

**EXPLICIT SOLUTION FOR THE ONE-PHASE STEFAN
PROBLEM WITH LATENT HEAT DEPENDING ON
THE POSITION AND A CONVECTIVE BOUNDARY
CONDITION AT THE FIXED FACE**

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ABSTRACT: An explicit solution of a similarity type is obtained for a one-phase Stefan problem in a semi-infinite material using Kummer functions. It is considered a phase-change problem with a latent heat defined as a power function of the position with a non-negative real exponent and a convective boundary condition at the fixed face $x = 0$. Existence and uniqueness of the solution is proved. Relationship between this problem and the problems with temperature and flux boundary condition is also analysed. Furthermore it is studied the limit behaviour of the solution when the coefficient which characterizes the heat transfer at the fixed boundary tends to infinity. Computing this limit allows to demonstrate that the problem proposed in this paper with a convective boundary condition generalizes the problem with Dirichlet boundary condition. Numerical computation of the solution is done over certain examples, with a view to comparing this results with those obtained by general algorithms that solve Stefan problems.

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Key Words: Stefan problem, phase-change processes, variable latent heat, convective boundary condition, Kummer functions, similarity solution

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1. NOMENCLATURE

c	Coefficient that characterizes the heat flux at the fixed face, $[kg/s^{(5+\alpha)/2}]$.
d	Diffusivity coefficient, $[m^2/s]$.
h_0	Coefficient that characterizes the heat transfer in condition (4), $[kg/({}^\circ C s^{5/2})]$.
k	Thermal conductivity, $[W/(m{}^\circ C)]$.
q, r, s, s_∞	Position of the free front, $[m]$.
t	Time, $[s]$.
T	Temperature, $[{}^\circ C]$.
T_0	Coefficient that characterizes the temperature at the fixed face, $[{}^\circ C/s^{\alpha/2}]$.
T_∞	Coefficient that characterizes the bulk temperature, $[{}^\circ C/s^{\alpha/2}]$.
x	Spatial coordinate, $[m]$.

Greek symbols

α	Power of the position that characterizes the latent heat per unit volume, dimensionless.
γ	Coefficient that characterizes the latent heat per unit volume, $[kg/(s^2 m^{\alpha+1})]$.
$\lambda, \mu, \nu, \nu_\infty$	Coefficient that characterizes the free interface, dimensionless.
η	Similarity variable in expression (6), dimensionless.
$\Theta, \Psi, \Psi_\infty$	Temperature, $[{}^\circ C]$.

2. INTRODUCTION

The study of heat transfer problems with phase-change such as melting and freezing constitutes a broad field that has a wide engineering and industrial

applications. Stefan problems can be formulated as models that represents thermal processes in phase transitions, where these phase transitions are characterized by heat diffusion and an exchange of latent heat. Due to their importance, they have been largely studied since the last century [1], [3]-[6],[8],[13] and [17]. In [16] it was presented an extensive bibliography regarding this subject.

In the classical formulation of Stefan problems there are many assumptions on the physical factors involved in the phase-change that are taken into account in order to simplify the description of the process. One of this hypothesis, is to consider the latent heat as a constant. Although it is a reasonable assumption, it can be generalized assuming a variable latent heat. For example, it can be introduced a new kind of problems where the latent heat depends on the position. The physical bases of this particular problems can be found in the movement of a shoreline [19], in the ocean delta deformation [7] or in the cooling body of a magma [10].

The first one on working with a non-constant latent heat was Primicerio [11]. In 1970 he gave sufficient conditions for the existence and uniqueness of solution of a one-phase Stefan problem taking a latent heat as a general function of the position. Voller et al. [19] in 2004 found an exact solution for a one-phase Stefan problem considering the latent heat as a linear function of position. On one hand, Salva and Tarzia [14] extended Voller's work by considering the two-phase Stefan problem with a latent heat distributed linearly on the position and heat flux boundary conditions. On the other hand, Zhou et al. in [20] generalized [19] by considering the one-phase Stefan problem with the latent heat as a power function of the position with an integer exponent. Recently Zhou and Xia [21] found an explicit solution to the latter problem assuming a real non-negative exponent.

In order to obtain a well-posed mathematical problem for the heat-conduction equation, all the above mentioned papers have considered temperature (Dirichlet) or flux (Neumann) boundary conditions at the surface of the body. However, the most realistic boundary condition is the convective one, given by the Newton law (see [4]). This means of heat input is the true relevant physical condition due to the fact that it establishes that the incoming flux at the fixed face is proportional to the difference between the temperature at the surface of the material and the ambient temperature to be imposed (see

[2], [4], [16] and [18]). Therefore the goal of this work is to consider for the first time a convective boundary condition into a Stefan problem with variable latent heat.

Motivated by [18] and [21] we are going to analyse the existence and uniqueness of solution of a one-phase Stefan problem, considering an homogeneous semi-infinite material, with a latent heat as a power function of the position and a convective boundary condition at the fixed face $x = 0$. This problem can be formulated in the following way:

Problem (P1): Find the temperature $\Psi(x, t)$ and the moving interface $s(t)$ such that:

$$\Psi_t(x, t) = d\Psi_{xx}(x, t), \quad 0 < x < s(t), \quad t > 0, \quad (1)$$

$$s(0) = 0, \quad (2)$$

$$\Psi(s(t), t) = 0, \quad t > 0, \quad (3)$$

$$k\Psi_x(0, t) = h_0 t^{-1/2} \left[\Psi(0, t) - T_\infty t^{\alpha/2} \right] \quad t > 0, \quad (4)$$

$$k\Psi_x(s(t), t) = -\gamma s(t)^\alpha \dot{s}(t), \quad t > 0, \quad (5)$$

where Ψ is the temperature, $s(t)$ is the moving interface, d is the thermal diffusion coefficient, k is the thermal conductivity, γx^α is the variable latent heat per unit of volume and the phase-transition temperature is zero. Condition (4) represents the convective boundary condition at the fixed face. T_∞ characterizes the bulk temperature at a large distance from the fixed face $x = 0$ and h_0 represents the heat transfer at the fixed face. Moreover $\dot{s}(t)$ represents the velocity of the phase-change interface. We will work under the assumption that $\gamma > 0, h_0 > 0$ and $T_\infty > 0$ which corresponds to the melting case. In case of freezing it is sufficient to assume $h_0 > 0, \gamma < 0$ and $T_\infty < 0$.

The main objective of this article is to provide a detailed mathematical analysis of the heat transfer problem (P1). It is worth mentioning that the most important feature of this problem is related to the convective boundary condition. It can be seen that condition (4) generalizes the temperature boundary condition already used by Zhou and Xia in [21] due to the fact that if we take the limit when h_0 goes to infinity, we obtain $\Psi(0, t) = t^{\alpha/2} T_0$, recovering a Dirichlet condition.

In Section 2 we will use the similarity transformation technique in order to obtain an explicit solution for the problem under consideration. Other explicit

solutions for phase-change processes are given in [12] and [15], and a recent review on the subject can be found in [17].

In Section 3 we will present a relationship between the problem (P1) and the two related problems with temperature and heat flux boundary conditions on the fixed face $x = 0$ studied in [21]. Section 4 deals with the limit behaviour of the solution of (P1) when the coefficient that characterizes the heat transfer at the fixed face tends to infinity. It will be proved that the explicit solution of the problem (1)-(5) converges to the explicit solution of the problem solved in [21]. Finally some computational examples will be shown in Section 5.

3. EXPLICIT SOLUTION

3.1. GENERAL CASE WHEN α IS A NON-NEGATIVE REAL EXPONENT

The following lemma have already been developed by Zhou-Xia in [21]. It is going to be useful in order to find solutions for the differential heat equation (1).

Lemma 1. (see [21])

a. *Let*

$$\Psi(x, t) = t^{\alpha/2} f(\eta), \quad \text{with} \quad \eta = \frac{x}{2\sqrt{dt}} \quad (6)$$

then $\Psi = \Psi(x, t)$ is a solution of the heat equation $\Psi_t(x, t) = d\Psi_{xx}(x, t)$, with $d > 0$ if and only if $f = f(\eta)$ satisfies the following ordinary differential equation:

$$z \frac{d^2 f}{d\eta^2}(\eta) + 2\eta \frac{df}{d\eta}(\eta) - 2\alpha f(\eta) = 0. \quad (7)$$

b. *An equivalent formulation for equation (7), introducing the new variable $z = -\eta^2$, is given by:*

$$z \frac{d^2 f}{dz^2}(z) + \left(\frac{1}{2} - z\right) \frac{df}{dz}(z) + \frac{\alpha}{2} f(z) = 0. \quad (8)$$

c. The general solution of the ordinary differential equation (8), called Kummer's equation, is given by:

$$f(z) = \widehat{c}_{11}M\left(-\frac{\alpha}{2}, \frac{1}{2}, z\right) + \widehat{c}_{21}U\left(-\frac{\alpha}{2}, \frac{1}{2}, z\right). \tag{9}$$

where \widehat{c}_{11} and \widehat{c}_{21} are arbitrary real constants and $M(a, b, z)$ and $U(a, b, z)$ are the Kummer functions defined by:

$$M(a, b, z) = \sum_{s=0}^{\infty} \frac{(a)_s}{(b)_s s!} z^s, \quad b \text{ cannot be a non-positive integer}, \tag{10}$$

$$U(a, b, z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)}M(a, b, z) + \frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}M(a-b+1, 2-b, z). \tag{11}$$

where $(a)_s$ is the pochhammer symbol defined by:

$$(a)_s = a(a+1)(a+2)\dots(a+s-1), \quad (a)_0 = 1 \tag{12}$$

Remark 2. All the properties of Kummer's functions to be used in the following arguments can be found in [9].

Remark 3. Taking into account definition (11) we can rewrite the general solution of (8) as:

$$f(z) = c_{11}M\left(-\frac{\alpha}{2}, \frac{1}{2}, z\right) + c_{21}z^{1/2}M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, z\right), \tag{13}$$

where c_{11} and c_{21} are real constants.

Our main outcome is given by the following theorem which ensures the existence and uniqueness of solution of the problem (P1) providing in addition, the corresponding explicit solution.

Theorem 4. *There exists a unique solution of a similarity type for the one-phase Stefan problem (1)-(5) and it is given by:*

$$\Psi(x, t) = t^{\alpha/2} \left[c_{11}M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\eta^2\right) + c_{21}\eta M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\eta^2\right) \right] \tag{14}$$

$$s(t) = 2\nu\sqrt{dt} \tag{15}$$

where $\eta = \frac{x}{2\sqrt{dt}}$ and the constants c_{11} and c_{21} are given by:

$$c_{11} = \frac{-\nu M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2\right)}{M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu^2\right)} c_{21}, \tag{16}$$

$$c_{21} = \frac{-2h_0\sqrt{d}T_\infty M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu^2\right)}{\left[kM\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu^2\right) + 2\sqrt{d}h_0\nu M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2\right)\right]}. \tag{17}$$

and the dimensionless coefficient ν is obtained as the unique positive solution of the following equation:

$$\frac{h_0 T_\infty}{\gamma 2^\alpha d^{(\alpha+1)/2}} f_1(x) = x^{\alpha+1}, \quad x > 0. \tag{18}$$

in which:

$$f_1(x) = \frac{1}{\left[M\left(\frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2}, x^2\right) + 2\frac{\sqrt{d}h_0}{k} x M\left(\frac{\alpha}{2} + 1, \frac{3}{2}, x^2\right)\right]}. \tag{19}$$

Proof. The general solution of equation (1), based on Kummer functions is given by the Lemma 1 . According to Remark 3 we can write:

$$\Psi(x, t) = t^{\alpha/2} \left[c_{11} M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\eta^2\right) + c_{21} \eta M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\eta^2\right) \right], \tag{20}$$

where $\eta = \frac{x}{2\sqrt{dt}}$ and where c_{11} and c_{21} are coefficients that must be determined in order to ensure that Ψ satisfies conditions (3)-(5).

Furthermore, condition (3) together with (20) implies that the free boundary should take the following form:

$$s(t) = 2\nu\sqrt{dt}. \tag{21}$$

where ν is a constant that have to be determined.

From equations (3), (20) and (21) we obtain that:

$$\begin{aligned} \Psi(s(t), t) = t^{\alpha/2} & \left[c_{11} M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu^2\right) + \right. \\ & \left. + c_{21} \nu M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2\right) \right] = 0, \end{aligned} \tag{22}$$

and isolating c_{11} we arrive to (16).

On the other hand, we know that the derivative of the Kummer functions (see [9]) are :

$$\frac{d}{dz} M(a, b, z) = \frac{a}{b} M(a + 1, b + 1, z), \tag{23}$$

$$\frac{d}{dz}z^{b-1}M(a, b, z) = (b - 1)z^{b-2}M(a, b - 1, z), \tag{24}$$

and therefore by using (23) and (24) we get that:

$$\begin{aligned} \Psi_x(x, t) = & \frac{t^{(\alpha-1)/2}}{\sqrt{d}} \left[c_{11}\alpha\eta M\left(-\frac{\alpha}{2} + 1, \frac{3}{2}, -\eta^2\right) + \right. \\ & \left. + \frac{c_{21}}{2} M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2}, -\eta^2\right) \right], \end{aligned} \tag{25}$$

and in consequence, condition (4) is satisfied if and only if:

$$k \frac{t^{(\alpha-1)/2}}{2\sqrt{d}} c_{21} = h_0 t^{-1/2} \left[t^{\alpha/2} c_{11} - T_\infty t^{\alpha/2} \right], \tag{26}$$

that is:

$$k \frac{c_{21}}{2\sqrt{d}} = h_0 [c_{11} - T_\infty]. \tag{27}$$

Replacing c_{11} given by (16) into (27) we find (17).

Until now we have obtained c_{11} and c_{21} as functions of ν , arriving to the expressions (16)-(17). By combining equations (16), (17), (21) and (25) and using the following identities [21]:

$$M(a, b, z) = e^z M(b - a, b, -z), \tag{28}$$

$$\begin{aligned} e^{-\nu^2} = & -2\alpha\nu^2 M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2\right) M\left(-\frac{\alpha}{2} + 1, \frac{3}{2}, -\nu^2\right) + \\ & + M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2}, -\nu^2\right) M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu^2\right), \end{aligned} \tag{29}$$

we obtain that the Stefan condition (5) holds if and only if ν satisfies the equation:

$$\frac{kh_0T_\infty}{\left[kM\left(\frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2}, x^2\right) + 2\sqrt{d}h_0xM\left(\frac{\alpha}{2} + 1, \frac{3}{2}, x^2\right) \right]} = \gamma x^{\alpha+1} 2^\alpha d^{(\alpha+1)/2}, \quad x > 0. \tag{30}$$

It means that Ψ and s defined in (14) and (15) constitute a solution of problem (1)-(5), with c_{11} and c_{22} given by equations (16) and (17) if and only if ν , the unknown coefficient, verifies the equation (30). Thus we have deduced an equality that must be satisfied by the positive coefficient ν and that can be written as:

$$\frac{h_0T_\infty}{\gamma 2^\alpha d^{(\alpha+1)/2}} f_1(\nu) = \nu^{\alpha+1}, \tag{31}$$

where the real function f_1 is defined by (19).

The proof will be completed by showing the existence and uniqueness of solution to equation (18) i.e equation (31), analysing the monotonicity of the left and the right hand side of this equality.

By using properties (23) and (24) of Kummer functions we can observe that:

$$f_1'(x) = - \left[2(\alpha + 1)xM \left(\frac{\alpha}{2} + \frac{3}{2}, \frac{3}{2}, x^2 \right) + 2\frac{\sqrt{d}}{k}h_0M \left(\frac{\alpha}{2} + 1, \frac{1}{2}, x^2 \right) \right] f_1^2(x) < 0, \quad \forall x > 0. \tag{32}$$

Therefore we can assure that $f_1(x)$ is a decreasing function of x . Consequently, the left hand side of (18), is also a decreasing function of x that goes from $\frac{h_0T_\infty}{\gamma 2^\alpha d^{(\alpha+1)/2}} > 0$ to 0 when x increases from 0 to $+\infty$. Meanwhile the right hand side of (18) is an increasing function of x that increases from 0 to $+\infty$, when x goes from 0 to $+\infty$.

The above assertions allow us to conclude that there always exists a unique positive solution ν of (18) regardless of the data. Then we obtain that the problem (1)-(5) always has a unique solution given by (14)-(19). \square

3.2. SPECIAL CASE WHEN α IS AN INTEGER

In the special case that α is a positive integer, denoted by n , the Kummer functions are related with the iterated integral of the complementary error function and with the gamma function as follows (see [21], [9]):

$$M \left(-\frac{n}{2}, \frac{1}{2}, -z^2 \right) = 2^n \Gamma \left(\frac{n}{2} + 1 \right) E_n(z), \tag{33}$$

$$z M \left(-\frac{n}{2} + \frac{1}{2}, \frac{3}{2}, -z^2 \right) = 2^{n-1} \Gamma \left(\frac{n}{2} + \frac{1}{2} \right) F_n(z), \tag{34}$$

where :

$$E_n(z) = \frac{[i^n \operatorname{erfc}(z) + i^n \operatorname{erfc}(-z)]}{2}, \tag{35}$$

$$F_n(z) = \frac{[i^n \operatorname{erfc}(-z) - i^n \operatorname{erfc}(z)]}{2}. \tag{36}$$

Such properties allow us to transform the solution of the problem (1)-(5), given by Theorem 4, in case that $\alpha = n \in \mathbb{N}$ into:

$$\Psi(x, t) = \frac{-t^{n/2} 2^n h_0 T_\infty \sqrt{d} \Gamma \left(\frac{n}{2} + \frac{1}{2} \right) \Gamma \left(\frac{n}{2} + 1 \right) [F_n(\eta) E_n(\nu) - F_n(\nu) E_n(\eta)]}{k \Gamma \left(\frac{n}{2} + 1 \right) E_n(\nu) + \sqrt{d} h_0 \Gamma \left(\frac{n}{2} + \frac{1}{2} \right) F_n(\nu)}, \tag{37}$$

$$s(t) = 2\nu\sqrt{dt}, \tag{38}$$

where $\eta = \frac{x}{2\sqrt{dt}}$ and ν is the unique positive solution of the following equation:

$$\frac{h_0 T_\infty}{\gamma d^{(n+1)/2} 2^{2n} \left[\Gamma\left(\frac{n}{2}+1\right) E_n(x) + \sqrt{d} \frac{h_0}{k} \Gamma\left(\frac{n}{2}+\frac{1}{2}\right) F_n(x) \right]} = x^{n+1} e^{x^2}. \tag{39}$$

Remark 5. Taking into account that $E_0(x) = 1$ and $F_0(x) = erf(x)$, in the case $\alpha = 0$, functions (37)-(38) and equation (39) reduce to:

$$\Psi(x, t) = \frac{-h_0 T_\infty \sqrt{d} \sqrt{\pi} \left[erf\left(-\frac{x}{2\sqrt{dt}}\right) - erf(\nu) \right]}{k \left[1 + \frac{\sqrt{d\pi} h_0}{k} erf(\nu) \right]}, \tag{40}$$

$$s(t) = 2\nu\sqrt{dt}, \tag{41}$$

where ν is the unique positive solution of:

$$\frac{h_0 T_\infty}{\gamma \sqrt{d} \left[1 + \frac{\sqrt{d\pi} h_0}{k} erf(x) \right]} = x e^{x^2}, \quad x > 0. \tag{42}$$

It can be noted that this solution is in accordance with the solution given by Tarzia [18] in case that initial temperature $T_i = 0$ (reducing the two-phase Stefan problem into a one-phase Stefan problem).

4. EQUIVALENCE BETWEEN PROBLEMS WITH TEMPERATURE, FLUX AND CONVECTIVE BOUNDARY CONDITIONS AT THE FIXED FACE $X = 0$

We denote by (P1) the problem governed by (1)-(5). If we change the convective condition (4) by a temperature boundary condition we obtain a problem that will be denote by (P2) whose explicit solution was presented in [21]. Similarly we can define the problem (P3) changing condition (4) by a flux boundary condition, whose exact solution was also presented by Zhou and Xia in [21]. It means that we have defined:

Problem (P2): Find the temperature $\Theta(x, t)$ and the moving interface $r(t)$ that satisfies:

$$\Theta_t(x, t) = d\Theta_{xx}(x, t), \quad 0 < x < r(t), \quad t > 0, \tag{43}$$

$$r(0) = 0, \tag{44}$$

$$\Theta(r(t), t) = 0, \quad t > 0, \tag{45}$$

$$\Theta(0, t) = T_0 t^{\alpha/2} \quad t > 0, \tag{46}$$

$$k\Theta_x(r(t), t) = -\gamma r(t)^\alpha \dot{r}(t), \quad t > 0, \tag{47}$$

where the solution according to [21] is given by :

$$\Theta(x, t) = t^{\alpha/2} \left[c_{12} M \left(-\frac{\alpha}{2}, \frac{1}{2}, -\eta^2 \right) + c_{22} \eta M \left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\eta^2 \right) \right], \tag{48}$$

$$r(t) = 2\mu\sqrt{dt}, \tag{49}$$

where $\eta = \frac{x}{2\sqrt{dt}}$ and the constants c_{12} and c_{22} are given by:

$$c_{12} = T_0, \quad c_{22} = \frac{-T_0 M \left(-\frac{\alpha}{2}, \frac{1}{2}, -\mu^2 \right)}{\mu M \left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\mu^2 \right)} \tag{50}$$

and the parameter μ is the unique positive solution of the following equation:

$$\frac{kT_0}{2^{\alpha+1} d^{\alpha/2+1} \gamma} f_2(x) = x^{\alpha+1}, \quad x > 0 \tag{51}$$

with:

$$f_2(x) = \frac{1}{x M \left(\frac{\alpha}{2} + 1, \frac{3}{2}, x^2 \right)}. \tag{52}$$

Problem (P3): Find the temperature $T(x, t)$ and the moving interface $q(t)$ such as:

$$T_t(x, t) = dT_{xx}(x, t), \quad 0 < x < q(t), \quad t > 0, \tag{53}$$

$$q(0) = 0, \tag{54}$$

$$T(q(t), t) = 0, \quad t > 0, \tag{55}$$

$$kT_x(0, t) = -ct^{(\alpha-1)/2} \quad t > 0, \tag{56}$$

$$kT_x(q(t), t) = -\gamma q(t)^\alpha \dot{q}(t), \quad t > 0, \tag{57}$$

where the solution according to [21] is given by:

$$T(x, t) = t^{\alpha/2} \left[c_{13}M \left(-\frac{\alpha}{2}, \frac{1}{2}, -\eta^2 \right) + c_{23}\eta M \left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\eta^2 \right) \right] \tag{58}$$

$$q(t) = 2\lambda\sqrt{dt} \tag{59}$$

where $\eta = \frac{x}{2\sqrt{dt}}$ and the constants c_{13} and c_{23} are given by:

$$c_{13} = \frac{-\lambda M \left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\lambda^2 \right)}{M \left(-\frac{\alpha}{2}, \frac{1}{2}, -\lambda^2 \right)} c_{23}, \quad c_{23} = \frac{-2c\sqrt{d}}{k}. \tag{60}$$

and λ is the unique positive solution of the following equation:

$$\frac{c}{\gamma 2^{\alpha} d^{(\alpha+1)/2}} f_3(x) = x^{\alpha+1}, \quad x > 0 \tag{61}$$

where

$$f_3(x) = \frac{1}{M \left(\frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2}, x^2 \right)}. \tag{62}$$

Once we have defined our three problems, we are going to prove the equivalence between them. We refer to equivalence in the sense that if the data of both problems satisfy certain relationship then they have the same solution.

Theorem 6. *The free boundary problems (P1) and (P2) are equivalent. Moreover we have:*

- a) *the relationship between the datum T_0 of problem (P2) with the data T_∞ and h_0 of the problem (P1) is given by:*

$$T_0 = \frac{2\sqrt{d}h_0T_\infty\nu M \left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2 \right)}{kM \left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu^2 \right) + 2\sqrt{d}h_0\nu M \left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2 \right)}. \tag{63}$$

where ν is the parameter that characterizes the moving interface in problem (P1) and it is given as the unique solution of the equation (18).

- b) *the relationship between the data h_0 and T_∞ of problem (P1) with the datum T_0 of the problem (P2) is given by $T_\infty > T_0$ and :*

$$h_0 = \frac{-kT_0M \left(-\frac{\alpha}{2}, \frac{1}{2}, -\mu^2 \right)}{2\sqrt{d}(T_0 - T_\infty)\mu M \left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\mu^2 \right)}. \tag{64}$$

where μ is the parameter that characterizes the moving interface in problem (P2) and it is given as the unique solution of the equation (51).

Proof.

- a) First, we solve the free boundary problem (P1) and we obtain $\Psi(x, t)$, and $s(t)$ through equations (14)-(19). If we compute the temperature of this problem at the fixed face $x = 0$ we get:

$$\begin{aligned} \Psi(0, t) &= t^{\alpha/2} c_{11} \\ &= t^{\alpha/2} \frac{2\sqrt{d}h_0T_\infty\nu M\left(-\frac{\alpha}{2}+\frac{1}{2}, \frac{3}{2}, -\nu^2\right)}{kM\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu^2\right)+2\sqrt{d}h_0\nu M\left(-\frac{\alpha}{2}+\frac{1}{2}, \frac{3}{2}, -\nu^2\right)}, \end{aligned} \tag{65}$$

so it leads us to define T_0 as $\frac{\Psi(0,t)}{t^{\alpha/2}}$ arriving to (63). Observe that ν is the parameter which defines $s(t)$ (the moving interface of problem (P1)) and it is the unique solution of (18).

Considering the problem (P2) with this particular T_0 , defined by (63), we obtain that the temperature $\Theta(x, t)$ and the moving interface $r(t)$ are given by (48)-(52). From this equations we have that the parameter μ which characterizes $r(t)$ is the unique positive solution of :

$$\frac{k}{2^{\alpha+1}d^{\alpha/2+1}\gamma} \left[\frac{2\sqrt{d}h_0T_\infty\nu M\left(-\frac{\alpha}{2}+\frac{1}{2}, \frac{3}{2}, -\nu^2\right)}{kM\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu^2\right)+2\sqrt{d}h_0\nu M\left(-\frac{\alpha}{2}+\frac{1}{2}, \frac{3}{2}, -\nu^2\right)} \right] f_2(x) = x^{\alpha+1}. \tag{66}$$

If we replace x by ν in equation (66) we obtain equation (18) whose unique solution is ν . So we can conclude that ν is a solution of (66) . Therefore we get that $\mu = \nu$, and $r(t) = s(t)$. Working algebraically we obtain that the temperature of both problems are equal, i.e $\Theta(x, t) = \Psi(x, t)$. In other words, the problem (P1) has the same solution of problem (P2) when T_0 is defined in function of the data of (P1) as (63).

- b) Conversely, we consider the problem (P2), and we solve it using equations (48)-(52), we obtain $\Theta(x, t)$ and $r(t)$. If we compute $\Theta(0, t)$ and $\Theta_x(0, t)$, the coefficient h_0 can be defined in order that convective condition (4) is satisfied. That is to say:

$$\begin{aligned} h_0 &= \frac{k\Theta_x(0,t)}{t^{-1/2}[\Theta(0,t)-T_\infty t^{\alpha/2}]} \\ &= \frac{-kt^{(\alpha-1)/2}T_0M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\mu^2\right)}{2\sqrt{d}\mu M\left(-\frac{\alpha}{2}+\frac{1}{2}, \frac{3}{2}, -\mu^2\right)t^{-1/2}[t^{\alpha/2}T_0-t^{\alpha/2}T_\infty]}, \end{aligned} \tag{67}$$

arriving to definition (64), where μ is the parameter that characterizes the moving interface $r(t)$, and it is the unique solution of (51).

Imposing a $T_\infty > T_0$, it turns out that h_0 defined by (64) is positive, and hence we can solve the problem (P2) with this h_0 . By equations (14)-(19) we obtain the temperature $\Psi(x, t)$ and the moving interface $s(t) = 2\nu\sqrt{dt}$. From (18) and taking into account the form of h_0 we get that ν is the unique positive solution of:

$$\frac{-kT_0M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\mu^2\right)}{2\sqrt{d}(T_0-T_\infty)\mu M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\mu^2\right)} \frac{T_\infty}{\gamma^{2\alpha}d^{(\alpha+1)/2}} f_1(x) = x^{\alpha+1}. \tag{68}$$

If we replace x by μ in equation (68) we obtain equation (51). As μ is the unique solution of (51), we obtain that μ is a solution of (68). By uniqueness of solution of equation (68) we get that $\nu = \mu$. In consequence, it follows that $s(t) = r(t)$ and $\Psi(x, t) = \Theta(x, t)$. So we can claim to have for the problem (P2) the same solution as for the problem (P1) considering h_0 defined by (64) in function of the data of (P2). Therefore, we can conclude that problems (P1) and (P2) are equivalents.

□

It remains to prove that (P1) and (P3) are also equivalents in the same way we have done for Theorem 6.

Theorem 7. *The free boundary problems (P1) and (P3) are equivalents. Moreover we have:*

- a) *the relationship between the datum c of problem (P2) with the data T_∞ and h_0 of the problem (P1) is given by:*

$$c = \frac{h_0T_\infty M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu^2\right)}{\left[M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu^2\right) + \frac{2\sqrt{d}h_0}{k} \nu M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2\right)\right]} \tag{69}$$

where ν is the parameter that characterizes the moving interface in problem (P1).

- b) *the relationship between the data h_0 and T_∞ of problem (P1) with the datum c of the problem (P3) is given by :*

$$T_\infty > \frac{2c\sqrt{d}}{k} \frac{\lambda M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\lambda^2\right)}{M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\lambda^2\right)} \tag{70}$$

$$h_0 = \frac{-cM\left(-\frac{\alpha}{2}, \frac{1}{2}, -\lambda^2\right)}{\frac{2c\sqrt{d}}{k}\lambda M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\lambda^2\right) - T_\infty M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\lambda^2\right)}. \tag{71}$$

where λ is the parameter that characterizes the moving interface in problem (P3).

Proof.

- a) First, we solve the free boundary problem (P1) and we obtain $\Psi(x, t)$, and $s(t)$ through equations (14)-(19). If we compute the flux Ψ at the fixed face $x = 0$ we get:

$$\Psi_x(0, t) = \frac{-t^{(\alpha-1)/2}h_0T_\infty M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu^2\right)}{k\left[M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu^2\right) + \frac{2\sqrt{d}h_0}{k}M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2\right)\right]} \tag{72}$$

so it leads us to define $c = -\frac{k\Psi_x(0,t)}{t^{(\alpha-1)/2}}$ as in (69). Observe that ν is the parameter which defines $s(t)$ (the moving interface of problem (P1)) and it is the unique solution of (18).

If we consider the problem (P3) with this particular c defined by (69), we obtain that the solution, it means the temperature $T(x, t)$ and the moving interface $q(t)$ are given by (58)-(62). From this equations we have that the parameter λ which characterizes $q(t)$ is the unique positive solution of :

$$\frac{h_0T_\infty M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu^2\right)f_3(x)}{\left[M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu^2\right) + \frac{2\sqrt{d}h_0\nu}{k}M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2\right)\right]\gamma^{2\alpha}d^{(\alpha+1)/2}} = x^{\alpha+1}. \tag{73}$$

If we replace x by ν we can reduce equation (73) into (18), and as ν is the unique solution of (18), we deduce that ν is a solution of (73). . Therefore we get that $\lambda = \nu$, and $q(t) = s(t)$. Working algebraically we obtain that the temperature of both problems are equal, i.e $T(x, t) = \Psi(x, t)$. In other words, the problem (P1) has the same solution of problem (P3) considering a c defined by (69).

- b) Conversely, if we take the problem (P3), and we solve it using equations (58)-(62), we obtain $T(x, t)$ and $q(t)$. For convective condition (4) to happen, we compute $T(0, t)$ and $T_x(0, t)$ and define h_0 as:

$$h_0 = \frac{kT_x(0,t)}{t^{-1/2}[T(0,t) - T_\infty t^{\alpha/2}]}$$

$$= \frac{k}{2\sqrt{d}} \frac{(-2c\sqrt{d})}{k} \frac{M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\lambda^2\right)}{\left[\frac{2c\sqrt{d}}{k} \lambda M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\lambda^2\right) - T_\infty M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\lambda^2\right)\right]} \quad (74)$$

arriving to an h_0 given by (71). Observe that λ is the parameter that characterizes the moving interface $q(t)$, which is the unique solution of (61).

Prescribing a T_∞ as in (70), we are able to ensure that $h_0 > 0$. Hence we can pose the problem (P3) with h_0 defined by (71). By equations (14)-(19) we obtain the temperature $\Psi(x, t)$ and the moving interface $s(t) = 2\nu\sqrt{dt}$. From (18) and taking into account the form of h_0 we get that ν is the unique positive solution of:

$$\frac{-cM\left(-\frac{\alpha}{2}, \frac{1}{2}, -\lambda^2\right)}{\left[\frac{2c\sqrt{d}}{k} \lambda M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\lambda^2\right) - T_\infty M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\lambda^2\right)\right]} \frac{T_\infty}{\gamma^{2\alpha} d^{(\alpha+1)/2}} f_1(x) = x^{\alpha+1}. \quad (75)$$

If we replace x by λ , equation (75) reduces to equation (61). As λ is the unique solution of (61), we obtain that λ is a solution of (75). By uniqueness of solution of equation (75) we get that $\nu = \lambda$. In consequence, it follows that $s(t) = q(t)$ and $\Psi(x, t) = T(x, t)$. It yields that the problem (P3) has the same solution of the problem (P1) when h_0 and T_∞ are defined from the data of (P3) by equations (70)-(71).

Thus we can conclude that (P1) and (P3) are equivalents.

□

5. LIMIT BEHAVIOUR

In this section we are going to analyse the behaviour of the problem (P1) when the coefficient h_0 that characterizes the heat transfer at the fixed face $x = 0$ tends to infinity. Due to the fact that the solution of this problem, i.e the temperature and the free boundary depends on h_0 , we will rename them. Thus, we will consider $\Psi_{h_0}(x, t) := \Psi(x, t)$ and $s_{h_0}(t) := s(t)$ defined by equations (14)-(15), where $c_{11} = c_{11}(h_0)$, $c_{21} = c_{21}(h_0)$ and $\nu = \nu_{h_0}$ is the unique solution of the following equation:

$$\frac{h_0 T_\infty}{\gamma^{2\alpha} d^{(\alpha+1)/2}} f_1(x, h_0) = x^{\alpha+1}, \quad x > 0. \quad (76)$$

in which:

$$f_1(x, h_0) = \frac{1}{\left[M\left(\frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2}, x^2\right) + 2\frac{\sqrt{dh_0}}{k} x M\left(\frac{\alpha}{2} + 1, \frac{3}{2}, x^2\right) \right]}. \tag{77}$$

On the other hand, let us consider a new problem (P4) defined in the following way:

Problem (P4): Find the temperature $\Psi_\infty(x, t)$ and the moving interface $s_\infty(t)$ that satisfies:

$$\Psi_{\infty t}(x, t) = d\Psi_{\infty xx}(x, t), \quad 0 < x < s_\infty(t), \quad t > 0, \tag{78}$$

$$s_\infty(0) = 0, \tag{79}$$

$$\Psi_\infty(s_\infty(t), t) = 0, \quad t > 0, \tag{80}$$

$$\Psi_\infty(0, t) = T_\infty t^{\alpha/2} \quad t > 0, \tag{81}$$

$$k\Psi_{\infty x}(s_\infty(t), t) = -\gamma s_\infty(t)^\alpha \dot{s}_\infty(t), \quad t > 0, \tag{82}$$

As we can observe, this problem corresponds to a problem where a temperature boundary condition is imposed at the fixed face $x = 0$. Thus the solution according to [21] can be obtained from equations (48)-(52):

$$\begin{aligned} \Psi_\infty(x, t) = t^{\alpha/2} & \left[c_{11\infty} M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\eta^2\right) + \right. \\ & \left. + c_{21\infty} \eta M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\eta^2\right) \right], \end{aligned} \tag{83}$$

$$s_\infty(t) = 2\nu_\infty \sqrt{dt}, \tag{84}$$

where $\eta = \frac{x}{2\sqrt{dt}}$ and the constants $c_{12\infty}$ and $c_{22\infty}$ are given by:

$$c_{11\infty} = T_\infty, \quad c_{21\infty} = \frac{-T_\infty M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu_\infty^2\right)}{\nu_\infty M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu_\infty^2\right)}, \tag{85}$$

and the parameter ν_∞ is the unique positive solution of the following equation:

$$\frac{kT_\infty}{2^{\alpha+1} d^{\alpha/2+1} \gamma} f_2(x) = x^{\alpha+1}, \quad x > 0 \tag{86}$$

with:

$$f_2(x) = \frac{1}{xM\left(\frac{\alpha}{2} + 1, \frac{3}{2}, x^2\right)}. \tag{87}$$

Once we have introduced the problems (P1) and (P4) we are able to state the following convergence theorem.

Theorem 8. *The problem (P1) converges to problem (P4) when h_0 tends to infinity, i.e.:*

$$\lim_{h_0 \rightarrow +\infty} P1 = P4 \tag{88}$$

In this context the term “convergence” means that:

$$\left\{ \begin{array}{l} \lim_{h_0 \rightarrow +\infty} \nu_{h_0} = \nu_\infty, \\ \lim_{h_0 \rightarrow +\infty} s_{h_0}(t) = s_\infty(t), \quad \forall t > 0 \\ \lim_{h_0 \rightarrow +\infty} \Psi_{h_0}(x, t) = \Psi_\infty(x, t), \quad \forall t > 0, x > 0. \end{array} \right.$$

Proof. Let us consider the problem (P1). We know that the parameter that characterizes the free boundary, ν_{h_0} , is the unique solution of equation (76). In order to obtain the limit of ν_{h_0} it is necessary to study the convergence of equation (76) when h_0 goes to infinity. The limit of the left hand side function of (76) is:

$$\begin{aligned} & \lim_{h_0 \rightarrow +\infty} \left\{ \frac{h_0 T_\infty}{\gamma^{2\alpha} d^{(\alpha+1)/2}} \frac{1}{\left[\frac{1}{h_0} M\left(\frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2}, x^2\right) + 2 \frac{\sqrt{d}}{k} x M\left(\frac{\alpha}{2} + 1, \frac{3}{2}, x^2\right) \right]_{h_0}} \right\} \\ &= \frac{T_\infty}{\gamma^{2\alpha} d^{(\alpha+1)/2}} \frac{1}{\left[2 \frac{\sqrt{d}}{k} x M\left(\frac{\alpha}{2} + 1, \frac{3}{2}, x^2\right) \right]} \\ &= \frac{k T_\infty}{2^{\alpha+1} d^{\alpha/2+1} \gamma} f_2(x). \end{aligned} \tag{89}$$

This imply that equation (76) converges to equation (86) when $h_0 \rightarrow \infty$. On one hand, we have that the limit of ν_{h_0} must be a solution of equation (86). On the other hand, (86) has a unique solution ν_∞ . Thus it turns out that $\lim_{h_0 \rightarrow \infty} \nu_{h_0} = \nu_\infty$. Once obtained this convergence, it is immediately that $\lim_{h_0 \rightarrow +\infty} s_{h_0}(t) = s_\infty(t), \forall t > 0$. For the convergence of the temperature $\Psi_{h_0}(x, t)$ to $\Psi_\infty(x, t)$ when $h_0 \rightarrow \infty$, it can be easily proved that: $\lim_{h_0 \rightarrow \infty} c_{21}(h_0) = c_{21\infty}$ and $\lim_{h_0 \rightarrow \infty} c_{11}(h_0) = c_{11\infty}$. □

6. NUMERICAL COMPUTATION

From Theorem 4 the solution of the problem (P1) is characterized by a parameter ν defined as the unique solution of equation (18). This equation can be rewritten into the following way:

$$F(x) = \frac{h_0 T_\infty}{\gamma 2^\alpha d^{(\alpha+1)/2}} f_1(x) - x^{\alpha+1} = 0, \quad x > 0. \quad (90)$$

where $f_1(x)$ is defined by (19).

In order to approximate the unique root of the nonlinear equation defined above we can apply Newton's method. Beginning with an estimate ν_0 of ν , we define inductively:

$$\nu_{k+1} = \nu_k - \frac{F(\nu_k)}{F'(\nu_k)} \quad (91)$$

where

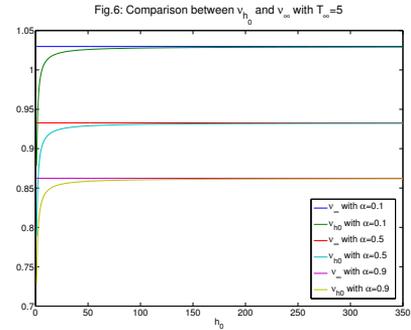
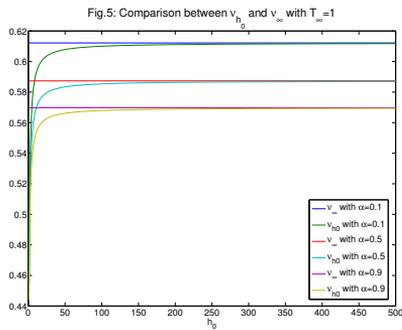
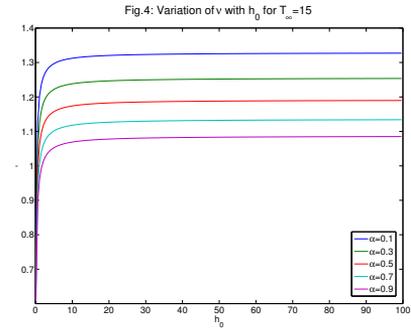
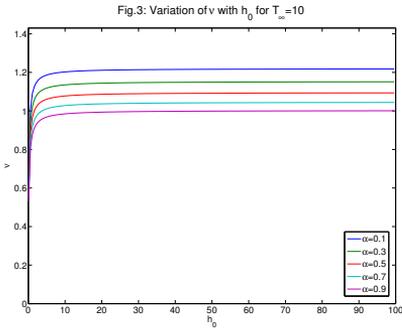
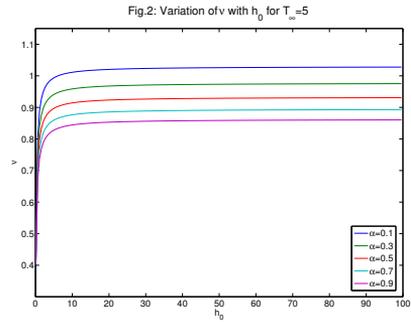
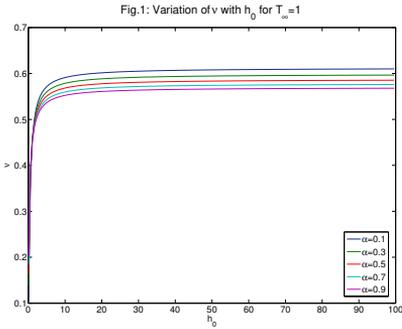
$$F'(x) = \frac{h_0 T_\infty}{\gamma 2^\alpha d^{(\alpha+1)/2}} f_1'(x) - (\alpha + 1)x^\alpha. \quad (92)$$

noting that:

$$\begin{aligned} f_1'(x) &= -f_1^2(x) \left[2(\alpha + 1)xM \left(\frac{\alpha}{2} + \frac{3}{2}, \frac{3}{2}, x^2 \right) + \right. \\ &\quad \left. + 2 \frac{\sqrt{d}h_0}{k} M \left(\frac{\alpha}{2} + 1, \frac{1}{2}, x^2 \right) \right]. \end{aligned} \quad (93)$$

We have implemented Newton's Method using Matlab software. The main reason for choosing this programming language is that the Kummer function $M(a, b, z)$ can be represented by the command 'hypergeom'. The stopping criterion used is the boundedness of the absolute error $|\nu_k - \nu_{k+1}| < 10^{-15}$. Without loss of generality we assume $\gamma = d = k = 1$. The following Figures 1 to 4 present the computational values obtained for ν versus h_0 corresponding to different values of T_∞ and α .

We can observe that, in all cases ν varies monotonically increasing with respect to h_0 . In addition it can be appreciated that as h_0 increases, ν tends to stabilize. This behaviour is in accordance with Theorem 8, which ensures the existence of a limit for $\nu := \nu_{h_0}$ when h_0 goes to infinity. For this reason, we also applied Newton's method to the problem (P4) taking into account equations (86)-(87), using the same stopping criterion as above and taking



$\gamma = d = k = 1$. In the next Figures 5 to 8, we compare the coefficients ν_{h_0} and ν_∞ corresponding to problems (P1) and (P4) respectively for different input data T_∞ and α .

In Figure 9 we show the variation of the temperature Ψ with respect to x and t taking the particular values of the data: $\gamma = k = d = 1$, $\alpha = 0.4$, $h_0 = 0.5$ and $T_\infty = 1$. As we are dealing with a melting problem, for every fixed value of the position (x) we can note when the phase-change takes place and observe how the temperature becomes greater over time once the phase-change have occurred.

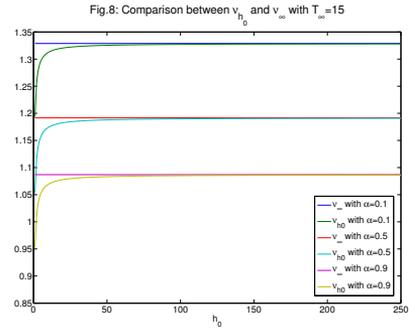
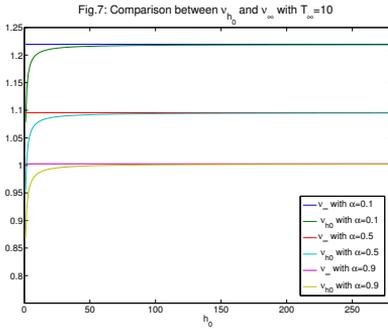
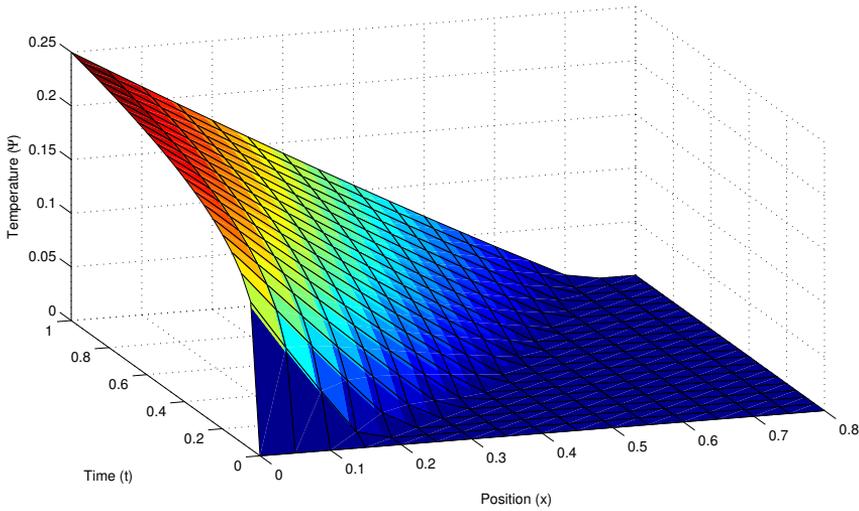


Fig.9: Variation of Ψ with x and t .



7. CONCLUSIONS

In this article a closed analytical solution of a similarity type have been obtained for a one-dimensional one-phase Stefan problem in a semi-infinite material using Kummer functions. The novel feature in the problem studied concerns a variable latent heat that depends on the position as well as a convective boundary condition at the fixed face $x = 0$ of the material which is considered for first time in these kind of problems. On one hand, assuming a latent heat defined as a power function of the position allows the generalization of some previous theoretical results, finding its physical base in problems related to the movement of a shoreline or the cooling body of a magma. On the other hand, the fact of considering a convective boundary condition at

the fixed boundary $x = 0$ reflects a more realistic way of heat input than an imposed temperature or flux, known as Dirichlet and Neumann conditions respectively.

The key contribution of this paper has been to present the exact solution of the problem (P1) which is worth finding not only to understand better the physical process involved but also to verify the consistency and estimate errors of numerical methods designed to solve Stefan problems.

We have demonstrated the equivalence between our problem and the problems defined by considering a temperature or a flux boundary condition instead of the convective one.

Besides, it has been analysed the limit behaviour of the solution when the coefficient h_0 that characterizes the heat transfer at the fixed face $x = 0$ tends to infinity. It has been shown that our problem (P1) converges pointwise to the problem (P4) solved by Zhou and Xia in [21]. Thus it can be said that the explicit solution obtained in this paper generalizes the one obtained in [21] since the latter solution can be obtained from the former one by computing a limit. Moreover, for particular coefficients of the problem (P1) we can obtain the explicit solution given recently for the classical two-phase Stefan problem with a convective boundary condition given in [18].

Finally, we have applied Newton's method to the closed formula obtained for our problem (P1), in order to estimate the parameter that characterizes the free front numerically. In the same way we did to problem (P4). The computations obtained help us to validate our convergence result.

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