq \int_\Omega \frac{1}{p(x)} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx + \int_\Omega \frac{1}{p'(x)} \left(\frac{|g(x)|}{\mu}\right)^{p'(x)} dx \\
&\leq \frac{1}{p_-} \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx + \frac{1}{p'_-} \int_\Omega \left(\frac{|g(x)|}{\mu}\right)^{p'(x)} dx \\
&= \frac{1}{p_-} + \frac{1}{p'_-}
\end{aligned}$$

Now, the result follows just observing that

$$\lambda = \|f\|_{p(\cdot)} \leq \max\left\{ \left(\int_\Omega |f(x)|^{p(x)}\,dx\right)^{1/p_-} ; \left(\int_\Omega |f(x)|^{p(x)}\,dx\right)^{1/p_+} \right\}. \qquad \square$$

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