International Journal of Geometric Methods in Modern Physics
Vol. 12 (2015) 1560023 (7 pages)

DOI: 10.1142/S0219887815600233

Intelligent spin states constructed from $\mathrm{SU}_q(2)$ coherent states

M. Reboiro* and O. Civitarese†

Department of Physics, University of La Plata
49 y 115. C.C.67 (1900) La Plata, Argentina
**reboiro@fisica.unlp.edu.ar*
†osvaldo.civitarese@fisica.unlp.edu.ar

Received 11 November 2014
Accepted 28 April 2015
Published

It is shown that q-deformed Coherent Spin States (CSS), belonging to the $\mathrm{SU}_q(2)$ representation, behave as Intelligent Spin States (ISS). The proof is based on the calculation of spin-squeezing observables, for both CSS and ISS.

Keywords: Coherent spin states; intelligent spin states; squeezing.

Mathematics Subject Classification 2010:

AQ: Please provide Mathematics Subject Classification codes.

1. Introduction

The study of spin-squeezed atoms and ions plays a main role in high precision measurements [1, 2]. Spin-squeezed states have been observed in recent experiments [3–6]. The authors of [4] have reported the observation of coherent squeezed states in Bose–Einstein condensates of rubidium atoms (^{87}Ru), and presented a direct experimental demonstration of interferometric phase-precision beyond the standard quantum-limit in a nonlinear Ramsey interferometer. Intelligent Spin States (ISS) are states which minimize Heisenberg Uncertainty Relations. The notion of ISS have been introduced by Aragone and coworkers in [7]. Following [7], the study of the properties of ISS [8], as well as the empirical realization of these states [9], have been pursued intensively. The authors of [9] have constructed ISS for an array of two-level atoms interacting with a broadband squeezed-radiation-field (see also [10]).

The interaction between spin-states and a radiation field has been described in terms of an effective q-deformed Hamiltonian [11] with spin–spin interactions, both for two- and three-level atoms [12, 13]. Here, we shall show that $\mathrm{su}_q(2)$ Coherent Spin States (CSS) behave as ISS [14], and that they can be used to modeled squeezed Dicke states [9]. In doing so, we shall follow the work of [15], and calculate $\mathrm{su}_q(2)$ uncertainty relations, for a q-deformed CSS.

2. Formalism

Agarwal and Puri [9] have solved the density matrix equation, for a system of N identical two-level atoms of frequency ω_0 interacting with broadband squeezed-radiation-field, and found an ISS as a steady-state solution of this equation. For such a system one can write the Hamiltonian

$$H = \omega_0 S_z + \int d\omega\, \omega a^\dagger(\omega)a(\omega) + \int d\omega\, g(\omega)(a^\dagger(\omega)S_- + S_+ a(\omega)). \tag{1}$$

In this expression, the operators $S_\pm$, S_z are angular momentum operators, acting on the set of states with angular momentum $S = N/2$. The atom–field interaction is given by the spectral function $g(\omega)$, and the annihilation and creation operators of field-quanta, $a(\omega)$ and $a^\dagger(\omega)$, obey the commutation relation $[a(\omega_1), a^\dagger(\omega_2)] = \delta(\omega_1 - \omega_2)$. These operators are acting on the squeezed vacuum-state. The reduced density matrix of the atomic system can be derived by using standard master equation methods. In [9] it is shown that the stationary-state solution of the master equation is the density $\rho = |\Phi_0\rangle\langle\Phi_0|$, with

$$|\Phi_0\rangle = A_0 e^{\theta \mathrm{S}_z} e^{-i\frac{\Pi}{2}\mathrm{S}_y}|0\rangle, \tag{2}$$

where $|0\rangle$ is eigenstate of S_z with eigenvalue $M = 0$ and A_0 is a normalization constant. The expectation values and directions of the components of the angular momentum S_i on $|\Phi_0\rangle$ obey the relations

$$\Delta^2 S_x \Delta^2 S_y = |\langle S_z\rangle|/2. \tag{3}$$

Thus, the state $|\Phi_0\rangle$ is an ISS, since [9]

- it is an eigenstate of the operator $\cosh(|\zeta|)S_- + \sinh(|\zeta|)S_+$ with zero eigenvalue, where ζ is the squeezing-parameter of the field, and,
- the component S_x is squeezed at the expense of the component S_y.

In terms of squeezing-parameters $\zeta^2_{x(y)}$, these conditions are expressed

$$\begin{aligned} \zeta_x^2 &= \frac{2\Delta^2 S_x}{|\langle S_z\rangle|} < 1, \\ \zeta_y^2 &= \frac{2\Delta^2 S_y}{|\langle S_z\rangle|} > 1, \\ \zeta_x^2\zeta_y^2 &= 1. \end{aligned} \tag{4}$$

As shown in [9], the squeezing-parameters of Eq. (4) can be written in terms of the squeezing-parameter of the radiation field, ζ, then the following relations

AQ: Please check whether the edit of the sentence "As shown ...." is okay?

$$\begin{aligned} \zeta_x^2 &= e^{-|\zeta|}, \\ \zeta_y^2 &= e^{|\zeta|} \end{aligned} \tag{5}$$

(see Ref. [10]) point-out to some consequences of this definition of the squeezing-parameters, as determined by experiments.

Now, we shall demonstrate that the state (2) can be constructed from an $\mathrm{su}_q(2)$ coherent state.

The generators of the $\mathrm{su}_q(2)$ algebra, S_+, S_-, S_z, obey the relations [16]

$$[S_z, S_\pm] = \pm S_\pm, \quad [S_+, S_-] = [2S_z]_q, \tag{6}$$

with

$$[x]_q = \frac{\sin(\lambda x)}{\sin(\lambda)}, \tag{7}$$

where λ is the deformation parameter (with this definition $q = e^{i\lambda}$ [16]). As in the case of the su(2) algebra, the irreducible representation of the $\mathrm{su}_q(2)$ are labeled by S. We shall denote the orthonormal basis of representation as $|SM\rangle$, with $M = -S, -S+1, \ldots, S$. The action of the generators of the $\mathrm{su}_q(2)$ on this basis obeys ($\hbar = 1$)

$$\begin{aligned} S_z|SM\rangle &= M|SM\rangle, \\ S_\pm|SM\rangle &= \sqrt{[S \mp M]_q[S \pm M + 1]_q}|SM \pm 1\rangle. \end{aligned} \tag{8}$$

For a given S-representation of the $\mathrm{su}_q(2)$ algebra, q-deformed CSS states, $|\eta\rangle$, are defined by

$$\begin{aligned} |\eta\rangle &= \mathcal{N} \mathrm{e}_q^{\eta S_+}|S - S\rangle \\ &= \mathcal{N} \sum_{k=0}^{2S} \eta^k \begin{bmatrix} 2S \\ k \end{bmatrix}_q^{1/2} |S - S + k\rangle, \end{aligned} \tag{9}$$

where e_q is the q-exponential function [16]

$$\mathrm{e}_q^x = \sum_k \frac{x^k}{[k]_q!} \tag{10}$$

and

$$\begin{bmatrix} 2S \\ k \end{bmatrix}_q = \frac{[2S]_q!}{[2S-k]_q![2S]_q!} \tag{11}$$

is the q-binomial coefficient [16]. The normalization constant $\mathcal{N}$ is written

$$\mathcal{N} = \frac{1}{\sqrt{(1+|\eta|^2)_q^{2S}}}, \tag{12}$$

with the q-binomial expansion [16]

$$(1+|\eta|^2)_q^{2S} = \sum_{k=0}^{2S} |\eta|^{2k} \begin{bmatrix} 2S \\ k \end{bmatrix}_q, \tag{13}$$

where $\eta = \tan\frac{\theta_0}{2} e^{i\phi_0}$, being (θ_0, ϕ_0) orientation angles.

The components x and y of the spin can be written, as usual, in terms of the ladder operators S_+ and S_-:

$$S_x = \frac{S_+ + S_-}{2}, \qquad S_y = \frac{S_+ - S_-}{2\mathrm{i}}. \tag{14}$$

The quadratic deviations of these operators are given by the uncertainty relations

$$\Delta^2 S_x \Delta^2 S_y \geq \frac{1}{4}|\langle[S_x, S_y]\rangle|^2. \tag{15}$$

Following the procedure of Kitagawa and Ueda [15], we shall look for the mean value of the spin operator, $\langle\mathbf{S}\rangle$, on the state of Eq. (9). We shall define a new system of axes, such that the z'-direction coincides with the direction of the mean value of the total spin $\langle\mathbf{S}\rangle$, that is the direction determined by the unitary vector $\{\breve{n}_{x'}, \breve{n}_{y'}, \breve{n}_{z'}\}$, with

$$\breve{n}_{z'} = \frac{\langle\mathbf{S}\rangle}{|\langle\mathbf{S}\rangle|}. \tag{16}$$

In this new system

$$\Delta^2 S_{x'} \Delta^2 S_{y'} = \frac{1}{4}|\langle[S_{x'}, S_{y'}]\rangle|^2. \tag{17}$$

For $\lambda = 0$, the standard (non-deformed) su(2) algebra is recovered. In this limit, the coherent spin state satisfies minimum uncertainty relations, with uncertainties equally distributed over any two orthogonal components normal to the vector defined by $\langle\mathbf{S}\rangle$. For $\lambda \neq 0$, $\Delta^2 S_x'$ and $\Delta^2 S_y'$ are, in general, not equal, thus the q-coherent spin state resembles the state of a correlated system. In view of the previous results we shall adopt, as indicators of the relative fluctuations, the quantities

$$\zeta^2_{x'q} = \frac{2\Delta^2 S_{x'}}{|\langle[S_{x'}, S_{y'}]\rangle|}, \qquad \zeta^2_{y'q} = \frac{2\Delta^2 S_{y'}}{|\langle[S_{x'}, S_{y'}]\rangle|}. \tag{18}$$

The deformation parameter λ, of Eq. (7), can be related to the value of the squeezing-parameter of the field state, ζ, by the comparison of the parameter ζ_x^2 for the ISS and $\zeta^2_{x'q}$ of the q-deformed CSS. As an example, we shall take the case of a system with $S = 1$ and orientation angles $(\theta_0, \phi_0 = \pi)$, for which one can find analytical solutions. It is straightforward to compute the parameters of Eq. (18), they read

$$\zeta^2_{x'q} = \sqrt{\cos^2(\theta_0) + \cos^2(\lambda)\sin^2(\theta_0)}, \qquad \zeta^2_{y'q} = \frac{1}{\sqrt{\cos^2(\theta_0) + \cos^2(\lambda)\sin^2(\theta_0)}}. \tag{19}$$

Thus, in order to model the stationary solution of the Hamiltonian of Eq. (1) for $N = 2$, the parameter λ should be taken as $\lambda = \arcsin(\sqrt{(1 - \mathrm{e}^{-2|\zeta|})\sin^{-2}(\theta_0)})$. The definitions of Eq. (18) are consistent with the property (17). However, due to the fact that in constructing the new system of orthogonal axes we have not performed a q-rotational transformation, the commutation relations between the rotated components of the spin do not coincide with the commutation relations valid in the original system where the inequality (15) holds.

3. Results and Discussion

In this section, we shall present some numerical results to test the validity of our conjecture. We shall take an arbitrary large value of the spin, in order to get larger values of the spin components and their fluctuations. Figure 1 shows the values of the parameters of Eq. (18), as a function of the deformation. We have chosen a q-deformed CSS with total spin $S = 10$. The orientation angles of the state are $\theta_0 = \pi/3$ and $\phi_0 = 0$. These values are arbitrary ones, a choice which is not affecting the validity of our conclusions, as we shall see later on. As it can be seen from Fig. 1, as $\zeta^2_{y'q}$ increases $\zeta^2_{x'q}$ decreases, while the product $\zeta^2_{y'q}\zeta^2_{x'q}$ is constant and it equals unity. Thus, the q-deformed CSS behaves as an intelligent spin state. Due to the properties of the q-deformed coherent state $|\eta\rangle$ of Eq. (9), it can be used as the stationary solution of the Hamiltonian of Eq. (1). The correspondence between the value ζ, and the value of q of the q-deformed CSS, for the state of Fig. 1, is shown in Fig. 2. As seen from the figure, the parameter λ is a smooth function of $|\zeta|$. In order to show that the features displayed in Fig. 1 remain, regardless the particular choice of the orientation angle θ_0, we have calculated the parameters (18) for a set of values of θ_0. Indeed, the curves displayed in Fig. 3 show that the product $\zeta^2_{y'q}\zeta^2_{x'q}$

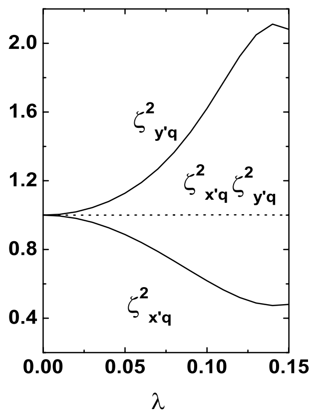

Fig. 1. Parameters, $\zeta^2_{y'q}$ and $\zeta^2_{x'q}$ of Eq. (18), as a function of the deformation parameter, λ. The figure shows the results obtained for a q-coherent spin state. For this calculation we have adopted the values $S = 10$, $\theta_0 = \pi/3$ and $\phi_0 = 0$.

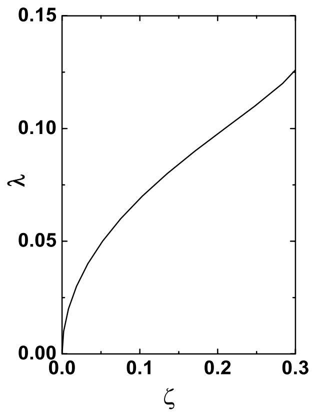

Fig. 2. Corresponding deformation parameter, λ, of the CSS $|\eta\rangle$ of Eq. (9), for different values of the vacuum squeezing field, ζ. The figure shows the results obtained for a q-coherent spin state. For this calculation we have adopted the values $S = 10$, $\theta_0 = \pi/3$ and $\phi_0 = 0$.

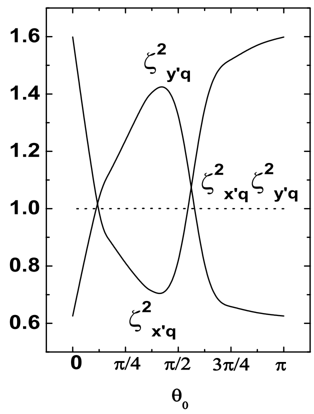

Fig. 3. ζ_i^2, as a function of the orientation angle θ_0 of the initial q-coherent spin-state. We have considered a system with total spin $S = 10$, and an initial state with $\lambda = 0.08$ and $\phi_0 = 0$.

equals unity, as it is the case of the results shown in Fig. 1. The present results may indicate that q-deformed coherent spin states can be regarded as intelligent spin states, when the orientation of the components of the spin are properly defined [17]. Because of the fact that q-deformed coherent spin states can be used to model real squeezed states, they should be of interest in spectroscopy [1, 2]. In a recent experiment [4], collisions in ultracold atomic gases have been used to induce quadrature spin squeezing in two-component Bose condensates. The analysis of [4] shows that a nonlinear atom interferometer surpasses classical precision limit. These type of states have potential application to mesoscopic systems, as for instance molecular spin clusters which are paradigmatic cases to study the cross over between quantum and classical domains.

4. Conclusions

In this work, we have reported on the realization of q-deformed coherent spin states as intelligent spin states. We have proved that they can be used to modeled the stationary solutions of a system of two level atoms in a squeezed radiation bath. From the point of view of practical applications, q-deformed $\mathrm{su}_q(2)$ coherent states should be of interest in spectroscopy, as well as in preparation of entangled states beam splitters. Work is in progress concerning formal aspects of the conjecture, as well as the application of this type of states to mesoscopic systems.

Acknowledgments

This work was partially supported by the National Research Council of Argentine (CONICET) and by the ANPCYT (Argentina).

References

AQ: Please provide article tittle for all journal references.

[1] D. J. Wineland *et al.*, *Phys. Rev. A* **46** (1992) 6797.
[2] C. Ospelkaus *et al.*, *Nature* **476** (2011) 181.
[3] J. Estève *et al.*, *Nature* **455** (2008) 1216.
[4] C. Gross *et al.*, *Nature* **464** (2010) 1165.
[5] K. Maussang *et al.*, *Phys. Rev. Lett.* **105** (2010) 080403.
[6] R. Auccaise *et al.*, *Phys. Rev. Lett.* **114** (2015) 043604.
[7] C. Aragone *et al.*, *J. Phys. A*: *Math. Nucl. Gen.* **7** (1974) L149.
[8] M. A. Rashid *et al.*, *J. Math Phys.* **19** (1978) 1391.
[9] G. S. Agarwal and R. R. Puri, *Phys. Rev. A* **41** (1990) 3782.
[10] R. A. Campos and C. C. Gerry, *Phys. Rev. A* **60** (1999) 157.
[11] O. Civitarese and M. Reboiro, *Phys. Lett. A* **374** (2010) 4664.
[12] C. Gómez, G. Sierra and M. Ruiz-Altaba, *Quantum Groups in Two-Dimensional Physics* (Cambridge University Press, Cambridge, MA, 1996).
[13] A. Ballesteros and S. M. Chumakov, *J. Phys. A: Math. Gen.* **32** (1999) 6261.
[14] M. M. Milks and H. de Guise, *J. Opt. B*: *Quantum Semiclass. Opt.* **7** (2005) S622.
[15] M. Kitagawa and M. Ueda, *Phys. Rev. A* **47** (1993) 5138.
[16] S. Majid, *Foundations of Quantum Group Theory* (Cambridge University Press, 2000).
[17] M. Reboiro and O. Civitarese, *Phys. Lett. A* **376** (2011) 14.