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Non-deterministic conditionals and transparent truth

Abstract. Theories where truth is a naive concept fall under the following dilemma: either the theory is subject to Curry's Paradox, which engenders triviality, or the theory is not trivial but the resulting conditional is too weak. In this paper we explore a number of theories which arguably do not fall under this dilemma. In these theories the conditional is characterized in terms of (possibly infinite) non-deterministic matrices. These non-deterministic theories are similar to Łukasiewicz logic in that they are consistent and their conditionals are quite strong. The difference is the following: while Łukasiewicz logic is ω-inconsistency, the non-deterministic theories might turn out to be ω-consistent.

Keywords: Naive truth theory, Łukasiewicz logic, Curry's paradox, non-deterministic semantics, ω-inconsistency.

1. Introduction

In this paper we want to address the problem of finding a strong conditional connective for a naive theory of truth[1]. This problem is not really new. Usually, theories where truth is treated as a naive concept fall under the following dilemma: either the theory is subject to Curry's Paradox, which engenders triviality, or the theory is not trivial but the resulting conditional is too weak[2]. Recently there have been many attempts to avoid this dilemma by the introduction of rather complicated conditionals (see for example [3], [5], [6], and [11]).

One relatively familiar and uncomplicated conditional which does not fall under the dilemma is Łukasiewicz conditional (as long as the semantics is continuum-valued). However, discussing Łukasiewicz logic, Hartry Field [6], p.94 claims that:

Presented by **Name of Editor**; *Received* December 1, 2005

[1]By a *naive truth theory* we mean a theory that contains all instances of the schema $Tr\ulcorner\phi\urcorner \leftrightarrow \phi$ (where $\ulcorner\phi\urcorner$ is a name for the sentence ϕ). By a *transparent truth theory* we mean a theory where $Tr\ulcorner\phi\urcorner$ and ϕ are everywhere intersubstitutable. In the non-deterministic theories we introduce below, this distinction will not matter.

[2]Some substructural theories of truth do not fall under this dilemma, but here we will only consider theories with a consequence relation that satisfies all the usual structural properties.

> (...) the clear inadequacy of the continuum-valued semantics for languages with quantifiers should not blind us to its virtues in a quantifier-free context. Indeed, one might well hope that some simple modification of it would work for languages with quantifiers. In fact, this does not seem to be the case: major revisions in the approach seem to be required.

Now, we do not know exactly what Field means by 'major revisions', but here we will consider several close cousins of Łukasiewicz logic and argue that most of them are at least not clearly inadequate. Although this can be done proof-theoretically -by analyzing which axioms and rules must be satisfied by the target conditional- we employ a semantic approach. In particular, we will use non-deterministic matrices to obtain (relatively) strong subtheories of Łukasiewicz logic.

The rest of the paper is structured as follows. As the technique of non-deterministic matrices might be unfamiliar for philosophers interested in semantic paradoxes, the next section gives a brief and sketchy tutorial on the topic. In Section 3, after showing the inadequacy of finitely-valued Łukasiewicz logic (whether deterministic or not), we present the well-known continuum-valued version of Łukasiewicz logic. Section 4 contains several ways of making this logic non-deterministic and shows how strong the resulting theories are. Section 5 contains some speculative remarks on whether the non-deterministic theories we consider are ω-inconsistent, and section 6 shows how it is possible to define a determinately operator in these theories. Section 7 compares the present proposal to a similar but slightly different approach developed recently and section 8 contains some closing remarks.

2. Non-deterministic matrices

The idea of a non-truth-functional connective is quite old and well-known. Recently, though, this idea has been studied by using what is sometimes called 'non-deterministic matrices'. This formal tool has been rigorously developed by those - specially computer scientist - who wish to study a number of properties of proof systems from a semantic point of view. A very complete introduction to the topic can be found in [1][3]. Intuitively,

[3] In any case, we think non-deterministic matrices have not been sufficiently explored as a tool to study *semantic paradoxes* (nor, for that matter, to study other kinds of philosophical puzzles). For a very brief survey on possible applications of non-deterministic matrices, see [1]. To the best of our knowledge, the application to semantic paradoxes has not been discussed anywhere.

in a non-deterministic framework there is at least one connective such that you cannot completely determine the value of a compound formula involving that connective even if you know the values of all the atomic formulas of the language. We can give a more formal definition in the following way:

DEFINITION 2.1. *($\mathcal{ND}Matrix$) A* non-deterministic matrix *for a language $\mathcal{L}$ is a tuple $\mathcal{M} = <\mathcal{V}, \mathcal{D}, \mathcal{O}>$, where:*

- *$\mathcal{V}$ is a non-empty set of truth values,*
- *$\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, and*
- *$\mathcal{O}$ is a set of functions such that for every n-ary connective $\diamond$ in $\mathcal{L}$, there is a corresponding function $\diamond^{\mathcal{M}}$ in $\mathcal{O}$ such that $\diamond^{\mathcal{M}}$: $\mathcal{V}^n \longrightarrow \mathcal{P}(\mathcal{V}) - \varnothing$.*[4]

The interesting part of the definition has to do with the set $\mathcal{O}$ of functions for the non-deterministic connectives. In a deterministic matrix, for each n-ary connective $\diamond$ in $\mathcal{L}$, there is a corresponding function $\diamond^{\mathcal{M}}$ such that $\diamond^{\mathcal{M}}$: $\mathcal{V}^n \longrightarrow \mathcal{V}$. The function takes a certain n-tuple of values in $\mathcal{V}^n$ and assigns a value in $\mathcal{V}$. In the case of non-deterministic connectives, the co-domain of the corresponding function is the set of sets of values $\mathcal{P}(\mathcal{V}) - \varnothing$, rather than the set of values $\mathcal{V}$.

Also notice that deterministic matrices are a special case of non-deterministic matrices. More specifically, for each n-ary connective $\diamond$ in a deterministic matrix $\mathcal{M}$ which is interpreted as a function $\diamond^{\mathcal{M}}$: $\mathcal{V}^n \longrightarrow \mathcal{V}$, we can build a non-deterministic matrix $\mathcal{M}'$ where that connective can be taken as a function that only outputs singleton values, that is, $\diamond^{\mathcal{M}'}$: $\mathcal{V}^n \longrightarrow \{\mathcal{A} \subseteq \mathcal{V}$: $|\mathcal{A}| = 1\}$. By doing this we obtain a non-deterministic matrix with connectives that mimic the behavior of the deterministic connectives.

It is straightforward to characterize the usual notions of *valuation*, *satisfaction*, *validity*, and so on, for non-deterministic matrices:

DEFINITION 2.2. *(Valuation) Let $Form_{\mathcal{L}}$ denote the set of formulae of the language $\mathcal{L}$. A* valuation *in $\mathcal{M}$ is a function v: $Form_{\mathcal{L}} \longrightarrow \mathcal{V}$ such that for each n-ary connective $\diamond$ of $\mathcal{L}$, the following holds for all $\phi_1, ..., \phi_n \in Form_{\mathcal{L}}$: $v(\diamond(\phi_1, ..., \phi_n)) \in \diamond^{\mathcal{M}}(v(\phi_1), ..., v(\phi_n))$*

Notice that since $\diamond^{\mathcal{M}}(v(\phi_1), ..., v(\phi_n))$ gives a set of values rather than a single value, we use $\in$ instead of $=$ in the previous definition. With this

[4] The reason for excluding the empty set is that it is not straightforward how to compute the value of compound formulae where at some step of the computation we have as input the empty set.

new notion of *valuation*, the concepts of *satisfaction* and *validity* can be defined in the usual way.

To see how this actually works, we will provide an example of a non-deterministic matrix that might be relevant for the study of semantic paradoxes.

EXAMPLE 2.3. *Let $\mathcal{L}$ be a propositional language with one unary connective $\neg$ and two binary connective $\vee$ and $\wedge$. Let $\mathcal{M}_1 = <\mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1>$, where:*

- $\mathcal{V}_1 = \{1, 0\}$,
- $\mathcal{D}_1 = \{1\}$, *and*
- $\mathcal{O}_1$ *is defined in the following way:*

	$\neg^{\mathcal{M}_1}$
1	{*0*}
0	{*1,0*}

		$\vee^{\mathcal{M}_1}$
1	*1*	{*1*}
1	*0*	{*1*}
0	*1*	{*1*}
0	*0*	{*0*}

		$\wedge^{\mathcal{M}_1}$
1	*1*	{*1*}
1	*0*	{*0*}
0	*1*	{*0*}
0	*0*	{*0*}

A reason for employing this matrix is that it is compatible with a naive theory of truth. It is well-known that the paracomplete three-valued logic K_3 can support a transparent truth predicate. But it is easy to see that $\nvDash_{\mathcal{M}_1} \phi \vee \neg\phi$. Moreover, every K_3-countermodel can be turned into an $\mathcal{M}_1$-countermodel by replacing all assignments of the value $\frac{1}{2}$ by 0, and leaving everything else untouched. This means that $\mathcal{M}_1$ is a sublogic of K_3. Hence, $\mathcal{M}_1$ is a (paracomplete two-valued!) consistent non-deterministic matrix that supports a transparent truth predicate[5].

From this example it should be clear that taking a deterministic matrix and transforming it into a non-deterministic matrix (possibly) weakens the resulting logic. This makes sense: a non-deterministic matrix considers more valuations than a deterministic one, at least *ceteris paribus*. So if we are in the business of solving paradoxes by weakening logic, non-deterministic matrices might be a good tool to see what sort of paradox-immune logics we can obtain.

[5]The logic characterized by this matrix is usually known as CLaN. Notice also that the dual of the logic characterized by $\mathcal{M}_1$ is the paraconsistent logic $CLuN$, which is a sublogic of the paraconsistent logic LP, dual to K_3. For more details on these logics see [4].

3. Łukasiewicz logic

Since we want to discuss semantic paradoxes, we assume, as usual, that the language we are working with has some way to talk about itself. More specifically, for each formula ϕ of the language there is a term $\ulcorner\phi\urcorner$ that is the name of that formula. So, if the language contains a truth predicate, we can construct a Liar sentence λ such that λ is $\neg\, Tr\ulcorner\lambda\urcorner$, a Curry sentence δ such that δ is $Tr\ulcorner\delta\urcorner \to \perp$, and so on[6].

There are some multivalued logics that could arguably be considered as plausible solutions to the Liar Paradox (K_3 and LP are usually seen as natural candidates). The problems with this kind of views are well-known. The material conditional exhibits an odd behavior in these logics: LP does not validate Modus Ponens ($\phi, \phi \supset \psi \nvDash_{LP} \psi$), while K_3 does not validate Identity ($\nvDash_{K_3} \phi \supset \phi$). Both Field [6] and Beall [5] have worked on supplementing such theories with a suitable conditional. However, Curry's paradox makes this task quite complicated: any naive truth theory cannot have a conditional connective validating, for example, Modus Ponens and Contraction, or Modus Ponens and Conditional Proof.

Is there any other way of supplementing these logics with a suitable conditional? That depends, of course, on what we take a suitable conditional to be. One option is to say that a suitable conditional is one that satisfies certain principles and rules of inferences, for instance, Modus Ponens, Identity, and so on. Another option is to impose more general constraints regarding the way in which a valuation should behave with respect to a conditional connective. These are not mutually exclusive approaches. We could impose constraints on the valuations in such a way that the conditional validates the principles and rules we want. Nonetheless, it might be that in certain contexts one of the approaches is more illuminating than the other.

If we are working in a *linearly ordered* space of values[7], it seems useful to embrace the second approach and to say that a conditional $\to$ is *suitable* if it is not subject to Curry's Paradox and it satisfies the following constraints:

1. If $v(\phi) \leq v(\psi)$, then $v(\phi \to \psi) \in \mathcal{D}$
2. If $v(\phi) > v(\psi)$, then $v(\phi \to \psi) \in \mathcal{V} - \mathcal{D}$

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[6]We won't concern ourselves at this point on how to get self-reference. If the reader prefers, she can take λ to be *equivalent* to the sentence $\neg Tr\ulcorner\lambda\urcorner$ and so on. As far as we can tell, nothing we say depends on this.

[7]We'll assume further that the space of values satisfies the following condition:
for all x and for all y, if $x \in \mathcal{D}$ and $y \in \mathcal{V} - \mathcal{D}$, then $x > y$.

Can a transparent truth theory be supplemented with a suitable conditional in this sense? Unfortunately, for any finitely-valued linearly-ordered matrix, regardless of whether it is deterministic or not, the following can be proved:

THEOREM 3.1. *No n-valued linearly ordered matrix containing a transparent truth predicate can have a suitable conditional, provided that for each non-designated value i there is a sentence ϕ such that $v(\phi) = i$*[8].

PROOF. Assume that $\mathcal{V}$ has n elements. Since the elements in $\mathcal{V}$ are linearly ordered by some relation $<$, we can list them in the following way: $r_1, ..., r_n$, where $r_1 < .... < r_n$ and $\varnothing \neq \mathcal{D} \subseteq \{r_2, ..., r_n\}$. Since $\mathcal{D}$ is finite, there must be a greatest non-designated element $r_i \notin \mathcal{D}$. Now consider a sentence γ such that γ is $Tr\ulcorner\gamma\urcorner \rightarrow \phi$, where $v(\phi) = r_i$. Now we reason as follows: if $v(\gamma) \in \mathcal{D}$, then $v(\gamma) > v(\phi)$. So by constraint 2, $v(\gamma) \in \mathcal{V} - \mathcal{D}$; if $v(\gamma) \in \mathcal{V} - \mathcal{D}$, then $v(\gamma) \leq v(\phi)$. So by constraint 1, $v(\gamma) \in \mathcal{D}$. Either way, we have a contradiction. ∎

So the problem with finite non-deterministic matrices is that you can use the greatest non-designated value to construct a version of Curry's Paradox[9]. With infinite matrices the problem does not necessarily arise. There might *not* be a greatest non-designated value precisely because there might be infinitely many increasing non-designated values. However, there is a different problem with infinitely valued theories that include a naive truth predicate. The best such theory in the market is Łukasiewicz's theory $Ł_\infty$[10], which can be semantically characterized as follows (see [8] for more details on $Ł_\infty$):

DEFINITION 3.2. *(*Łukasiewicz logic $Ł_\infty$*) Let $Ł_\infty$ be the theory characterized by the matrix $< d, I, \mathcal{V}, \mathcal{D}, \mathcal{O} >$, where*

- *d is a non-empty set,*
- *I is an interpretation function for the non-logical vocabulary,*
- $\mathcal{V} = \{x \in \mathbb{R} : 0 \leq x \leq 1\} = [0, 1]$,

[8] In fact, we can prove that there is such a sentence, so we can dispose of this assumption. However, the proof is simpler this way.

[9] See [9] for a different version of this result not involving non-deterministic matrices. Restall's theorem is in a way stronger than what we have just proved, since it also applies to some non-linearly ordered space of values. However, in a different sense, it is weaker, since it only considers deterministic matrices.

[10] Usually, if theorists want to stress that the truth predicate is in the language, this goes under the label $Ł_\infty Tr$, and moreover, if there is a base syntax theory present such as Peano Arithmetic, it becomes $Ł_\infty^{PA}Tr$. To ease the notation we will use $Ł_\infty^+$ for the theory with the truth predicate plus some sort of naming system.

- $\mathcal{D} = \{1\}$, *and*
- $\mathcal{O}$ *is defined in the following way:*
  - $v(\neg\phi) = 1 - v(\phi)$,
  - $v(\phi \vee \psi) = max(v(\phi), v(\psi))$,
  - $v(\exists x\phi) = sup\{v'(\phi) : v' \text{ is an } x\text{-variant of } v\}$, *and*
  -
$$v(\phi \to \psi) = \begin{cases} 1 & \text{if } v(\phi) \leq v(\psi) \\ 1 - (v(\phi) - v(\psi)) & \text{otherwise} \end{cases}$$

We obtain the theory $Ł^{+}_{\infty}$ by considering only those valuations that in addition to this satisfy $v(Tr\ulcorner\phi\urcorner) = v(\phi)$, for every ϕ. This theory has some attractive properties, specially regarding the truth predicate and the conditional. For instance, like K_3 and LP, the Liar and other problematic sentences receive the value $\frac{1}{2}$, but unlike K_3 both Identity and all T-biconditionals are valid in $Ł^{+}_{\infty}$, and unlike LP, Modus Ponens is valid in $Ł^{+}_{\infty}$. Also, it is possible to provide a weakly complete axiomatization for its propositional Tr-free fragment. More specifically, all $Ł_{\infty}$-tautologies and inferences with finitely many premises are provable from the following four axioms (together with the rule of Modus Ponens):

$\phi \to (\psi \to \phi)$
$(\neg\psi \to \neg\phi) \to (\phi \to \psi)$
$(\phi \to \psi) \to ((\chi \to \phi) \to (\chi \to \psi))$
$((\phi \to \psi) \to \psi) \to ((\psi \to \phi) \to \phi)$

The problem is that $Ł^{+}_{\infty}$ has some unpleasant properties as well. First, the previous axiomatization is only *weakly* complete. There are semantically valid Tr-free *inferences* that are not provable in it.

Secondly, the natural way to extend this theory to a first-order language is by introducing the following two axioms (together with the rule of Generalization):

$\forall x\phi(x) \to \phi(t)$ (where t is free for x in ϕ)
$\forall x(\phi \to \psi) \to (\phi \to \forall x\psi)$ (where x is not free in ϕ)

However, once we do so, the theory is not even weakly complete[11]. There are

[11] [8] refers the reader to a proof by Scarpellini.

some first-order semantically valid Tr-free *sentences* which are not provable in this axiomatization.

These two properties are not really disturbing if you do not care much about proof-theory. However, there is a third unpleasant property: $Ł_\infty^+$ is "nearly" inconsistent. A bit more rigorously:

DEFINITION 3.3. *(*ω-inconsistency*) We say that a theory* $\mathcal{T}$ *is* ω-inconsistent *if and only if for some formula* $\phi(x)$ *and each object* o*,* $\mathcal{T} \vDash \phi[\boldsymbol{o}/x]$ *but* $\mathcal{T} \vDash \exists x \neg\phi(x)$ *(where* $\boldsymbol{o}$ *is a name for* o*)*[12].

Assuming, for instance, that the base theory of $Ł_\infty^+$ is Peano arithmetic, the following can be proved:

THEOREM 3.4. (See [10], [7], [2]) *$Ł_\infty^+$ is ω-inconsistent*[13].

So a question naturally arises: are there interesting subtheories of $Ł_\infty$ with a strong conditional which are not ω-inconsistent? This question has been raised by Bacon in [2]. Nevertheless, our approach will be different from Bacon's[14].

4. Making Łukasiewicz logic non-deterministic

The first logic we'll talk about is $\mathcal{ND}Ł_\infty^+$, which is just like $Ł_\infty^+$ with the only difference that the conditional behaves in the following way:

$$v(\phi \to \psi) \in \begin{cases} \{1\} & \text{if } v(\phi) \leq v(\psi) \\ \mathcal{V} - \mathcal{D} & \text{otherwise} \end{cases}$$

[12]We should point out that being ω-inconsistent is different from being inconsistent in ω-logic (i.e. from lacking an ω-model), although the latter might be regarded as an undesirable property as well. A theory is inconsistent in ω-logic if the theory together with the ω-rule is inconsistent. A theory that is consistent in ω-logic is ω-consistent, but the converse might fail.

[13]In addition to this, in [7] it is proved that adding compositional axioms for truth to $Ł_\infty^+$ makes the theory inconsistent, and not just ω-inconsistent. More precisely, let $Sent_{Ł_\infty^+}(x)$ be a predicate holding of all and only the (names of) sentences of $Ł_\infty^+$, and let $\dot\vee$ be a function that when applied to the codes of two formulas gives the code of their disjunction. It can be shown that $Ł_\infty^+$ already validates every instance of axiom-schemas such as $Tr\ulcorner\phi\dot\vee\psi\urcorner \leftrightarrow (Tr\ulcorner\phi\urcorner \vee Tr\ulcorner\psi\urcorner)$. However, the addition of $\forall x\forall y(Sent_{Ł_\infty}(x) \wedge Sent_{Ł_\infty}(y) \to (Tr(x\dot\vee y) \leftrightarrow (Tr(x) \vee Tr(y))))$ makes the theory inconsistent.

[14]His approach is proof-theoretic. He analyses what axioms for the conditional can we endorse while avoiding ω-inconsistency. In section 5, after presenting our approach, we will briefly compare it to Bacon's.

Clearly, while Łukasiewicz's conditional is deterministic, $\mathcal{NDL}^+_\infty$'s conditional is not[15]. Of course, both conditionals behave exactly the same when the value of the antecedent is less or equal to the value of the consequent, but they differ when this is not the case. Also notice that $\mathcal{NDL}^+_\infty$'s conditional flows very naturally from the two constraints imposed above on any suitable conditional. Here is an incomplete list of principles and inferences which are valid in this theory (in some cases the names are somewhat arbitrary):

$\phi \to \psi, \phi \vDash \psi$	(Modus Ponens)
$\vDash \phi \to \phi$	(Identity)
$\vDash \neg\neg\phi \to \phi$	(Double Negation)
$(\phi \to \psi) \wedge \neg\psi \vDash \neg\phi$	(Modus Tollens)
$\vDash (\phi \wedge (\psi \vee \chi)) \to (\phi \wedge \psi) \vee (\phi \wedge \chi)$	(Distribution)
$\vDash \phi \to (\phi \vee \psi)$	(Disj. Intr.)
$\vDash (\phi \wedge \psi) \to \phi$	(Conj. Elim.)
$\vDash (\phi \to \psi) \vee (\psi \to \phi)$	(Connectivity)
$\phi \vDash \psi \to \phi$	(Positive Weakening)
$\neg\phi \vDash \phi \to \psi$	(Explosion)
$(\phi \to \psi) \wedge (\psi \to \chi) \vDash \phi \to \chi$	(Weak Transitivity)
$\neg\phi \vee \psi \vDash \phi \to \psi$	(Material Conditional)

The problem with this theory is that the conditional is still too weak. It does not validate many plausible inferences and principles. So it is interesting to see how much the conditional can be strengthened without making the theory inconsistent or ω-inconsistent[16]. To investigate this issue we now introduce a number of restrictions on the valuations over which the notion of validity will be characterized.

A very odd feature of $\mathcal{NDL}^+_\infty$ is that a conditional might not get the value 0 even if its antecedent gets value 1 and its consequent value 0. A way to avoid this is by imposing the following straightforward restriction:

Definition 4.1. *(*Semiclassical*) A valuation v in a matrix $\mathcal{M}$ is* semiclassical *if and only if for any pair of formulae ϕ_1 and ϕ_2, if $v(\phi_1)$ and $v(\phi_2)$ are both in $\{0, 1\}$, then $v(\phi_1 \to \phi_2) = v(\neg\phi_1 \vee \phi_2)$.*

[15] Actually, some minor additional adjustments need to be made. The other logical expressions are defined non-deterministically too, but in a non-interesting way. For example, negation and disjunction are characterized as follows: $v(\neg\phi) \in \{1 - v(\phi)\}$, and $v(\phi \vee \psi) \in \{max(v(\phi), v(\psi))\}$.

[16] Actually, we have no proof of its ω-consistency, but we strongly suspect that it is in fact ω-consistent. As we will show in section 5, the usual ways to prove ω-inconsistency will not apply.

Let's call this new theory $\mathcal{NDŁ}^{+}_{\infty}(S)$, for non-deterministic infinitely-valued Łukasiewicz logic with a transparent truth predicate and *semiclassical valuations*. With this new restriction we obtain several inferences that were not validated in $\mathcal{NDŁ}^{+}_{\infty}$ (we give examples below).

This is not the only odd feature of $\mathcal{NDŁ}^{+}_{\infty}$. Consider two conditionals $\phi_1 \to \phi_2$ and $\phi_3 \to \phi_4$ (where the value of the antecedent is greater than the value of the consequent) such that in $\phi_1 \to \phi_2$ the "distance" between ϕ_1 and ϕ_2 is close to 0 and in $\phi_3 \to \phi_4$ it is close to 1. Nothing so far prevents a valuation from assigning to the first a value close to 0 and to the second a value close to 1. To avoid this unpleasant consequence, we can impose the following restriction:

Definition 4.2. *(*Uniform$_1$*) A valuation v in a matrix $\mathcal{M}$ is* uniform$_1$ *if and only if for any formulae ϕ_1, ϕ_2, ϕ_3 and ϕ_4 such that $v(\phi_1) > v(\phi_2)$ and $v(\phi_3) > v(\phi_4)$, if $v(\phi_1) - v(\phi_2) > v(\phi_3) - v(\phi_4)$, then $v(\phi_1 \to \phi_2) < v(\phi_3 \to \phi_4)$.*

Intuitively this says that if we consider two conditional claims, such that the difference between (the value of) the antecedent and (the value of) the consequent in the first conditional is greater than the difference between (the value of) the antecedent and (the value of) the consequent in the second, the value of the second conditional should be greater than the value of the first conditional. For example, if $v(\phi_1) = .8$, $v(\phi_2) = .6$, $v(\phi_3) = .3$ and $v(\phi_4) = .2$, then $v(\phi_1 \to \phi_2) < v(\phi_3 \to \phi_4)$. We'll call the resulting theory $\mathcal{NDŁ}^{+}_{\infty}(U_1)$.

Yet another unsatisfactory feature of $\mathcal{NDŁ}^{+}_{\infty}$ is the following. We might have the following two conditionals: $\top \to \lambda$ and $\lambda \to \bot$. Since λ is a Liar sentence, its value is $\frac{1}{2}$ in every valuation. Hence, its "distance" from $\top$ is the same as its "distance" from $\bot$. But so far nothing prevents a valuation from assigning very different values (less than 1) to these two formulae. This issue can be dealt with by imposing the following restriction:

Definition 4.3. *(*Uniform$_2$*) A valuation v in a matrix $\mathcal{M}$ is* uniform$_2$ *if and only if for any formulae ϕ_1, ϕ_2, ϕ_3 and ϕ_4 such that $v(\phi_1) > v(\phi_2)$ and, $v(\phi_3) > v(\phi_4)$, if $v(\phi_1) - v(\phi_2) = v(\phi_3) - v(\phi_4)$, then $v(\phi_1 \to \phi_2) = v(\phi_3 \to \phi_4)$.*

This informally says that if we have two conditional claims, such that the difference between (the value of) the antecedent and (the value of) the consequent is the same in both, the value of the conditionals should be the same. For example, if $v(\phi_1) = .8$, $v(\phi_2) = .6$, $v(\phi_3) = .3$ and $v(\phi_4) = .1$, then $v(\phi_1 \to \phi_2) = v(\phi_3 \to \phi_4)$. We'll call the resulting theory $\mathcal{NDŁ}^{+}_{\infty}(U_2)$.

Now let's introduce one final restriction. We think this last restriction is plausible enough, but perhaps not as easy to justify as the others (for the record, the restriction holds in $Ł^+_\infty$):

DEFINITION 4.4. *(*Bounded from below*) A valuation v in a matrix $\mathcal{M}$ is* bounded from below *if and only if for any pair of formulae ϕ_1 and ϕ_2, if $v(\phi_1) > v(\phi_2)$, then $v(\phi_1 \to \phi_2) \geq v(\phi_2)$.*

So if a conditional has a value other than 1, its value has to be greater than the value of its consequent. In other words, an untrue conditional cannot be more untrue than its own consequent. The resulting theory is $\mathcal{ND}Ł^+_\infty(B)$[17].

Naturally, it might be desirable to impose these conditions simultaneously. The strongest theory obtainable is this framework is $\mathcal{ND}Ł^+_\infty(SU_{1,2}B)$, in which all our restrictions play a role[18]. These four restrictions seem fairly natural to us and in fact they all hold in $Ł^+_\infty$, but we are not claiming that there are no other plausible restrictions that could be imposed without making the conditional fully deterministic[19].

Observe that a valuation for untrue conditionals is acceptable in the theory $\mathcal{ND}Ł^+_\infty(SU_{1,2}B)$ just in case it can be characterized by a strictly decreasing function[20] f such that $f(x) \geq 1-x$, where the value of x is given by the difference between the value of the antecedent and the value of the consequence of the conditional.

Let's now define validity for each of the possible theories obtainable by imposing one or more of the restrictions above:

DEFINITION 4.5. *(*Validity*) An argument from the set of formulas Γ to the formula ϕ is valid$_i$ ($\Gamma \vDash_i \phi$) if and only if every* i *valuation* v *in $\mathcal{M}$ that satisfies γ for every $\gamma \in \Gamma$, also satisfies ϕ, where i might be* Semiclassical*,* Uniform$_1$*,* Uniform$_2$*,* Bounded from Below *or any combination of them.*[21]

[17] It has been pointed out to us that this last restriction is too strong for conditionals where the content of the antecedent has nothing to do with the content of the consequent. However, this sort of relevant-oriented worry its out of place in this context. $Ł^+_\infty$'s conditional is not intended as a model of relevant reasoning, and nor are the subtheories of it we are considering.

[18] A similar algebraic approach can be found in [6], chapter 15.

[19] A natural additional restriction that could be imposed is that the valuation functions representing the conditional have to be continuous functions. However, we do not see a strong reason to reject valuation functions that do not fulfill this restriction.

[20] We say that a function f is *strictly decreasing* if and only if for all $x_1, x_2 \in dom f, f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

[21] The definition also contemplates the case where Γ is the empty set, so we ambiguously apply 'valid' to both arguments and sentences.

In what follows we are going to state a number of facts about the theories obtainable by applying one or more of these restrictions. For instance, it is not hard to see that by requiring all valuations to be *Semiclassical*, we obtain:

$\phi \wedge \neg\psi \vDash_S \neg(\phi \to \psi)$ (Negative Material Conditional)

If we demand all valuations to respect the *Uniform*$_1$ requirement, the following become valid:

$\phi \to \psi \vDash_{U_1} (\psi \to \chi) \to (\phi \to \chi)$ (Transitivity$_1$)
$\phi \to \psi \vDash_{U_1} (\chi \to \phi) \to (\chi \to \psi)$ (Transitivity$_2$)

It is also straightforward to check that $\mathcal{NDL}^{U_2}_{\infty}$ validates:

$\vDash_{U_2} (\neg\psi \to \neg\phi) \to (\phi \to \psi)$ (Contraposition)
$\vDash_{U_2} ((\phi \to \psi) \wedge (\phi \to \chi)) \to (\phi \to (\psi \wedge \chi))$ (Conj. Intro.)
$\vDash_{U_2} ((\phi \to \chi) \wedge (\psi \to \chi)) \to ((\phi \vee \psi) \to \chi)$ (Disj. Elim.)

Finally, if we consider $\mathcal{NDL}^{B}_{\infty}$, we get:

$\vDash_B \phi \to (\psi \to \phi)$ (Strong Positive Weakening)
$(\phi \wedge \psi) \to \chi \vDash_B \phi \to (\psi \to \chi)$ (Exportation)

It should be clear that each restriction imposed on the set of valuations gives a strictly stronger notion of validity. One could say that the more restrictions we impose on the set of valuations, the more deterministic the conditional is (or, equivalently, the stronger it gets)[22]. A rough picture of how the restrictions can be put to work together is provided in figure 4.1:

[22]This idea is not very different to the one encountered when providing a possible world semantics for relevant logic. There we obtain stronger and stronger conditionals by imposing different constraints on the ternary accessibility relation. See [8], chapter 10 for the details.

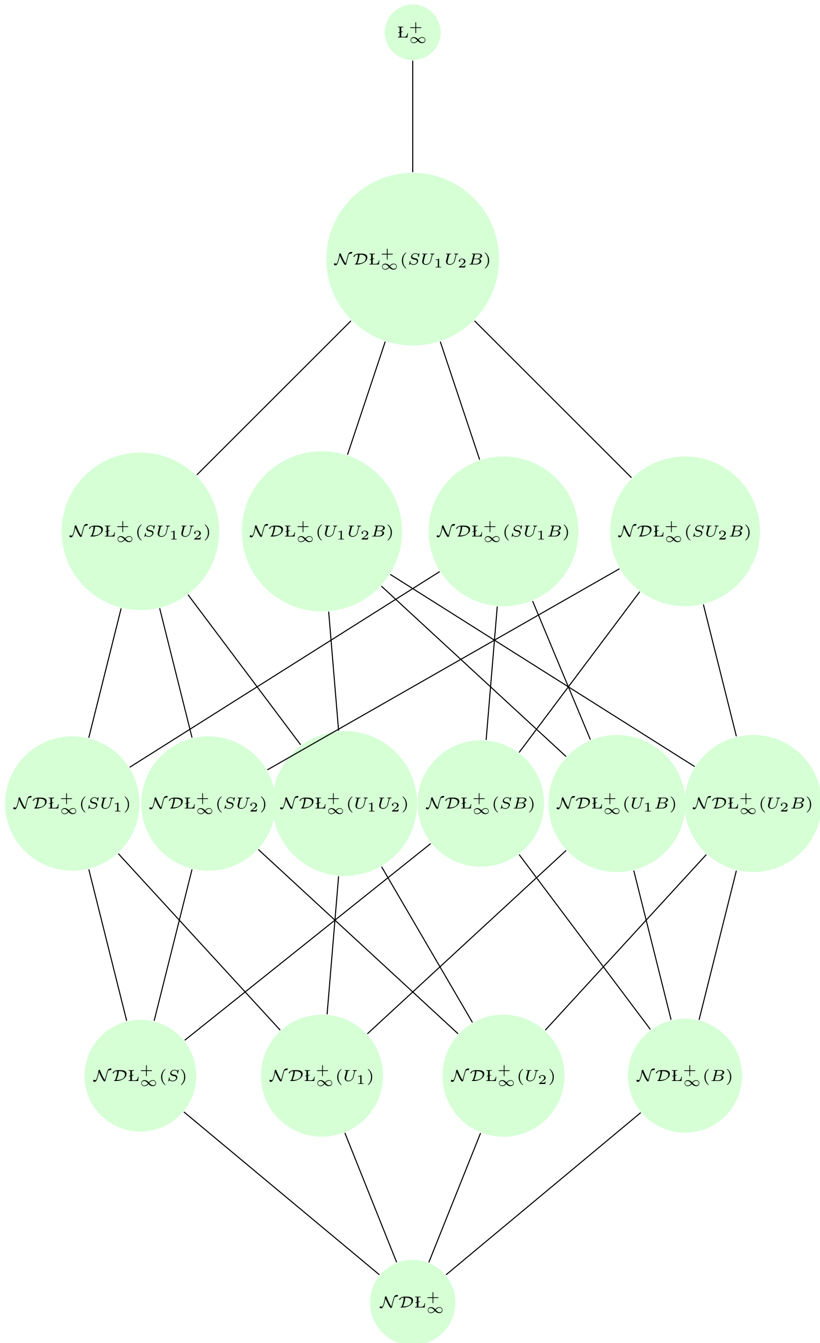

Figure 4.1: All the theories.

A way of spelling out this idea formally is by using the notion of a *refinement*:

DEFINITION 4.6. *(See [1]) A non-deterministic matrix* $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ *is a* refinement *of a non-deterministic matrix* $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ *(notationally,* $\mathcal{M}_1 \preccurlyeq \mathcal{M}_2$*) if and only if*

- $\mathcal{V}_2 \subseteq \mathcal{V}_1$,
- $\mathcal{D}_2 = \mathcal{D}_1 \cap \mathcal{V}_2$,
- $\diamond^{\mathcal{M}_2}(x_1, ..., x_n) \subseteq \diamond^{\mathcal{M}_1}(x_1, ..., x_n)$ *for every n-ary conective* $\diamond$ *and every* $x_1, ..., x_n \in \mathcal{V}_1$.

As we impose more restrictions on the set of valuations, we get refinements of the previous theories. For example, the following obtains:

$\mathcal{ND}\text{Ł}^+_\infty \preccurlyeq \mathcal{ND}\text{Ł}^+_\infty(S) \preccurlyeq \mathcal{ND}\text{Ł}^+_\infty(SU_1) \preccurlyeq \mathcal{ND}\text{Ł}^+_\infty(SU_{1,2}) \preccurlyeq \mathcal{ND}\text{Ł}^+_\infty(SU_{1,2}B) \preccurlyeq \text{Ł}^+_\infty$.[23]

In [1], the authors prove that for any pair of non-deterministic matrices $\mathcal{M}_1$ and $\mathcal{M}_2$, if $\mathcal{M}_1 \preccurlyeq \mathcal{M}_2$, then $\vDash_{M_1} \subseteq \vDash_{\mathcal{M}_2}$. So it immediately follows that:

$\vDash_{\mathcal{ND}\text{Ł}^+_\infty} \subseteq \vDash_S \subseteq \vDash_{SU_1} \subseteq \vDash_{SU_{1,2}} \subseteq \vDash_{SU_{1,2}B} \subseteq \vDash_{\text{Ł}^+_\infty}$.

Of course, this is just one example. All the ways in which one of our theories can refine another can be found in Figure 4.1. More specifically, if there is an upward path from a theory $\mathcal{T}_1$ to a theory $\mathcal{T}_2$, then $\mathcal{T}_1 \preccurlyeq \mathcal{T}_2$ and hence $\vDash_{\mathcal{T}_1} \subseteq \vDash_{\mathcal{T}_2}$.

Of particular interest is the fact that the strongest non-deterministic theory we have considered $\mathcal{ND}\text{Ł}^+_\infty(SU_{1,2}B)$ is such that $\vDash_{SU_{1,2}B} \subsetneq \vDash_{\text{Ł}_\infty}$. Moreover, we know that $\mathcal{ND}\text{Ł}^+_\infty(SU_{1,2}B)$ is a *proper* sublogic of Ł^+_∞, since for instance $\vDash_{\text{Ł}^+_\infty} ((\phi \to \psi) \to \psi) \to ((\psi \to \phi) \to \phi)$, but $\nvDash_{SU_{1,2}B} ((\phi \to \psi) \to \psi) \to ((\psi \to \phi) \to \phi)$.

Although we will not offer the proof here, the following is a well-known result:

THEOREM 4.7. *(See [7])* Ł^+_∞ *is consistent.*

Since all subtheories of Ł^+_∞ will be consistent as well, as a corollary of the previous theorem, we can infer:

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[23] Although the definition of a refinement is only meant to be applied to non-deterministic matrices, the comparison with Ł^+_∞ is legitimate, since we have shown that every deterministic matrix can be mimicked using some non-deterministic matrix.

COROLLARY 4.8. *$\mathcal{ND}L^+_\infty(SU_{1,2}B)$ is consistent.*

Naturally, it also follows that all subtheories of $\mathcal{ND}Ł^+_\infty(SU_{1,2}B)$ are consistent[24].

5. Proving ω-inconsistency?

What about ω-consistency? It is not at all obvious to us how to construct a nice model for the non-deterministic theories. Clearly, we cannot show that a nice model exists by defining a monotone jump operator on the interpretations of the truth predicate, as is usually done in a number of many-valued theories. The reason is that $\mathcal{ND}Ł^+_\infty(SU_{1,2}B)$'s conditional (just as $Ł^+_\infty$'s conditional) is not a monotone operation on the set of values.

Another strategy would be to use Brower's fixed point theorem, according to which (roughly) every continuous function in the interval $[0,1]$ has a fixed point. In fact, in [6] Field uses this theorem to show that the propositional fragment of $Ł^+_\infty$ has a nice model. However, his proof applies only to the propositional part of the language and it depends on the conditional being a continuous function, something which holds in $Ł^+_\infty$ but does not hold in our theories[25].

So although we will not offer a proof of ω-consistency, we will show that the usual strategies for proving ω-inconsistency do not work for these theories.

Restall's [10] proof of $Ł^+_\infty$'s ω-inconsistency rests on the possibility of defining a fusion operator using Łukasiewicz's conditional. But this cannot be done –at least in the same way- in the sort of non-deterministic frameworks we have presented. Restall defines a fusion operator $\circ$ in the following way

[24] What about the possibility of axiomatizing these theories? Although $Ł^+_\infty$ is not fully axiomatizable, it is still an open problem whether the non-deterministic theories have a complete axiomatization. As an anonymous referee points out, if this were the case, we would have a strong argument in favor of these theories. As far as we can see, the issue is not at all trivial. For example, it is quite easy to show that $Ł^+_\infty$ is not compact and that, as a consequence, it is not axiomatizable (a proof of this claim can be found in [8], p. 240, exercises 8-9). However, the most direct proof of this fact depends on the possibility of using the conditional to define a *fission* operator -dual to the fusion operator (see below)- something that cannot be done in the usual way with the non-deterministic conditionals.

[25] Perhaps it is possible to modify the conditional so that it is a continuous function (or more precisely, a continuous function relative to each valuation). But even in that case it is still unclear to us whether this strategy would be successful for our non-deterministic theories.

$\phi^\circ\psi =_{def} \neg(\phi \to \neg\psi)$

Restall takes 0 to represent truth and 1 to represent falsity. So 0 is the only designated value and the conditional is defined as restricted substraction: $v(\phi \to \psi) = v(\psi) \dot{-} v(\phi)$. This means that $v(\phi^\circ\psi) = min(1, v(\phi) + v(\psi))$.

Since we are working with 1 as the only designated value, Łukasiewicz's conditional is defined as follows: $v(\phi \to \psi) = min(1, 1–v(\phi) + v(\psi))$. This means that $v(\phi^\circ\psi) = 1–min(1, (1–\phi) + (1–\psi))$, which simplifies to

$v(\phi^\circ\psi) = max(0, \psi–(1–\phi)))$

The key aspect of $^\circ$'s behavior is that, for any formula ϕ and any valuation v such that $v(\phi) \neq 1$, there is some finite number n such that the n-fold fusion of ϕ with itself will receive the value 0.

$$v(\underbrace{\phi^\circ(\phi^\circ \ldots (\phi^\circ\phi)...))}_{n-times}) = 0.$$

Why is that so? Because, as we have already stated, $v(\phi^\circ\phi) = max(0, \phi –(1–\phi))$. So $\phi^\circ\phi$ (that is, $\neg(\phi \to \neg\phi)$) is designed to give a value such that, if $v(\phi) = 1$, then $v(\phi^\circ\phi) = 1$; but if $v(\phi) \neq 1$, then $v(\phi^\circ\phi)$ is strictly less than $v(\phi)$. Moreover, if we fusion $\phi^\circ\phi$ with ϕ, we get a formula whose value is strictly less than $v(\phi^\circ\phi)$; and if we fusion $\phi^\circ(\phi^\circ\phi)$ with ϕ we get a formula whose value is strictly less than $v(\phi^\circ(\phi^\circ\phi))$; and so on until we reach a formula with a value less or equal to $\frac{1}{2}$. In that event, one more fusion with ϕ is enough to reach a formula with value 0.

Using the fusion operator Restall constructs a sequence of sentences S_0, S_1, S_2,... such that S_0 says that not every S_i is true for $i > 0$, and S_{n+1} is the $n + 1$-fold fusion of S_0. Then he goes on to show, using a semantic argument, that $Ł_\infty$ is ω-inconsistent, as it declares S_0 true, but also declares, for each i, that S_i is true. The reader is encouraged to see [9] for the details.

Things are not so easy in the non-deterministic frameworks we have been considering. For example, if we try using the conditional as Restall does it to define a fusion operator, we will not obtain the desired result. To see why, we will consider $\mathcal{ND}Ł^+_\infty$ first. Take any formula ϕ such that $\frac{1}{2} < v(\phi) < 1$. For definiteness, assume that $v(\phi) = .8$. Whereas in $Ł^+_\infty$, $v(\phi \to \neg\phi) = .4$ and hence $v(\neg(\phi \to \neg\phi)) = v(\phi^\circ\phi) = .6$, this formula can receive any undesignated truth-value in $\mathcal{ND}Ł^+_\infty$, which means that its negation can receive any undesignated truth-value as well. Therefore, the fusion of this particular formula with itself, not only does not decrease the value of ϕ, but it might get a higher value than that of ϕ. So $^\circ$, defined in this way, does not work in $\mathcal{NDL}^+_\infty$ as it is supposed to. So $\mathcal{NDL}^+_\infty$'s presumed ω-inconsistency cannot be proved using this method.

This comes as no surprise, since $\mathcal{NDL}^+_\infty$ is a very weak theory. What about the stronger theories? It turns out that the situation is more or less the same. We will consider $\mathcal{NDL}^+_\infty(SU_{1,2}B)$, the strongest non-deterministic theory we have presented. Assume once again that $\frac{1}{2} < v(\phi) < 1$ and for definiteness let $v(\phi) = .8$. This time it would be inaccurate to say that $\phi^\circ\phi$ can take any undesignated value. Since valuations are bounded from below, $v(\phi \to \neg\phi) \geq .2$, and hence $v(\phi^\circ\phi) = v(\neg(\phi \to \neg\phi)) \leq .8$[26]. So $v(\phi^\circ\phi)$ is indeed less or equal than $v(\phi)$. In fact, it holds for any formula ϕ and any valuation v that the value of the n-fold fusion of ϕ with itself is less or equal than the value of the $n-1$-fold fusion of ϕ with itself. The problem, of course, is that the restrictions are not enough to guarantee that there is a finite number n such that the n-fold fusion of ϕ with itself will have value 0. It is perfectly possible for the fusion operation to decrease the values "too slowly", in the sense that the repeated application of this operator produces formulas whose values do not decrease or strictly decrease but with a limit different from 0.

A different way of proving ω-inconsistency is offered by Andrew Bacon in [2]. Bacon shows a proof-theoretic version of the following result:

Theorem 5.1. *(See [2]) Any transparent truth theory closed under the following rules is ω-inconsistent:*[27]
if $\phi \vDash \psi$, *then* $\exists x\phi \vDash \exists x\psi$
$\phi \to \exists x\psi \vDash \exists x(\phi \to \psi)$ *(where x is not free in ϕ)*

Proof. See [2] for the proof. Even though his proof is proof-theoretic, it can be mimicked semantically. ∎

So if any of the non-deterministic theories we have developed satisfies both these principles, then it is ω-inconsistent. However, we can show that the second rule does not hold in $\mathcal{NDL}^+_\infty(SU_{1,2}B)$ (and *a fortiori* that it does not hold in any of the weaker theories). Just consider a formula $\psi(x)$ with (at least) x free such that for no x-variant v' of v it holds that $v(\phi) \leq v'(\psi)$ but $sup\{v'(\psi) : v' \text{ is an } x\text{-variant of } v\} = v(\phi)$. In $Ł^+_\infty$, this valuation will make both the premise and the conclusion true. However, in $\mathcal{NDL}^+_\infty(SU_{1,2}B)$ or

[26] The other conditions seem to be of no help in restricting the valuations further.

[27] Actually, Bacon makes a distinction between strongly ω-inconsistent theories and weakly ω-inconsistent theories. However, for our purposes this distinction will be unnecessary.

in any of the weaker non-deterministic theories, there is no guarantee that the conclusion is true.

There might be some other way to define fusion, or to prove ω-inconsistency, but currently we are not aware of any.

6. Determinate truth in non-deterministic semantics

One of the known problems of paracomplete theories like K_3 is that their languages are too weak to express the idea that certain sentences are not determinately true. An additional virtue of $Ł^+_\infty$ that has been pointed out by Field in [6], ch.4 is that it overcomes this defect. Its conditional can be used to define a nice determinately operator:

$$D\phi =_{df} \neg(\phi \rightarrow \neg\phi).$$

The reader will notice that $D\phi$ is the same as $\phi^\circ\phi$. So in $Ł^+_\infty$ the determinately operator is just a case of the fusion operator.

Now, in the previous section we have seen that the fusion operator cannot be used -at least not in the standard way- to prove the ω-inconsistency of the non-deterministic theories we have been considering. Since the determinately operator is just a limit case of the fusion operator, a natural worry is that these theories loose one very attractive feature of $Ł^+_\infty$, the ability to consistently add such an operator, and *a fortiori*, the ability to express the idea that certain sentences are not determinately true.

However, we will show that there is no reason for concern. Following [6], we will say that we should expect the following from a nice determinately operator:

1. If $v(\phi) = 1$, then $v(D\phi) = 1$.
2. If $v(\phi) \leq v(\neg\phi)$, then $v(D\phi) = 0$.
3. If $0 < v(\phi) < 1$, then $v(D\phi) \leq v(\phi)$[28].

[28]In [6], p. 235-36, Field flirts with the stronger:
If $0 < v(\phi) < 1$, then $v(D\phi) < v(\phi)$,
which will not hold in our non-deterministic theories, but ends up using 3. For the stronger version to hold we would need to strengthen *bounded from below* by imposing that $v(\phi \rightarrow \psi) > v(\psi)$. As an anonymous referee suggests, it is this what is responsible for $Ł^+_\infty$'s ω-inconsistency, because it allows us to define an operator $D^*\phi$ expressing the idea that ϕ is determinate at all countable ordinals. This in turn would be enough to define classical negation, since $v(D^*\phi) = 1$ whenever $v(\phi) = 1$, and $v(D^*\phi) = 0$, otherwise. However, as far as we can see, this condition is not enough to define classical negation in our non-

4. If $v(\phi) \leq v(\psi)$, then $v(D\phi) \leq v(D\psi)$.

Fortunately, we can show that if $D\phi$ is defined as $\neg(\phi \to \neg\phi)$ all these conditions hold for D in $\mathcal{NDL}^+_\infty(SU_{1,2}B)$, the strongest of the theories we have been considering. More precisely,

THEOREM 6.1 (A determinately operator). *$\mathcal{NDL}^+_\infty(SU_{1,2}B)$ contains a nice determinately operator D.*

PROOF. Just as before, let $D\phi$ be $\neg(\phi \to \neg\phi))$. It is clear that all valuations satisfy condition 2. and in addition, the following is the case:
If it satisfies *semiclassical*, then condition 1. holds;
if it satisfies *bounded from below*, condition 3 holds; and
if it satisfies both $uniform_1$ and $uniform_2$, condition 4 holds. ■

These properties are not only nice in themselves. Once D is shown to satisfy them, we can prove that D is non-idempotent, in the sense that sometimes $v(D\phi) \neq v(DD\phi)$. This is specially important in the case of Liar-like sentences λ_n such that $v(\lambda_n) = v(\neg D^n Tr\ulcorner\lambda_n\urcorner)$, where D^n stands for n iterations of the D operator. We not only have $v(D\lambda) = 0$, but for any λ_n such that λ_n is $\neg D^n Tr\ulcorner\lambda_n\urcorner$, we can prove $v(D^{n+1}\lambda_n) = 0$. In other words, for any Liar sentence expressible in the language we can say in the language that there is a sense in which it is not determinately true[29].

7. Conclusion

We'll finish by mentioning a similar approach explored recently by Bacon in [2]. Bacon considers two theories: BCKN and BCKD. BCKN has models where the space of values is not linearly ordered, so this feature can be exploited to show that $\phi \to \exists x\psi \nvDash \exists x(\phi \to \psi)$. But since we are not considering non-linearly ordered matrices in this paper, we will ignore it. In the case of BCKD, the presence of the Dummett axiom D guarantees that the theory is true only on linearly ordered spaces of values. In particular, Bacon considers the following semantic definition for BCKD's conditional:

$$v(\phi \to \psi) = \begin{cases} 1 & \text{if } v(\phi) \leq v(\psi) \\ v(\psi) & \text{otherwise} \end{cases}$$

deterministic theories, because we can always pick a value r in $(\frac{1}{2}, 1)$ for $D^*\phi$. So if λ^* is the sentence $\neg D^*\lambda$, there might be valuations v such that $v(\lambda^*) = v(\neg D^*\lambda) \in (0, \frac{1}{2})$.

[29]The reader can see [6] for more details.

Then he goes on to show that the inference from $\phi \to \exists x\psi$ to $\exists x(\phi \to \psi)$ fails for BCKD's conditional. However, this inference fails only at the cost of using models that are not compatible with a transparent truth predicate. Consider again a Curry sentence δ such that δ is $\mathrm{T}(\delta) \to \bot$. It is straightforward to see that the above definition for the conditional cannot consistently assign a truth value to δ[30].

In the case of the non-deterministic theories presented above, there is no analogous problem. Curry's Paradox is blocked (every Curry sentence receives a stable non-designated value) and hence triviality is avoided. But are these theories ω-consistent? We can conjecture that that they are but, certainly, the need for an ω-consistency proof remains.

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Acknowledgements. Acknowledgements.

[30]It also seems costly to drop the axiom N, since it is equivalent to Double Negation Elimination in most theories.

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