CONVERGENCE IN THE CESÀRO SENSE OF ERGODIC OPERATORS ASSOCIATED WITH A FLOW

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Abstract. We study the a.e. convergence in the Cesàro-α sense of the ergodic averages and the ergodic Hilbert transform associated with a Cesàro bounded flow.

1. Introduction.

Let $(X, \mathcal{F}, \nu)$ be a finite measure space. By a flow $\{\tau_t : t \in \mathbb{R}\}$ we mean a group of measurable transformations $\tau_t : X \to X$ with τ_0 the identity and $\tau_{t+s} = \tau_t \circ \tau_s, (t, s \in \mathbb{R})$. The flow is said to be measure preserving if the τ_t are measure-preserving, i.e., if $\nu(\tau_{-t}E) = \nu(E)$ for all $E \in \mathcal{F}$. The flow is said to be nonsingular if $\nu(\tau_{-t}E) = 0$ for all $t \in \mathbb{R}$ and all $E \in \mathcal{F}$ with $\nu(E) = 0$. Finally, the flow is said to be measurable if the map $(x, t) \to \tau_t x$ from $X \times \mathbb{R}$ into X is $\widetilde{\mathcal{F}}$-$\mathcal{F}$-measurable where $\widetilde{\mathcal{F}}$ is the completion of the product-σ-algebra $\mathcal{F} \otimes \mathcal{B}$ of $\mathcal{F}$ with the Borel sets, and the completion is taken with respect to the product measure of ν on $\mathcal{F}$ and the Lebesgue measure on $\mathcal{B}$. Analogously we can define what we mean by semiflow $\{\tau_t : t > 0\}$, a measure-preserving semiflow, a nonsingular semiflow and a measurable semiflow.

Y. Deniel studied in [4] the convergence in the Cesàro-α sense ((C,α) sense) of the ergodic averages associated with a measure-preserving semiflow on a probability space $(\Omega, \mathcal{F}, \mu)$. More precisely, he proved the following result.

Theorem A [4]**.** *Let $\{\tau_t : t > 0\}$ be a measure-preserving semiflow of a probability space $(\Omega, \mathcal{F}, \mu)$. Let $-1 < \alpha < 0$, $\frac{1}{1+\alpha} < p < \infty$ and $f \in L^p(d\mu)$. Then, the (C,α) ergodic averages*

$$A^{+}_{T,1+\alpha} f(x) = \frac{1}{T^{1+\alpha}} \int_0^T f(\tau_t x)(T - t)^{\alpha} \, dt$$

converge, when $T \to \infty$, almost everywhere and in the $L^p(d\mu)$-norm.

Theorem A does not hold in the limit case $p = \frac{1}{1+\alpha}$ [4]. However a positive result was obtained in [2] in this limit case. Their result is the following.

Key words and phrases. Cesàro convergence, ergodic averages, ergodic Hilbert transform, Cesàro bounded flows.

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Theorem B [2]**.** *Let $\{\tau_t : t > 0\}$, $(\Omega, \mathcal{F}, \mu)$ and α be as in Theorem A. Then, there exists the limit* $\lim_{T\to\infty} A^+_{T,1+\alpha} f(x)$ *a.e. for all f in the Lorentz space* $L_{\frac{1}{1+\alpha},1}(d\mu) = \{f : ||f||_{\frac{1}{1+\alpha},1;\mu} = \int_0^\infty \left(\lambda_f(t)\right)^{1+\alpha} dt < \infty\}$, *where* $\lambda_f(t) = \mu(\{x : |f(x)| > t\})$ *is the distribution function of f.*

On the other hand, it was studied in [6] the convergence of the ergodic averages

$$A_{T,1} f(x) = \frac{1}{2T} \int_{-T}^{T} f(\tau_t x)\, dt,$$

and the ergodic Hilbert transform $Hf(x) = \lim_{\varepsilon\to 0} H_\varepsilon f(x)$, where

$$H_\varepsilon f(x) = \int_{\varepsilon<|t|<1/\varepsilon} \frac{f(\tau_t x)}{t} dt,$$

associated with a Cesàro bounded flow on a finite measure space $(X, \mathcal{F}, \nu)$ (notice that the flow does not need to preserve the measure ν). The result proved in [6] is as follows.

Theorem C [6]**.** *Let $(X, \mathcal{F}, \nu)$ be a finite measure space, $1 \le p < \infty$ and let $\{\tau_t : t \in \mathbb{R}\}$ be a nonsingular measurable flow on X such that for some positive constant C and all $f \in L^p(d\nu)$*

$$\sup_{T>0} ||A_{T,1} f||_{p;\nu} \le C ||f||_{p;\nu}.$$

(i) If $1 < p < \infty$ and $f \in L^p(d\nu)$, then there exist the limits $\lim_{T\to\infty} A_{T,1} f(x)$ *and* $\lim_{\varepsilon\to 0} H_\varepsilon f(x)$ *a.e. and in the $L^p(d\nu)$-norm.*
(ii) If $p = 1$ and $f \in L^1(d\nu)$, then there exist the limit $\lim_{T\to\infty} A_{T,1} f(x)$ *a.e. and in the $L^1(d\nu)$-norm and there exist the limit* $\lim_{\varepsilon\to 0} H_\varepsilon f(x)$ *a.e..*

The aim of this paper is to study the (C,α) convergence of the ergodic averages and the ergodic Hilbert transform in the setting of Theorem C, i.e., for Cesàro bounded flows. More precisely, for the (C,α) ergodic averages, we shall prove the following theorem.

Theorem 1.1. *Let $(X, \mathcal{F}, \nu)$ be a finite measure space, $-1 < \alpha \le 0$ and $\frac{1}{1+\alpha} \le p < \infty$. Let $\{\tau_t : t \in \mathbb{R}\}$ be a nonsingular measurable flow on X such that for some positive constant C and all $f \in L^{p(1+\alpha)}(d\nu)$*

$$\sup_{T>0} ||A^+_{T,1} f||_{p(1+\alpha);\nu} \le C ||f||_{p(1+\alpha);\nu}. \tag{1.2}$$

(i) If $\frac{1}{1+\alpha} < p < \infty$ and $f \in L^p(d\nu)$, then there exists the limit $\lim_{T\to\infty} A^+_{T,1+\alpha} f(x)$ *a.e. and in the $L^p(d\nu)$-norm.*
(ii) If $p = \frac{1}{1+\alpha}$ and $f \in L_{\frac{1}{1+\alpha},1}(d\nu)$, then there exists the limit $\lim_{T\to\infty} A^+_{T,1+\alpha} f(x)$ *a.e..*

Now, we precise what we mean as (C,α) convergence of the ergodic Hilbert transform. Following Hardy [5,§5.14 and Notes on Chapter V] we wish to study the existence of the limit $Hf(x) = \lim_{\varepsilon\to 0} H_\varepsilon f(x) = \lim_{t\to\infty} H_{1/t}f(x)$ in the (C,α) sense; in the case $\alpha > 0$ that means that we want to study the limit

$$\lim_{T\to\infty} \frac{\alpha}{T^\alpha} \int_0^T H_{1/t}f(x)(T-t)^{\alpha-1}\,dt.$$

Interchanging the integrals we can easily see that to study the above limit is equivalent to study the limit of

$$H_{\varepsilon,\alpha}f(x) = \int_{\varepsilon<|t|\le 1} \frac{f(\tau_t x)}{t}\left(1-\frac{\varepsilon}{|t|}\right)^\alpha dt + \int_{1<|t|\le 1/\varepsilon} \frac{f(\tau_t x)}{t}\left(1-\varepsilon|t|\right)^\alpha dt,$$

when $\varepsilon \to 0$. We shall see that for suitable f the above integrals make sense not only for $\alpha \ge 0$ but also for $\alpha > -1$. Since the convergence of $H_{\varepsilon,0}f(x)$ implies the convergence of $H_{\varepsilon,\alpha}f(x)$ for $\alpha > 0$ (see §4: claim (d)), we are interested in the study of the limit $\lim_{\varepsilon\to 0} H_{\varepsilon,\alpha}f$, for $-1 < \alpha \le 0$. In particular, we shall prove the following theorem.

Theorem 1.3. *Let $(X, \mathcal{F}, \nu)$ be a finite measure space, $-1 < \alpha \le 0$ and $\frac{1}{1+\alpha} \le p < \infty$. Let $\{\tau_t : t \in \mathbb{R}\}$ be a nonsingular measurable flow on X such that for some positive constant C and all $f \in L^{p(1+\alpha)}(d\nu)$*

$$\sup_{T>0} ||A_{T,1}f||_{p(1+\alpha);\nu} \le C||f||_{p(1+\alpha);\nu}. \tag{1.4}$$

(i) If $\frac{1}{1+\alpha} < p < \infty$ and $f \in L^p(d\nu)$, then there exists the limit $\lim_{\varepsilon\to 0} H_{\varepsilon,\alpha}f(x)$ a.e. and in the $L^p(d\nu)$-norm.

(ii) If $p = \frac{1}{1+\alpha}$ and $f \in L_{\frac{1}{1+\alpha},1}(d\nu)$, then there exists the limit $\lim_{\varepsilon\to 0} H_{\varepsilon,\alpha}f(x)$ a.e..

On one hand, notice that Theorem 1.3 for $\alpha = 0$ is contained in Theorem C. On the other hand, Theorem 1.3 also holds for the (C,α) ergodic averages

$$A_{T,1+\alpha}f(x) = \frac{1}{(2T)^{1+\alpha}} \int_{-T}^T f(\tau_t x)(T-|t|)^\alpha\,dt,$$

but this result is an easy consequence of Theorem 1.1 applied to the flows $\{\tau_t : t \in \mathbb{R}\}$ and $\{\tilde\tau_t : t \in \mathbb{R}\}$, where $\tilde\tau_t = \tau_{-t}$.

Throughout this paper α will be a number such that $-1 < \alpha \le 0$ and if $1 < p < \infty$ then p' will be denote its conjugate exponent, i.e., $1/p + 1/p' = 1$. The letter C will mean a positive constant non necessarily the same at each ocurrence.

2. Preliminary results

In order to prove the theorems we will need results about some maximal operators which were studied in [9] and [1]. First we introduce the following definitions about weights.

Definition 2.1 [10]**.** *Let w be a positive measurable function on the real line. It is said that w satisfies the Muckenhoupt A_p condition, $1 \le p < \infty$, if there exists a constant $C > 0$ such that*

$$\sup_{a<b}\left(\frac{1}{b-a}\int_a^b w(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b w^{1-p'}(t)\,dt\right)^{p-1} \le C \qquad \text{if} \quad 1<p<\infty$$

and

$$\sup_{r>0}\left(\frac{1}{2r}\int_{-r}^r w(x-t)\,dt\right) \le Cw(x) \quad \text{a.e.} \qquad \text{if} \quad p=1.$$

Definition 2.2 [12]. *Let w be a positive measurable function on the real line. It is said that w satisfies A_p^+, $1 \le p < \infty$, if there exists a constant $C > 0$ such that*

$$\sup_{a<b<c} \left(\frac{1}{c-a} \int_a^b w(t)\, dt \right) \left(\frac{1}{c-a} \int_b^c w^{1-p'}(t)\, dt \right)^{p-1} \le C \qquad \textit{if} \quad 1 < p < \infty$$

and

$$\sup_{r>0} \left(\frac{1}{r} \int_0^r w(x-t)\, dt \right) \le C w(x) \quad a.e. \qquad \textit{if} \quad p = 1.$$

The A_p^- classes are defined in the obvious way, reversing the orientation in the real line.

In order to study the (C,α) ergodic averages we will need the boundedness of the following maximal operator:

$$M_{1+\alpha}^+ f(x) = \sup_{T>0} \frac{1}{T^{1+\alpha}} \int_0^T |f(x+t)|(T-t)^\alpha\, dt.$$

For this operator it was proved in [9] the following result (see Theorem 2.5, Theorem 3.5 and Final Remarks in [9]).

Theorem D [9]. *Let $-1 < \alpha \le 0$, $\frac{1}{1+\alpha} \le p < \infty$ and let w be a positive measurable function on the real line.*
(i) If $\frac{1}{1+\alpha} < p < \infty$ and $w \in A_{p(1+\alpha)}^+$, then there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}} \left[M_{1+\alpha}^+ f(t) \right]^p w(t)\, dt \le C \int_{\mathbb{R}} |f(t)|^p w(t)\, dt$$

for all $f \in L^p(w(t)dt)$.
(ii) If $p = \frac{1}{1+\alpha}$ and $w \in A_1^+$, then there exists a constant $C > 0$ such that

$$w(\{t \in \mathbb{R} : M_{1+\alpha}^+ f(t) > \lambda\}) \le \frac{C}{\lambda^{\frac{1}{1+\alpha}}} ||f||_{\frac{1}{1+\alpha},1;w}^{1/1+\alpha}$$

for all $f \in L_{\frac{1}{1+\alpha},1}(w(t)dt)$ and all $\lambda > 0$.

Remark 2.3. Actually, in [9] it was proved Theorem D (ii) only for characteristic functions but, for $-1 < \alpha < 0$, applying Theorem 3.13 in [13, p.195] which also holds for the sublinear operator $M_{1+\alpha}^+$, we easily obtain the result for all $f \in L_{\frac{1}{1+\alpha},1}(w(t)dt)$. On the other hand, if $\alpha = 0$, statement (ii) is the known result that $w \in A_1^+$ implies the weak type (1,1) inequality for the one-sided Hardy-Littlewood maximal function with respect to $w(t)dt$ that was proved by E. Sawyer [12] (see also [8] and [7]).

Obviously, a result analogous to Theorem D hold for the other one-sided maximal operator $M_{1+\alpha}^- f(x) = \sup_{T>0} \frac{1}{T^{1+\alpha}} \int_{-T}^0 |f(x+t)|(T+t)^\alpha\, dt$ and the corresponding $A_{p(1+\alpha)}^-$ classes. Now, taking into account that the maximal operator

$$M_{1+\alpha} f(x) = \sup_{T>0} \frac{1}{(2T)^{1+\alpha}} \int_{-T}^T |f(x+t)|(T-|t|)^\alpha\, dt,$$

is pointwise equivalent to the sum of the operators $M^+_{1+\alpha}$ and $M^-_{1+\alpha}$ and that $A_{p(1+\alpha)} = A^+_{p(1+\alpha)} \cap A^-_{p(1+\alpha)}$ we see that Theorem D is valid for $M_{1+\alpha}$, changing the $A^+_{p(1+\alpha)}$ classes by the $A_{p(1+\alpha)}$ classes.

On the other hand, in the study of the ergodic Hilbert transform in the Cesàro-α sense appears (see Section 3) the following maximal operator

$$N_{1+\alpha}f(x) = \sup_{T>0} \frac{1}{(2T)^{1+\alpha}} \int_{T<|t|<2T} |f(x+t)|(|t|-T)^\alpha \, dt.$$

This operator was studies in [1] (Theorems 2.1 and 2.4), obtaining analogous results to those ones for the operator $M_{1+\alpha}$. In the following theorem we collect these results and the corresponding ones for $M_{1+\alpha}$.

Theorem E ([9] and [1]). *Let $-1 < \alpha \le 0$, $\frac{1}{1+\alpha} \le p < \infty$ and let w be a positive measurable function on the real line. Let us denote by $\mathcal{M}$ either $M_{1+\alpha}$ or $N_{1+\alpha}$.*

(i) If $\frac{1}{1+\alpha} < p < \infty$ and $w \in A_{p(1+\alpha)}$, then there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}} [\mathcal{M}f(t)]^p \, w(t) \, dt \le C \int_{\mathbb{R}} |f(t)|^p w(t) \, dt,$$

for all $f \in L^p(w(t)dt)$.

(ii) If $p = \frac{1}{1+\alpha}$ and $w \in A_1$, then there exists a constant $C > 0$ such that

$$w(\{t \in \mathbb{R} : \mathcal{M}f(t) > \lambda\}) \le \frac{C}{\lambda^{\frac{1}{1+\alpha}}} ||f||^{1/1+\alpha}_{\frac{1}{1+\alpha},1;w},$$

for all $f \in L_{\frac{1}{1+\alpha},1}(w(t)dt)$ and all $\lambda > 0$.

3. Boundedness for the ergodic maximal operators

This section is devoted to establish the boundedness of the maximal operators

$$S^+_{1+\alpha}f(x) = \sup_{T>0} \frac{1}{T^{1+\alpha}} \int_0^T |f(\tau_t x)|(T-t)^\alpha \, dt,$$

associated to the (C,α) ergodic averages $A^+_{1+\alpha}f$, and

$$R_{1+\alpha}f(x) = \sup_{T>0} \frac{1}{(4T)^{1+\alpha}} \int_{-2T}^{2T} |f(\tau_t x)| \, |T - |t||^\alpha \, dt,$$

which appears in the study of the maximal operator $H^*_\alpha = \sup_{\varepsilon>0} |H_{\varepsilon,\alpha}|$ associated with the Hilbert transform in the Cesàro-α sense. More precisely we shall prove the following theorems.

Theorem 3.1. *Let $(X, \mathcal{F}, \nu)$, α, p and $\{\tau_t : t \in \mathbb{R}\}$ be as in Theorem 1.1.*

(i) If $\frac{1}{1+\alpha} < p < \infty$, then there exists a constant $C > 0$ such that for all $f \in L^p(d\nu)$

$$||S^+_{1+\alpha}f||_{p;\nu} \le C||f||_{p;\nu}.$$

(ii) If $p = \frac{1}{1+\alpha}$, then there exists a constant $C > 0$ such that for all $f \in L_{\frac{1}{1+\alpha},1}(d\nu)$ and all $\lambda > 0$

$$\nu(\{x \in X : S^+_{1+\alpha}f(x) > \lambda\}) \le \frac{C}{\lambda^{\frac{1}{1+\alpha}}} ||f||^{1/1+\alpha}_{\frac{1}{1+\alpha},1;\nu}.$$

Theorem 3.2. *Let $(X, \mathcal{F}, \nu)$, α, p and $\{\tau_t : t \in \mathbb{R}\}$ be as in Theorem 1.3.*

(i) If $\frac{1}{1+\alpha} < p < \infty$, then there exists a constant $C > 0$ such that for all $f \in L^p(d\nu)$

$$||R_{1+\alpha} f||_{p;\nu} \leq C||f||_{p;\nu}.$$

(ii) If $p = \frac{1}{1+\alpha}$, then there exists a constant $C > 0$ such that for all $f \in L_{\frac{1}{1+\alpha},1}(d\nu)$ and all $\lambda > 0$

$$\nu(\{x \in X : R_{1+\alpha} f(x) > \lambda\}) \leq \frac{C}{\lambda^{\frac{1}{1+\alpha}}} ||f||^{1/1+\alpha}_{\frac{1}{1+\alpha},1;\nu}.$$

In order to prove these theorems, we need two lemmas. The proof of the first one is very similar to the proof of the claim in the proof of Theorem 1 in [6]; therefore we omit it.

Lemma 3.3. *Let $(X, \mathcal{F}, \nu)$, α, p and $\{\tau_t : t \in \mathbb{R}\}$ be as in Theorem 1.1 or in Theorem 1.3. Then, there exists a measure μ equivalent to ν such that the flow $\{\tau_t : t \in \mathbb{R}\}$ preserves the measure μ.*

In what follows, the measure μ will be fixed and w will be the Radon-Nikodym derivate of ν with respect to μ. It is clear that $0 < w < \infty$ a.e.. Let us call w^x to the functions $w^x : \mathbb{R} \to \mathbb{R}$ such that $w^x(t) = w(\tau_t x)$.

Lemma 3.4. *Let $(X, \mathcal{F}, \nu)$ be a finite measure space, $-1 < \alpha \leq 0$ and $\frac{1}{1+\alpha} \leq p < \infty$. Let $\{\tau_t : t \in \mathbb{R}\}$ be a nonsingular measurable flow on X.*

(i) If (1.2) holds, then $w^x \in A^+_{p(1+\alpha)}$ for almost every $x \in X$ and with the same constant.

(ii) If (1.4) holds, then $w^x \in A_{p(1+\alpha)}$ for almost every $x \in X$ and with the same constant.

Proof. We only sketch the proof of (i), since the proof of (ii) is similar (notice that (ii) was already used in [6]). First, observe that if $p = \frac{1}{1+\alpha}$, then (i) holds by using that the flow preserves the measure μ given in Lemma 3.3.

Assume now that $q = p(1+\alpha) > 1$ and let q' be its conjugated exponent. Taking into account Lemma 3.3 we get, by (1.2), that

$$\int_X \left|A^+_{T,1} f(x)\right|^q w(x)\, d\mu(x) \leq C \int_X |f(x)|^q w(x)\, d\mu(x),$$

for all $T > 0$. Then by duality and calling $\sigma = w^{1-q'}$ we can write the above inequality as

$$\int_X \left|\left(A^+_{T,1}\right)^* f(x)\right|^{q'} \sigma(x)\, d\mu(x) \leq C \int_X |f(x)|^{q'} \sigma(x)\, d\mu(x),$$

where $\left(A^+_{T,1}\right)^* f(x) = \frac{1}{T}\int_{-T}^0 f(\tau_t x)\, dt$ is the adjoint operator of $A^+_{T,1}$ with respect to the measure μ. Let us define the following operators:

$$P_T g = \left[A^+_{T,1}\left(|g|^{q'} w^{-1/q}\right)\right]^{1/q'} w^{\frac{1}{qq'}} \quad \text{and} \quad Q_T g = \left[\left(A^+_{T,1}\right)^*\left(|g|^q \sigma^{-1/q'}\right)\right]^{1/q} \sigma^{\frac{1}{qq'}}.$$

P_T and Q_T are sublinear operators and $P_T,\ Q_T : L^{qq'}(d\mu) \to L^{qq'}(d\mu)$ with $||P_T||, ||Q_T|| \le C$, where C is the constant in (1.2). Clearly, the same holds for the operator $P_T + Q_T$ and $||P_T + Q_T|| \le 2C$. Now, given $f \in L^{qq'}(d\mu)$, $f > 0$, let us define

$$g_T = \sum_{i=0}^{\infty} \frac{(P_T + Q_T)^{(i)} f}{(4C)^i},$$

where $(P_T + Q_T)^{(i)}$ denotes the i-th iteration of $P_T + Q_T$. Clearly $g_T \in L^{qq'}(d\mu)$ and verifies that

$$P_T(g_T)(x) \le 4Cg_T(x) \quad \text{and} \quad Q_T(g_T)(x) \le 4Cg_T(x).$$

From these inequalities we can see that, if $v_T = g_T^{q'} w^{-1/q}$ and $u_T = g_T^{q} \sigma^{-1/q'}$, we get that

$$A^+_{T,1}(v_T) \le Cv_T \quad \text{and} \quad \left(A^+_{T,1}\right)^*(u_T) \le Cu_T. \tag{3.5}$$

The lemma follows since $w(x) = u_T(x)v_T^{1-q}(x)$ for almost every $x \in X$ and as a consequence we can prove that $w^x \in A^+_q$. In fact, let a, b and c be real numbers such that $a < b < c$. If $t \in (a, b)$, by the inequality for v_T in (3.5) with $T = c - a$ we get

$$\begin{aligned} \frac{1}{c-a}\int_b^c v_T(\tau_s x)\,ds &= \frac{1}{c-a}\int_{b-t}^{c-t} v_T(\tau_r\tau_t x)\,dr \\ &\le \frac{1}{c-a}\int_0^{c-a} v_T(\tau_r\tau_t x)\,dr \\ &\le Cv_T(\tau_t x). \end{aligned}$$

In the same way, by using the inequality for u_T in (3.5) with $T = c - a$ and for $t \in (c, d)$, we get that

$$\frac{1}{c-a}\int_a^b u_T(\tau_s x)\,ds \le Cu_T(\tau_t x).$$

Then, from the last inequalities

$$\int_a^b u_T(\tau_t x)v_T^{1-q}(\tau_t x)\,dt \left(\int_b^c u_T^{1-q'}(\tau_t x)v_T(\tau_t x)\,dt\right)^{q-1} \le C(c-a)^q.$$

Proof of Theorem 3.1. We only prove (ii) since (i) follows in a similar way. Assume $\alpha < 0$. As we observe in Remark 2.3, in order to prove (ii) we only need to consider characteristic functions, i.e., we need to prove that

$$\nu(\{x \in X : S^+_{1+\alpha}(\chi_E)(x) > \lambda\}) \le \frac{C}{\lambda^{\frac{1}{1+\alpha}}}\int_X \chi_E(x)\,d\nu(x),$$

for all $\lambda > 0$ and all measurable sets E. We shall use a transference argument. For fixed $L > 0$ we define

$$S^+_{1+\alpha,L}f(x) = \sup_{0<T<L} \frac{1}{T^{1+\alpha}}\int_0^T |f(\tau_t x)|(T-t)^\alpha\,dt.$$

Then, for all $N > 0$ we have

$$\begin{aligned}\nu(\{x \in X : S^+_{1+\alpha}(\chi_E)(x) > \lambda\}) &= \frac{1}{N}\int_0^N \int_X \chi_{\{x:S^+_{1+\alpha}(\chi_E)(x)>\lambda\}}(\tau_t x) w(\tau_t x)\, d\mu(x)\, dt \\ &= \frac{1}{N}\int_0^N \int_{\{x\in X:S^+_{1+\alpha}(\chi_E)(\tau_t x)>\lambda\}} w(\tau_t x)\, d\mu(x)\, dt.\end{aligned}$$

Since $S^+_{1+\alpha}(\chi_E)(\tau_t x) \le M^+_{1+\alpha}\left(\chi^x_E \chi_{(0,N+L)}\right)(t)$, where $\chi^x_E(t) = \chi_E(\tau_t x)$ and w^x satisfies A^+_1 for almost all x with the same constant (Lemma 3.4 (ii)), then Theorem D (ii) implies that

$$\begin{aligned}\nu(\{x \in X : S^+_{1+\alpha}(\chi_E)(x) > \lambda\}) &\le \frac{1}{N}\int_X \int_{\{t:M^+_{1+\alpha}\left(\chi^x_E\chi_{(0,N+L)}\right)(t)>\lambda\}} w^x(t)\, dt\, d\mu \\ &\le \frac{C}{N\lambda^{\frac{1}{1+\alpha}}}\int_X \int_0^{N+L} \chi_E(\tau_t x) w(\tau_t x)\, dt\, d\mu \\ &= \frac{C(N+L)}{N\lambda^{\frac{1}{1+\alpha}}}\int_X \chi_E(x)\, d\nu(x)\end{aligned}$$

because the flow preserves the measure μ. Letting $N \to \infty$ and then $L \to \infty$ we finish the proof for $-1 < \alpha < 0$. The case $\alpha = 0$ is proved in the same way but using general functions $f \in L^1(d\mu)$.

Proof of Theorem 3.2. The proof of Theorem 3.2 is completely similar to the proof of Theorem 3.1. We only need to notice that the operator $R_{1+\alpha}$ is pointwise equivalent to the sum of the following two maximal operators:

$$R^1_{1+\alpha} f(x) = \sup_{T>0} \frac{1}{(2T)^{1+\alpha}} \int_{-T}^{T} |f(\tau_t x)|(T - |t|)^\alpha\, dt$$

and

$$R^2_{1+\alpha} f(x) = \sup_{T>0} \frac{1}{(2T)^{1+\alpha}} \int_{T<|t|<2T} |f(\tau_t x)|(|t| - T)^\alpha\, dt.$$

Then, when we apply the transference arguments we shall need to use the results of Theorem E for the operators $M_{1+\alpha}$ and $N_{1+\alpha}$.

Now, we establish the boundedness of the ergodic maximal operator $H^*_\alpha = \sup_{\varepsilon>0} |H_{\varepsilon,\alpha}|$. First, we easily see that the ergodic truncations $H_{\varepsilon,\alpha} f$ are well defined. In fact, by Theorem 3.2, we get that

$$\int_{\varepsilon<|t|\le 1} \frac{|f(\tau_t x)|}{|t|}\left(1 - \frac{\varepsilon}{|t|}\right)^\alpha dt + \int_{1<|t|\le 1/\varepsilon} \frac{|f(\tau_t x)|}{|t|}(1 - \varepsilon|t|)^\alpha\, dt \le C_\varepsilon\, R_{1+\alpha}(f)(x) < \infty,$$

for almost every x and $f \in L^p(d\nu)$ if $\frac{1}{1+\alpha} < p < \infty$ or $f \in L_{\frac{1}{1+\alpha},1}(d\nu)$ if $p = \frac{1}{1+\alpha}$. In order to prove the boundedness of the operator H^*_α we start proving the following pointwise estimate.

Lemma 3.6. *Let $(X, \mathcal{F}, \nu)$ be a finite measure space, $-1 < \alpha \leq 0$ and let $\{\tau_t : t \in \mathbb{R}\}$ be a nonsingular measurable flow on X. Then, there exists a constant $C > 0$ such that*

$$H_\alpha^* f(x) \leq C\left[R_{1+\alpha} f(x) + H_0^* f(x)\right].$$

Proof. First, we write

$$\begin{aligned} H_{\varepsilon,\alpha} f(x) = & \int_{\varepsilon<|t|\leq 2\varepsilon} \frac{f(\tau_t x)}{t}\left(1-\frac{\varepsilon}{|t|}\right)^\alpha dt + \int_{2\varepsilon<|t|\leq 1} \frac{f(\tau_t x)}{t}\left[\left(1-\frac{\varepsilon}{|t|}\right)^\alpha - 1\right] dt \\ & + \int_{2\varepsilon<|t|<1/2\varepsilon} \frac{f(\tau_t x)}{t}\, dt + \int_{1<|t|<1/2\varepsilon} \frac{f(\tau_t x)}{t}\left[(1-\varepsilon|t|)^\alpha - 1\right] dt \\ & + \int_{1/2\varepsilon\leq|t|\leq 1/\varepsilon} \frac{f(\tau_t x)}{t}(1-\varepsilon|t|)^\alpha\, dt = I + II + III + IV + V. \end{aligned}$$

Clearly, $|III| \leq H_0^* f(x)$. We can easily see also that $|I|, |V| \leq C R_{1+\alpha} f(x)$. On the other hand, by the mean value Theorem and decomposing the integral in II as the sum of integrals over the sets $\{t : 2^k\varepsilon < |t| \leq 2^{k+1}\varepsilon\}$, we can see that $|II|$ and $|IV|$ are bounded by a constant times the usual ergodic maximal operator $M_0 f(x)$. Then the lemma follows since for $-1 < \alpha \leq 0$, $M_0 f(x) \leq R_{1+\alpha} f(x)$.

Now, from the above lemma, Theorem 3.2 and Theorem 1 in [6], we obtain the following result for the operator H_α^*.

Theorem 3.7. *Let $(X, \mathcal{F}, \nu)$, α, p and $\{\tau_t : t \in \mathbb{R}\}$ be as in Theorem 1.3.*

(i) If $\frac{1}{1+\alpha} < p < \infty$, then there exists a constant $C > 0$ such that for all $f \in L^p(d\nu)$

$$||H_\alpha^* f||_{p;\nu} \leq C||f||_{p;\nu}.$$

(ii) If $p = \frac{1}{1+\alpha}$, then there exists a constant $C > 0$ such that for all $f \in L_{\frac{1}{1+\alpha},1}(d\nu)$ and all $\lambda > 0$

$$\nu(\{x \in X : H_\alpha^* f(x) > \lambda\}) \leq \frac{C}{\lambda^{\frac{1}{1+\alpha}}}||f||^{1/1+\alpha}_{\frac{1}{1+\alpha},1;\nu}.$$

4. Proofs of Theorems 1.1 and 1.3

From Theorem B and Theorem 3.1 we can easily prove Theorem 1.1.

Proof of Theorem 1.1. We only prove (i) since the proof if (ii) is similar. By Theorem 3.1, the Banach Principle and the dominated convergence Theorem it will suffice to prove the a.e. convergence of the averages $A^+_{1+\alpha} f$ for f in a dense subset of $L^p(d\nu)$. Using Theorem B we have the a.e. convergence of $A^+_{1+\alpha} f$ for $f \in L^p(d\nu) \cap L_{\frac{1}{1+\alpha},1}(d\mu)$ which is a dense subset of $f \in L^p(d\nu)$. Then, the theorem follows.

Proof of Theorem 1.3. As in the proof of Theorem 1.1 we only prove (i) and we only have to show that the a.e. convergence holds for the functions in a dense subset of $L^p(d\nu)$.

Let us fix β and q such that $p < \frac{1}{1+\beta} < q$ and let μ be the measure given in Lemma 3.3. On one hand, the set $D = L^p(d\nu) \cap L^q(d\mu)$ is a dense subset of $L^p(d\nu)$.

On the other hand, for all $f \in D$ and since μ is preserved by the flow, we have the following results:

(a) By the classical result by Cotlar [3] (see also [11]) or by Theorem C, there exists the limit $\lim_{\varepsilon\to 0} H_{\varepsilon,0} f(x) = Hf(x)$ for almost every $x \in X$.

(b) By Theorem 3.7, $H^*_\beta f$ is a.e. finite, because $q > \frac{1}{1+\beta}$ and the ergodic averages $A_{T,1}$ are uniformly bounded on $L^{q(1+\beta)}(d\mu)$.

In what follows we will prove that (a) and (b) imply, for all $f \in D$, the a.e. existence of the limit $\lim_{\varepsilon\to 0} H_{\varepsilon,\alpha} f(x)$ and that $\lim_{\varepsilon\to 0} H_{\varepsilon,\alpha} f(x) = Hf(x)$. The proof is an adaptation of Lemma 2.27 in [14].

For fixed $f \in D$, let $x \in X$ such that there exists the limit $\lim_{\varepsilon\to 0} H_{\varepsilon,0} f(x) = Hf(x)$ and $H^*_\beta f(x)$ is finite. We may assume without loss of generality that $Hf(x) = 0$. Applying the formula

$$(x-u)^{\alpha+\delta} = C \int_u^x (t-u)^\alpha (x-t)^{\delta-1}\, dt, \qquad \delta > 0, \tag{4.1}$$

with $\delta = \alpha - \beta$, where C depends only on α and δ (in fact, $C = \frac{\Gamma(\alpha+\delta+1)}{\Gamma(\alpha+1)\Gamma(\delta)}$ where Γ is the Gamma function), we obtain that

$$H_{\varepsilon,\alpha} f(x) = C\, \varepsilon^\alpha \int_1^{1/\varepsilon} (1/\varepsilon - t)^{\alpha-\beta-1}\, t^\beta\, H_{1/t,\beta} f(x)\, dt. \tag{4.2}$$

Given $\eta > 0$, let us fix θ with $1/2 < \theta < 1$ and $(1-\theta)^{\alpha-\beta} < \eta$. Then,

$$\begin{aligned} H_{\varepsilon,\alpha} f(x) &= C\, \varepsilon^\alpha \int_1^{\theta/\varepsilon} (1/\varepsilon - t)^{\alpha-\beta-1}\, t^\beta\, H_{1/t,\beta} f(x)\, dt \\ &\quad + C\, \varepsilon^\alpha \int_{\theta/\varepsilon}^{1/\varepsilon} (1/\varepsilon - t)^{\alpha-\beta-1}\, t^\beta\, H_{1/t,\beta} f(x)\, dt = I + II. \end{aligned}$$

First, we estimate II and we obtain that

$$|II| \le C\, \varepsilon^\alpha\, H^*_\beta f(x)\, (\theta/\varepsilon)^\beta\, (1/\varepsilon - \theta/\varepsilon)^{\alpha-\beta} \le C\, H^*_\beta f(x)\, \eta.$$

To estimate I we integrate by parts and we use (4.2) with $\alpha = \beta + 1$. Then we obtain

$$\begin{aligned} I &= C\, \varepsilon^\alpha\, (1/\varepsilon - \theta/\varepsilon)^{\alpha-\beta-1} \int_1^{\theta/\varepsilon} s^\beta H_{1/s,\beta} f(x)\, ds \\ &\quad + C\, \varepsilon^\alpha \int_1^{\theta/\varepsilon} (\alpha-\beta-1)(1/\varepsilon - t)^{\alpha-\beta-2} \int_1^t s^\beta H_{1/s,\beta} f(x)\, ds\, dt \\ &= C\, \varepsilon^\alpha \left(\frac{1-\theta}{\varepsilon}\right)^{\alpha-\beta-1} (\theta/\varepsilon)^{\beta+1}\, H_{\varepsilon/\theta,\beta+1} f(x) \\ &\quad + C\, \varepsilon^\alpha\, (\alpha-\beta-1) \int_1^{\theta/\varepsilon} (1/\varepsilon - t)^{\alpha-\beta-2} t^{\beta+1} H_{1/t,\beta+1} f(x)\, dt = III + IV. \end{aligned}$$

Now, we claim that

(c) $H^*_{\beta+\delta} f(x)$ is finite for all $\delta > 0$.

(d) There exists the limit $\lim_{\varepsilon\to 0} H_{\varepsilon,\beta+1} f(x) = Hf(x) = 0$.

The above claims follow from (4.1), (4.2), (a) and (b). Taking into account the claims (c) and (d) we obtain that

$$|III| \le C \ (1-\theta)^{\alpha-\beta-1}\, \theta^{\beta+1} |H_{\varepsilon/\theta,\beta+1} f(x)| < \eta,$$

for ε small enough.

On the other hand, since $\alpha - \beta - 2 \in (-2,-1)$ and $\beta > -1$, we have $(1/\varepsilon - t)^{\alpha-\beta-2} < (1/\varepsilon - \theta/\varepsilon)^{\alpha-\beta-2}$ and $t^{\beta+1} < (\theta/\varepsilon)^{\beta+1}$ for all $t \in (1, \theta/\varepsilon)$. Then,

$$|IV| \le C\ \varepsilon \int_1^{\theta/\varepsilon} |H_{1/t,\beta+1} f(x)|\, dt,$$

which tends to zero as ε goes to zero because $\lim_{t\to\infty} H_{1/t,\beta+1} f(x) = 0$ and $H^*_{\beta+1} f(x) < \infty$. Therefore we are done.

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