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Singular Integrals in the Cesàro Sense

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ABSTRACT. *The existence of the singular integral $\int K(x, y) f(y)\, dy$ associated to a Calderón–Zygmund kernel where the integral is understood in the principal value sense $Tf(x) = \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} K(x, y) f(y)\, dy$ has been well studied. In this paper we study the existence of the above integral in the Cesàro-α sense. More precisely, we study the existence of*

$$\lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} f(y) K(x, y) \left(1 - \frac{\epsilon}{|x-y|}\right)^{\alpha} dy \quad a.e.$$

for $-1 < \alpha < 0$ in the setting of weighted spaces.

1. Introduction

Let $K(x, y)$ be a Calderón–Zygmund kernel ($x, y \in \mathbb{R}^n$), defined as in [7] (see Section 2 in this paper). If K satisfies that

$$\text{there exists} \quad \lim_{\epsilon \to 0^+} \int_{\epsilon < |x-y| < 1} K(x, y)\, dy \quad \text{for almost every } x\,, \tag{1.1}$$

then there exists the integral $\int K(x, y) f(y)\, dy$ in the principal value sense, i.e., there exists the singular integral

$$Tf(x) = \lim_{\epsilon \to 0^+} T_\epsilon f(x) \quad \text{a.e.,} \quad \text{where} \quad T_\epsilon f(x) = \int_{|x-y|>\epsilon} K(x, y) f(y)\, dy\,,$$

for every $f \in L^p(\omega\, dx)$ if ω belongs to the A_p class of Muckenkoupt, $1 \le p < \infty$, i.e., if ω is a nonnegative measurable function and there exists a positive constant C such that for every ball B

$$\left(\int_B \omega\right) \left(\int_B \omega^{-\frac{1}{p-1}}\right)^{p-1} \le C|B|^p, \quad \text{when } 1 < p < \infty\,,$$

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where $|B|$ denotes the Lebesgue measure of B, and

$$\frac{1}{|B|}\int_B \omega \leq C\omega(x) \quad \text{a.e. } x \in B, \quad \text{when } p = 1\,.$$

In order to study the existence of the above limit, it is proved that the maximal operator

$$T^* f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|$$

is of strong type (p, p), $1 < p < \infty$, i.e.,

$$\int |T^* f|^p w \leq C \int |f|^p \omega, \quad 1 < p < \infty\,, \tag{1.2}$$

whenever $\omega \in A_p$, and it is of weak type (1,1), i.e.,

$$\int_{\{T^* f > \lambda\}} \omega \leq \frac{C}{\lambda} \int |f|\,\omega, \quad \lambda > 0\,, \tag{1.3}$$

if $\omega \in A_1$ (see [7, 2, 6], and [1] for these results). These inequalities can be proved controlling $T^* f$ by the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{R>0} \frac{1}{|B(x,R)|} \int_{B(x,R)} |f|\,,$$

where $B(x, R)$ is the ball of center x and radius R. In fact, one uses that M verifies (1.2) and (1.3) under the same assumptions on ω (see [7, 2, 6], and [1]).

The aim of this paper is to study the existence of the singular integral $\int K(x, y) f(y)\,dy$ in the Cesàro-α (C_α) sense; that is to study the existence of the limit $\lim_{\epsilon\to 0^+} T_\epsilon f(x) = \lim_{R\to\infty} T_{1/R} f(x)$ in the C_α sense (see [3, Section 5.14 and Notes on Chapter V]). This means that in the case $\alpha > 0$ we want to study the limit

$$\lim_{R\to\infty} \frac{\alpha}{R^\alpha} \int_0^R (R - t)^{\alpha-1} T_{1/t} f(x)\,dt\,.$$

If $f \in L^p$, $1 \leq p < \infty$, we can interchange the integrals and the parameter R by $1/\epsilon$ obtaining that the above limit equals

$$\lim_{\epsilon\to 0^+} \int_{|x-y|>\epsilon} f(y) K(x, y) \left(1 - \frac{\epsilon}{|x-y|}\right)^\alpha dy\,.$$

In what follows and throughout the paper we shall write

$$K_{\epsilon,\alpha}(x, y) = K(x, y) \left(1 - \frac{\epsilon}{|x-y|}\right)^\alpha \chi_{\{|x-y|>\epsilon\}}(y)\,.$$

It turns out that if f is in the Schwartz class, then the integrals

$$T_{\epsilon,\alpha} f(x) = \int f(y) K_{\epsilon,\alpha}(x, y)\,dy$$

make sense not only for $\alpha > 0$ but also for $\alpha > -1$. In this way, we reach the goal of this paper, i.e., to determine spaces of functions for which the limit

$$\lim_{\epsilon\to 0^+} T_{\epsilon,\alpha} f(x) = \lim_{\epsilon\to 0^+} \int_{|x-y|>\epsilon} f(y) K(x, y) \left(1 - \frac{\epsilon}{|x-y|}\right)^\alpha dy$$

exists for almost every x. If that limit exists a.e., we shall say that the singular integral $\int K(x, y) f(y)\,dy$ exists a.e. in the C_α sense.

It can be proved that the existence in the C_0 sense (i.e., in the principal value sense) implies the existence in the C_α sense for $\alpha > 0$. Since the case $\alpha = 0$ has been well studied, we shall restrict ourselves to the case $\alpha < 0$, although the statements of the theorems hold also for $\alpha = 0$.

Throughout the paper the letter C means a positive constant not necessarily the same at each occurrence and if $1 < p < \infty$ then p' denotes its conjugated exponent, i.e., $1/p + 1/p' = 1$. If $E \subset \mathbb{R}^n$ is a measurable set and g is a nonnegative measurable function, then $|E|$ and $g(E)$ stand for the Lebesgue measure of E and $\int_E g(x)\,dx$, respectively. Finally, if ω is a nonnegative measurable function we shall consider the Lorentz spaces

$$L_{p,1}(\omega dx) = \left\{ f : \|f\|_{p,1;\omega dx} = \int_0^\infty [\omega(\{x : |f(x)| > t\})]^{1/p}\,dt < \infty \right\}$$

and

$$L_{p,\infty}(\omega dx) = \left\{ f : \|f\|_{p,\infty;\omega dx} = \sup_{t>0} t[\omega(\{x : |f(x)| > t\})]^{1/p} < \infty \right\} .$$

If $\omega = 1$, we shall omit $\omega\,dx$.

2. Statements of the Results

In order to state the results, we begin establishing that a Calderón–Zygmund kernel (see [7, p. 293, 305–306]) is a function K defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$, where $\Delta = \{(x, y) : x = y\}$ is the diagonal, such that there exist constants $C > 0$ and γ, $0 < \gamma \le 1$, so that

$$|K(x, y)| \le C|x - y|^{-n} , \tag{2.1}$$

$$\left|K(x, y) - K(x', y)\right| \le C\frac{|x - x'|^\gamma}{|x - y|^{n+\gamma}}, \quad \text{if } \; 2|x - x'| \le |x - y| , \tag{2.2}$$

$$\left|K(x, y) - K(x, y')\right| \le C\frac{|y - y'|^\gamma}{|x - y|^{n+\gamma}}, \quad \text{if } \; 2|y - y'| \le |x - y| , \tag{2.3}$$

$$\int_{|x-x_0|<N} \left|I_{\epsilon,N}(x)\right|^2 dx \le CN^n \quad \text{and} \quad \int_{|x-x_0|<N} \left|I^*_{\epsilon,N}(x)\right|^2 dx \le CN^n , \tag{2.4}$$

for all ϵ, N and x_0, where

$$I_{\epsilon,N}(x) = \int_{\epsilon<|x-y|<N} K(x, y)\,dy \quad , \quad I^*_{\epsilon,N}(x) = \int_{\epsilon<|x-y|<N} K^*(x, y)\,dy ,$$

and $K^*(x, y) = \bar{K}(y, x)$ is the adjoint kernel of K.

To prove the almost everywhere existence of the limit $\lim_{\epsilon\to0^+} T_{\epsilon,\alpha} f(x)$ we study the operator $T^*_\alpha f = \sup_{\epsilon>0} |T_{\epsilon,\alpha} f|$ which is controlled by $T^* = T^*_0$ and the maximal operator

$$M_\alpha f(x) = \sup_{\epsilon>0} \frac{1}{\epsilon^{n+\alpha}} \int_{\epsilon<|x-y|\le2\epsilon} |f(y)|(|x - y| - \epsilon)^\alpha\,dy .$$

More precisely, we have the following proposition.

Proposition 1.

Let $-1 < \alpha < 0$ and let K be a Calderón–Zygmund kernel. If f is a measurable function such that $T_{\epsilon,\alpha}f(x)$ is defined for every $\epsilon > 0$, then there exists $C > 0$ independent of f such that

$$T_\alpha^* f(x) \le C\left[M_\alpha f(x) + T^* f(x)\right] .$$

As we pointed out in the introduction, the strong type (p, p) inequalities, $1 < p < \infty$, and the weak type (1,1) inequality for T^* with respect to ωdx hold if ω satisfies A_p and A_1, respectively. However, as far as we know the boundedness of M_α has been studied only for the Lebesgue measure ($\omega = 1$). It follows from [5, Theorem 1] that M_α is of restricted weak type $(\frac{1}{1+\alpha}, \frac{1}{1+\alpha})$ and, consequently, it is bounded in $L^p(dx)$ if $p > \frac{1}{1+\alpha}$. Since we shall work with measures $\omega\, dx$, we have studied the boundedness of M_α in weighted spaces, obtaining the following theorem for Muckenhoupt weights.

Theorem 1.

Let $-1 < \alpha < 0$, and assume that ω is a nonnegative measurable function.

(i) If $\omega \in A_1$, then there exists a constant C such that

$$\lambda\,[\omega(\{x : M_\alpha f(x) > \lambda\})]^{1+\alpha} \le C\|f\|_{\frac{1}{1+\alpha},1;\omega dx} ,$$

for all $\lambda > 0$ and all $f \in L_{\frac{1}{1+\alpha},1}(\omega dx)$.

(ii) If $p(1+\alpha) > 1$ and $\omega \in A_{p(1+\alpha)}$, then there exists a constant C such that

$$\int_{\mathbb{R}^n} |M_\alpha f|^p\, \omega \le C \int_{\mathbb{R}^n} |f|^p \omega \quad \textit{for all } f \in L^p(\omega dx) .$$

Remark. It is worth noting that M_α is not of weak type $(\frac{1}{1+\alpha}, \frac{1}{1+\alpha})$ for $\alpha < 0$ and $\omega = 1$. If $f(y) = |y|^{-1-\alpha}|\log y|^\gamma \chi_{(0,1/2)}(y)$, with $-1 \le \gamma < -1-\alpha$, we see that $f \in L^{\frac{1}{1+\alpha}}(dx)$ but $M_\alpha f(x) = \infty$ for all $x < 0$.

With this theorem and Proposition 1 we shall be ready to prove the main result of the paper.

Theorem 2.

Let $-1 < \alpha < 0$ and let K be a Calderón–Zygmund kernel. Assume that ω is a nonnegative measurable function.

(i) If $\omega \in A_1$, then there exists C such that

$$\lambda\left[\omega\left(\{x : T^* f(x) > \lambda\}\right)\right]^{1+\alpha} \le C\|f\|_{\frac{1}{1+\alpha},1;\omega dx} ,$$

for all $\lambda > 0$ and all $f \in L_{\frac{1}{1+\alpha},1}(\omega dx)$.

(ii) If $p(1+\alpha) > 1$ and $\omega \in A_{p(1+\alpha)}$, then there exists C such that

$$\int_{\mathbb{R}^n} |T^* f|^p \omega \le C \int_{\mathbb{R}^n} |f|^p \omega \quad \textit{for all } f \in L^p(\omega dx) .$$

(iii) If K satisfies that

$$\textit{there exists} \quad \lim_{\epsilon\to 0^+} \int_{\epsilon<|x-y|<1} K(x,y)\left(1 - \frac{\epsilon}{|x-y|}\right)^\alpha dy \quad \textit{a.e.} \tag{2.5}$$

then the singular integral exists a.e. in the C_α sense if $f \in L^p(\omega dx)$ with $\omega \in A_{p(1+\alpha)}$ and $p(1+\alpha) > 1$ or if $f \in L_{\frac{1}{1+\alpha},1}(\omega dx)$ with $\omega \in A_1$.

Remark. Observe that (2.5) is the natural substitute of (1.1).

The rest of the paper is organized as follows. We prove Theorem 1 in Section 3 while the proofs of Proposition 1 and Theorem 2 are in Section 4. We finish the paper providing examples of Calderón–Zygmund kernels satisfying (2.5).

3. Proof of Theorem 1

Proof of Theorem 1(i). Applying Theorem 3.13 in [8, p. 195], we get that it is enough to prove the inequality for characteristic functions. Therefore, we shall prove that

$$\lambda^{\frac{1}{1+\alpha}} \omega(\{x : M_\alpha \chi_E(x) > \lambda\}) \le C\omega(E)\,, \tag{3.1}$$

for all $\lambda > 0$ and all measurable set E. Inequality (3.1) is an easy consequence of the following lemma together with the weak type (1,1) inequality with respect to $\omega(x)\,dx$ of the maximal operator

$$M_\omega g(x) = \sup_{R>0} \frac{1}{\omega(B(x,R))} \int_{B(x,R)} |g|\,\omega\,.$$

Lemma 1.

Let $-1 < \alpha < 0$ and $\omega \in A_1$. Then there exists C such that $M_\alpha \chi_E \le C\,(M_\omega \chi_E)^{1+\alpha}$ for all measurable set E.

Proof of Lemma 1. It suffices to prove that there exists C such that

$$R^{\alpha(n-1)} \left(\frac{1}{|B_{2R}|}\int_{B_{2R}} \omega\right)^{1+\alpha} \int_{C_R} \chi_E(y)(|x-y|-R)^\alpha\,dy \le C\left(\int_{E\cap B_{2R}} \omega\right)^{1+\alpha}, \tag{3.2}$$

where $C_R = \{y : R < |x-y| \le 2R\}$ and $B_{2R} = B(x, 2R)$. In order to prove (3.2) we apply A_1 and the Hölder inequality for the Lorentz spaces [4]: $||fg||_1 \le C||f||_{\frac{1}{1+\alpha},1}||g||_{\frac{1}{-\alpha},\infty}$. Then the left-hand side of (3.2) is dominated by

$$CR^{\alpha(n-1)} \left(\operatorname{ess\,inf}_{B_{2R}} \omega\right)^{1+\alpha} \left\|\chi_{E\cap B_{2R}}\right\|_{\frac{1}{1+\alpha},1} ||g||_{\frac{1}{-\alpha},\infty} \le CR^{\alpha(n-1)} \left(\omega(E\cap B_{2R})\right)^{1+\alpha} ||g||_{\frac{1}{-\alpha},\infty}\,,$$

where $g(y) = (|x-y|-R)^\alpha \chi_{C_R}$. Therefore, we only need to prove that $||g||_{\frac{1}{-\alpha},\infty} \le CR^{-\alpha(n-1)}$. To prove this last inequality, we write

$$\begin{aligned} ||g||_{\frac{1}{-\alpha},\infty} &= \sup_{t>0} t\,\left|\left\{y \in C_R : (|x-y|-R)^\alpha > t\right\}\right|^{-\alpha} \\ &\le \sup_{0<t<R^\alpha} t\,|\{\dots\}|^{-\alpha} + \sup_{t\ge R^\alpha} t\,|\{\dots\}|^{-\alpha} = I + II\,. \end{aligned}$$

Since $\{y \in C_R : (|x-y|-R)^\alpha > t\} \subset C_R$ we get that $I \le CR^{-\alpha(n-1)}$. On the other hand, if we call $s = t^{1/\alpha}R^{-1}$, we have that

$$II \le CR^{-\alpha(n-1)} \sup_{0<s<1} \left[\frac{(1+s)^n - 1}{s}\right]^{-\alpha} \le CR^{-\alpha(n-1)}\,. \qquad \square$$

Proof of Theorem 1(ii). We shall need the following lemma.

Lemma 2.

Let $-1 < \alpha < 0$, $p > \frac{1}{1+\alpha}$ and $\omega \in A_{p(1+\alpha)}$. Then, there exists a constant C such that

$$M_\alpha f(x) \leq C\left[M_\omega\left(|f|^p\right)\right]^{1/p}(x).$$

for all measurable function f.

We postpone the proof of Lemma 2 and continue with the proof of (ii).

Let $\omega \in A_{p(1+\alpha)}$. Then, there exists r, $1 < r < p(1+\alpha)$, such that $\omega \in A_r$ (see [2], for instance). By Lemma 2 we have that $M_\alpha f(x) \leq C(M_\omega(|f|^s)(x))^{1/s}$ with $s = \frac{r}{1+\alpha}$. Then, since M_ω is of weak type (1,1) with respect to $\omega(x)dx$, we get that M_α is of weak type (s, s) with respect to $\omega(x)dx$. By interpolation, (ii) follows. □

Proof of Lemma 2. Let C_R and B_{2R} be as in the proof of Lemma 1. Assume that $\omega \in A_{p(1+\alpha)}$. Then there exists r, $1 < r < p(1+\alpha)$, such that $\omega \in A_r$. Applying the Hölder inequality with exponent $\frac{p-1}{r-1}$ and using that $\omega \in A_r$ we have that

$$\begin{aligned}
\int_{C_R} \omega^{-\frac{1}{p-1}}(y)(|x-y|-R)^{\alpha p'}\,dy &\leq \left(\int_{B_{2R}} \omega^{-\frac{1}{r-1}}\right)^{\frac{r-1}{p-1}} \left(\int_{C_R} (|x-y|-R)^{\frac{\alpha p}{p-r}}\,dy\right)^{\frac{p-r}{p-1}} \\
&\leq C\left[R^{nr}\left(\int_{B_{2R}} \omega\right)^{-1}\right. \\
&\qquad \left.\left(\int_{C_R} (|x-y|-R)^{\frac{\alpha p}{p-r}}\,dy\right)^{p-r}\right]^{\frac{1}{p-1}} \\
&\leq C\left[R^{p(n+\alpha)}\left(\int_{B_{2R}} \omega(y)\,dy\right)^{-1}\right]^{\frac{1}{p-1}}.
\end{aligned} \tag{3.3}$$

Applying again the Hölder inequality and (3.3), we obtain that

$$\begin{aligned}
\int_{C_R} |f(y)|(|x-y|-R)^\alpha\,dy &= \int_{C_R} |f(y)|\omega^{1/p}(y)\omega^{-1/p}(y)(|x-y|-R)^\alpha\,dy \\
&\leq \left(\int_{B_{2R}} |f(y)|^p\omega(y)\,dy\right)^{1/p} \\
&\qquad \left(\int_{C_R} \omega^{-p'/p}(y)(|x-y|-R)^{\alpha p'}\,dy\right)^{1/p'} \\
&\leq CR^{n+\alpha}\left(\int_{B_{2R}} |f(y)|^p\omega(y)\,dy\right)^{1/p} \\
&\qquad \left(\int_{B_{2R}} \omega(y)\,dy\right)^{-1/p}.
\end{aligned}$$

Taking supremum over $R > 0$ we are done. □

4. Proofs of Proposition 1 and Theorem 2

Proof of Proposition 1. Let us write

$$\begin{aligned} T_{\epsilon,\alpha}f(x) &= \int_{|x-y|\le 2\epsilon} f(y)K_{\epsilon,\alpha}(x,y)\,dy + \int_{|x-y|>2\epsilon} f(y)K(x,y)\,dy \\ &+ \int_{|x-y|>2\epsilon} f(y)K(x,y)\left[\left(1-\frac{\epsilon}{|x-y|}\right)^{\alpha}-1\right]dy = I+II+III\,. \end{aligned}$$

Observe that $|II| = |T_{2\epsilon}f(x)| \le T^*f(x)$. On the other hand, since K is a Calderón–Zygmund kernel

$$\begin{aligned} |I| &\le C\int_{\epsilon<|x-y|\le 2\epsilon} |f(y)|\frac{(|x-y|-\epsilon)^{\alpha}}{|x-y|^{n+\alpha}}\,dy \\ &\le C\left(\frac{1}{\epsilon^{n+\alpha}}\int_{\epsilon<|x-y|\le 2\epsilon}|f(y)|(|x-y|-\epsilon)^{\alpha}\,dy\right) \le CM_{\alpha}f(x)\,. \end{aligned}$$

Now, we estimate $|III|$. By the mean value theorem we get that

$$\left|f(y)K(x,y)\left[\left(1-\frac{\epsilon}{|x-y|}\right)^{\alpha}-1\right]\right| \le C|f(y)|\xi^{\alpha-1}\frac{\epsilon}{|x-y|^{n+1}}$$

where $\xi \in [1-\frac{\epsilon}{|x-y|}, 1]$. Then

$$\begin{aligned} |III| &\le C\int_{|x-y|>2\epsilon}|f(y)|\left(1-\frac{\epsilon}{|x-y|}\right)^{\alpha-1}\frac{\epsilon}{|x-y|^{n+1}}\,dy \\ &\le C\epsilon\sum_{k=1}^{\infty}\int_{2^k\epsilon<|x-y|\le 2^{k+1}\epsilon}|f(y)|\frac{(|x-y|-\epsilon)^{\alpha-1}}{|x-y|^{n+\alpha}}\,dy \\ &\le C\epsilon\sum_{k=1}^{\infty}\frac{\left[(2^k-1)\epsilon\right]^{\alpha-1}}{\left(2^k\epsilon\right)^{n+\alpha}}\int_{|x-y|\le 2^{k+1}\epsilon}|f(y)|\,dy \le CMf(x)\,, \end{aligned}$$

where M is the Hardy–Littlewood maximal function. Since $Mf(x) \le CM_{\alpha}f(x)$ we are done. □

Proof of Theorem 2. We claim that the truncations $T_{\epsilon,\alpha}f(x)$ are well defined for the functions considered in (i) and (ii), i.e., the functions $y \to f(y)K_{\epsilon,\alpha}(x,y) \in L^1(dy)$ for all x.

Proof of the Claim. Assume that $f \in L_{\frac{1}{1+\alpha},1}(\omega dx)$ and $\omega \in A_1$. We write

$$\int_{|x-y|>\epsilon}\left|f(y)K_{\epsilon,\alpha}(x,y)\right|\,dy = \int_{\epsilon<|x-y|\le 2\epsilon}\cdots\,dy + \int_{|x-y|>2\epsilon}\cdots\,dy = I+II\,.$$

Observe that $|x-y| > 2\epsilon$ implies that $\left(1-\frac{\epsilon}{|x-y|}\right)^{\alpha} < (1/2)^{\alpha}$. Then, $II < \infty$ since $f \in L_{\frac{1}{1+\alpha},1}(\omega dx) \subset L^{\frac{1}{1+\alpha}}(\omega dx)$ and $\omega \in A_1 \subset A_{\frac{1}{1+\alpha}}$.

To prove that I is finite, we apply the growth condition on K and the Hölder inequality in Lorentz spaces. Then we obtain

$$\int_{\epsilon<|x-y|\le 2\epsilon}\left|f(y)K_{\epsilon,\alpha}(x,y)\right|\,dy \le \left\|f\chi_{B(x,2\epsilon)}\right\|_{\frac{1}{1+\alpha},1}\|g\|_{\frac{1}{-\alpha},\infty}\,,$$

where $g(z) = \frac{(|z|-\epsilon)^\alpha}{|z|^{n+\alpha}} \chi_{\{|z|>\epsilon\}}(z)$. Since $\omega \in A_1$ we have that $|\{y \in B(x,2\epsilon) : |f(y)| > t\}| = \int_{\{y\in B(x,2\epsilon):|f(y)|>t\}} \omega^{-1}(y)\omega(y)\, dy \le C \left(\frac{1}{|B(x,2\epsilon)|}\int_{B(x,2\epsilon)} \omega\right)^{-1} \omega(\{y : |f(y)| > t\})$. Then

$$\left\| f\chi_{B(x,2\epsilon)}\right\|_{\frac{1}{1+\alpha},1} \le C\left(\frac{1}{|B(x,2\epsilon)|}\int_{B(x,2\epsilon)} \omega\right)^{-(1+\alpha)} \|f\|_{\frac{1}{1+\alpha},1(\omega dx)} < \infty .$$

On the other hand, the function g is radial decreasing in $\{z : |z| > \epsilon\}$, $g(z) \to \infty$ as $|z| \to \epsilon^+$ and $g(z) \to 0$ as $|z| \to \infty$. Therefore, for $\epsilon > 0$ and $t > 0$ fixed, there exists z_t, with $|z_t| > \epsilon$, such that $g(z_t) = t$. Since $|\{z : g(z) > t\}| = |\{z : \epsilon < |z| < |z_t|\}| = C(|z_t|^n - \epsilon^n)$, we have

$$||g||_{\frac{1}{-\alpha},\infty} \le C \sup_{t>0} t\left(|z_t|^n - \epsilon^n\right)^{-\alpha} \le C \sup_{t>0} \epsilon^{-n(1+\alpha)} \left(\frac{1-(\epsilon/|z_t|)^n}{1-\epsilon/|z_t|}\right)^{-\alpha} < \infty .$$

With this inequality we have that $T_{\epsilon,\alpha} f$ is defined for $f \in L_{\frac{1}{1+\alpha},1}(\omega dx)$ with $\omega \in A_1$.

Now assume that $f \in L^p(\omega dx)$, $\omega \in A_p$, and $p > \frac{1}{1+\alpha}$. Let I and II be as above. As before, II is finite. By (2.1), the Hölder inequality, and (3.3),

$$\begin{aligned} I &= \int_{\epsilon<|x-y|\le 2\epsilon} |f(y)||K(x,y)|\left(1-\frac{\epsilon}{|x-y|}\right)^\alpha \omega^{1/p}(y)\omega^{-1/p}(y)\, dy \\ &\le C\left(\int |f|^p\omega\right)^{1/p} \epsilon^{-(n+\alpha)}\left(\int_{\epsilon<|x-y|\le 2\epsilon} \omega^{-\frac{1}{p-1}}(y)(|x-y|-\epsilon)^{\alpha p'}\, dy\right)^{1/p'} \\ &\le C\left(\int |f|^p\omega\right)^{1/p}\left(\int_{|x-y|\le 2\epsilon} \omega(y)\, dy\right)^{-1/p} < \infty . \end{aligned}$$

Therefore, $T_{\epsilon,\alpha} f$ is defined for $f \in L^p(\omega dx)$, $\omega \in A_p$, and $p > \frac{1}{1+\alpha}$. □

Once we have proved that the truncations are well defined, it is clear that (i) and (ii) in Theorem 2 follow from Proposition 1, Theorem 1, and the inequalities for T^* in weighted-L^p spaces [see (1.2) and (1.3)]. Finally, (iii) in Theorem 2 is a consequence of (i) and (ii) and the a.e. convergence of $T_{\epsilon,\alpha} f$ for all f belonging to the Schwartz class $\mathcal{S}$. In order to prove the a.e. convergence for $f \in \mathcal{S}$, we write

$$\begin{aligned} T_{\epsilon,\alpha} f(x) &= \int_{\epsilon<|x-y|\le 2\epsilon} [f(y)-f(x)]K_{\epsilon,\alpha}(x,y)\, dy \\ &\quad + \int_{2\epsilon<|x-y|<1} [f(y)-f(x)]K_{\epsilon,\alpha}(x,y)\, dy \\ &\quad + f(x)\int_{|x-y|<1} K_{\epsilon,\alpha}(x,y)\, dy + \int_{|x-y|\ge 1} f(y)K_{\epsilon,\alpha}(x,y)\, dy \\ &= I + II + III + IV . \end{aligned}$$

By (2.5) there exists the limit of III a.e.. By the mean value theorem, we easily see that

$$|I| \le C||\nabla f||_\infty \int_{\epsilon<|x-y|\le 2\epsilon} \frac{(|x-y|-\epsilon)^\alpha}{|x-y|^{n+\alpha-1}}\, dy \le C\epsilon||\nabla f||_\infty .$$

Therefore, $\lim_{\epsilon\to 0^+} I = 0$. On the other hand, if $2\epsilon < |x-y| < 1$, we have that $|(f(y)-f(x)) K_{\epsilon,\alpha}(x,y)| \le C||\nabla f||_\infty |x-y|^{-n+1}\chi_{\{|x-y|<1\}}(y)$ and then, by the dominated convergence theorem, we conclude that there exists $\lim_{\epsilon\to 0^+} II$. Finally, if $|x-y| \ge 1$ and ϵ is small then $|f(y)K_{\epsilon,\alpha}(x,y)| \le C|f(y)|$. Keeping in mind that $f \in \mathcal{S}$ we obtain, from the last inequality and the dominated convergence theorem, that there exists $\lim_{\epsilon\to 0^+} IV$. □

5. Examples of Kernels

In this section we give examples of Calderón–Zygmund kernels satisfying (2.5). We only verify (2.5), when necessary, because it is well known that they are Calderón–Zygmund kernels.

Example 1. Let $K(x, y) = \frac{1}{x-y}$ be the kernel of the Hilbert transform or, with more generality, consider, in dimension n, the kernel $K(x, y) = \frac{\Omega(x-y)}{|x-y|^n}$, where $\Omega \in C^1(\mathbb{R}^n \setminus \{0\})$ is a homogeneous function of degree zero and $\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0$. It is clear that K satisfies (2.5). $\square$

Example 2. Let $K(x, y) = \frac{A(x)-A(y)}{(x-y)^2}$ be the kernel associated with the first Calderón commutator in one dimension, where A is a Lipschitz function such that there exists $A'(x)$ for all $x \in \mathbb{R}$. Let $x \in [-R, R]$, for some $R > 0$. Then for any $\epsilon > 0$ we get that

$$\begin{aligned} &\int_{\epsilon<|x-y|<1} \frac{A(x)-A(y)}{(x-y)^2}\left(1-\frac{\epsilon}{|x-y|}\right)^{\alpha} dy \\ &= \int_{x-1}^{x-\epsilon} \frac{A(x)-A(y)}{(x-y)^2}\left(1-\frac{\epsilon}{x-y}\right)^{\alpha} dy \\ &+ \int_{x+\epsilon}^{x+1} \frac{A(x)-A(y)}{(y-x)^2}\left(1-\frac{\epsilon}{y-x}\right)^{\alpha} dy = I\,. \end{aligned}$$

Integrating by parts we obtain that

$$\begin{aligned} I &= \frac{(1-\epsilon)^{1+\alpha}}{\epsilon(1+\alpha)}[2A(x)-A(x-1)-A(x+1)] \\ &- \frac{1}{\epsilon(1+\alpha)}\left[\int_{x-1}^{x-\epsilon} A'(y)\left(1-\frac{\epsilon}{x-y}\right)^{1+\alpha} dy \right.\\ &\left. - \int_{x+\epsilon}^{x+1} A'(y)\left(1-\frac{\epsilon}{y-x}\right)^{1+\alpha} dy\right]. \end{aligned}$$

Applying the L'Hopital rule we obtain

$$\begin{aligned} \lim_{\epsilon\to 0^+} I &= \lim_{\epsilon\to 0^+} -(1-\epsilon)^{\alpha}[2A(x)-A(x-1)-A(x+1)] \\ &+ \lim_{\epsilon\to 0^+} \int_{\epsilon<|x-y|<1} \frac{A'(y)\chi_{[-R-1,R+1]}(y)}{x-y}\left(1-\frac{\epsilon}{|x-y|}\right)^{\alpha} dy \\ &= -2A(x)+A(x-1)+A(x+1)+\lim_{\epsilon\to 0^+} H_{\epsilon,\alpha}\left(A'\chi_{[-R-1,R+1]}\right)(x) \\ &- \int_{|x-y|\geq 1} \frac{A'(y)\chi_{[-R-1,R+1]}(y)}{x-y}\, dy\,, \end{aligned}$$

where $H_{\epsilon,\alpha}$ is the C_α truncation of the Hilbert Transform, i.e.,

$$H_{\epsilon,\alpha} f(x) = \int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\left(1-\frac{\epsilon}{|x-y|}\right)^{\alpha} dy\,.$$

Now, since $K(x, y) = \frac{1}{x-y}$ satisfies (2.5) (Example 1) and $A'(y)\chi_{[-R-1,R+1]}(y) \in L^p$, for all $p \geq 1$, applying Theorem 2 we get that there exists $\lim_{\epsilon\to 0^+} H_{\epsilon,\alpha}\left(A'\chi_{[-R-1,R+1]}\right)(x)$ for almost all $x \in [-R, R]$. Then it follows that the kernel $K(x, y) = \frac{A(x)-A(y)}{(x-y)^2}$ satisfies (2.5). $\square$

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