Letters in Mathematical Physics **53**: 157–165, 2000.

On Gauge Invariant Regularization of Fermion Currents*

R. E. GAMBOA SARAVÍ[1], M. A. MUSCHIETTI[2] and J. E. SOLOMIN[2]
[1]*Departamento de Física, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, Argentina. e-mail: quique@dartagnan.fisica.unlp.edu.ar*
[2]*Departamento de Matemática, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, Argentina*

(Received: 18 July 2000)

Abstract. We compare Schwinger and complex powers methods to construct regularized fermion currents. We show that, although both of them are gauge invariant, they are not always yield the same result.

Mathematics Subject Classifications (2000). 81T16, 35S05.

Key words. Quantum field theories, pseudodifferential operators.

A difficulty specific to quantum field theories is the occurrence of infinities and hence the necessity of regularizing and renormalizing the theory. Whenever a field theory possesses a classical symmetry—and hence a conserved current—it is desirable to have at hand regularization procedures preserving that symmetry**.

The calculation of vacuum expectation values of vector currents involves the evaluation of the Green function for the particle fields at the diagonal, so a regularization is required. In a classic paper, Julian Schwinger introduced a point-splitting method to regularize fermion currents maintaining gauge symmetry on the quantum level [1].

More recently, the so-called ζ-function method, based on complex powers of pseudodifferential operators [2], has proved to be a very valuable gauge invariant regularizing tool (see, for example [3]). Some time ago we used it to obtain fermion currents in two and three dimensional models [4].

It is the aim of this Letter to compare the results obtained by the above-mentioned methods.

Let M be a n-dimensional spin closed manifold endowed with a Riemannian metric tensor $g_{\mu\nu}$. For any covector a_μ defined on M, we adopt the usual convention $\not a = \gamma^\mu a_\mu$, where the Dirac matrices γ satisfy $\gamma^\mu(x)\gamma^\nu(x) + \gamma^\nu(x)\gamma^\mu(x) = 2\, g^{\mu\nu}(x)$. Let $\not D = i\not\nabla + \not A$ be a Euclidean Dirac operator coupled to a gauge field A_μ, where the

*Partially supported by CONICET, Argentina.

**As it is well known, it is not always possible to preserve all the classical symmetries present simultaneously and *anomalies* can arise.

covariant derivative ∇ is given by $\nabla_\mu = \partial_\mu - \Gamma_\mu$, with Γ_μ the spin connection associated with Levi-Civita's. The operator $\not{D}$ is elliptic and, since its principal symbol has only real eigenvalues, it fulfills the Agmon cone condition [2]. Thus, the complex powers $\not{D}^s$ can be constructed following Seeley [2]. For Res < 0 we can write

$$\not{D}^s := \frac{i}{2\pi}\int_\Gamma \lambda^s(\not{D}-\lambda)^{-1}\mathrm{d}\lambda, \tag{1}$$

where Γ is a contour enclosing the spectrum of $\not{D}$, and we define $\not{D}^s$ for Res $\geqslant 0$ by using $\not{D}^{s+1} = \not{D}^s \circ \not{D}$.

For each $s \in \mathbb{C}$, $\not{D}^s$ turns out to be a pseudodifferential operator of order s and so, if *Res* $< -n$, its Schwartz kernel $K_s(x, y)$ is a continuous function. The evaluation at the diagonal $x = y$ of this kernel, $K_s(x, x)$, admits a meromorphic extension to the whole complex s-plane $\mathbb{C}$, with at most simple poles at $s \in \mathbb{Z}^-$. This extension will be also denoted by $K_s(x, x)$.

Since $K_{-1}(x, y)$ coincides with the Green function for $x \neq y$, the finite part of $K_s(x, x)$ at $s = -1$ can be used to define gauge-invariant regularized fermion currents [4]:

$$J^\mu(x) := -\mathrm{tr}\left(\gamma^\mu(x) \operatorname*{FP}_{s=-1} K_s(x, x)\right). \tag{2}$$

Notice that this definition makes sense. In fact, owing to the density character of $K_s(x, x)$ (see, for instance, [5]) and the vectorial nature of the γ matrices, the right-hand side in (2) is a vector density.

In order to compare this regularizing procedure with Schwinger's, it is convenient to consider the kernels $K_s(x, x)$ within the framework developed within [5]. Since we are interested in studying the behaviour of these kernels for $s \to -1$, we shall carry out our analysis just for $-1 \leqslant$ Res < 0.

By considering the finite expansion (see, for instance, [6])

$$\sigma(\not{D}^s) = \sum_{\ell=0}^{N} c_{s-\ell}(x, \xi) + r_N(x, \xi, s), \tag{3}$$

with $N = n - 1$, of the symbol of the operator $\not{D}^s$, with $c_{s-\ell}(x, \xi)$ positively homogeneous of degree $s - \ell$ for $|\xi| \geqslant 1$, we can write, for $s \neq -1$ the Schwartz kernel of this operator as

$$K_s(x, y) = \sum_{\ell=0}^{N} H_{-n-s+\ell}\,(x, x - y) + R_N(x, x - y, s), \tag{4}$$

where $H_{-n-s+\ell}(x, u)$ is the Fourier transform in the variable ξ of $\widetilde{c}_{s-\ell}(x, \xi)$, the homogeneous extension of $c_{s-\ell}(x, \xi)$, evaluated at $u = x - y$ (i.e. $H_{-n-s+\ell}(x, u) = \frac{1}{(2\pi)^n}\int \widetilde{c}_{s-\ell}(x, \xi)\, \mathrm{e}^{i\xi.u}\, d\xi$), and consequently u-homogeneous of degree $-n - s + \ell$

and $R_N(x, u, s)$ is that of $r_N(x, \xi, s) - \sum_{\ell=0}^{N}(\tilde{c}_{s-\ell} - c_{s-\ell})(x, \xi)$. Note that $(\tilde{c}_{s-\ell} - c_{s-\ell})(x, \xi) \equiv 0$ for $|\xi| \geqslant 1$.

Now, for $u \neq 0$, simple poles can arise at $s = -1$ in H_{-n-s+N} and in $R_N(x, u, s)$ [5]. Since $K_s(x, x-u)$ is holomorphic in the variable s for $u \neq 0$, these poles cancel each other. In fact, they are due to the singularity of $\tilde{c}_{s-N}(x, \xi)$ at $\xi = 0$ and then

$$\operatorname*{res}_{s=-1} R_N(x, u, s) = - \operatorname*{res}_{s=-1} H_{-n-s+N}\,(x, u). \tag{5}$$

Thus, for $u \neq 0$, we have for $G(x, y)$, the Green function of $\not{D}$,

$$G(x, y) = \lim_{s \to -1} K_s(x, y) = \sum_{\ell=0}^{N} G_{-n+1+\ell}\,(x, u) + R_G(x, u), \tag{6}$$

with

$$G_{-n+1+\ell}\,(x, u) = \lim_{s \to -1} H_{-n-s+\ell}\,(x, u) \quad \text{for} \quad \ell < N,$$

$$G_{-n+1+N}\,(x, u) = \operatorname*{FP}_{s=-1} H_{-n-s+N}\,(x, u) \quad \text{and} \quad R_G(x, u) = \operatorname*{FP}_{s=-1} R_N(x, u, s).$$

It is worth noticing that a logarithmic term can arise in $\mathrm{FP}_{s=-1}\, H_{-n-s+N}\,(x, u)$.

Then, taking into account that, for $s \neq -1$ [5],

$$K_s(x, x) = R_N(x, 0, s), \tag{7}$$

we have

$$\operatorname*{FP}_{s=-1} K_s(x, x) = R_G(x, 0). \tag{8}$$

As we shall see below, this last expression furnishes the link between the two regularization methods.

On the other hand, the fermionic currents regularized according to Schwinger's prescription are given by [1]

$$J^\mu(x) = - \operatorname*{Sch\text{-}lim}_{y \to x} \operatorname{tr} \left(\gamma^\mu G(x, y)\, e^{i \int_x^y A.\mathrm{d}z}\right), \tag{9}$$

where

$$\int_x^y A.\mathrm{d}z = - \int_0^1 A_\mu(x - tu)\, u^\mu\, \mathrm{d}t. \tag{10}$$

The Schwinger limit, $\text{Sch-lim}_{y \to x}$, is defined for each term in the expansion in u-homogeneous functions (and logarithmic ones if they appear) of $\gamma^\mu G(x, y)\, e^{i(\int_x^y A.\mathrm{d}z)}$ in the following way: the usual limit when the latter exists, vanishes for negative degrees and for logarithmic terms, and coincides with the mean value at $|u| = 1$ for terms of zero degree. The exponential factor was introduced by Schwinger [1] in order to maintain gauge invariance.

From (2), (8) and (9) we see that both methods yield the same result for J^μ if and only if

$$\underset{y\to x}{\text{Sch-lim}}\,\mathrm{tr}\left(\gamma^\mu \sum_{\ell=0}^{N} G_{-n+1+\ell}\ (x,u)\ \mathrm{e}^{i\int_x^y A.\mathrm{d}z}\right) = 0 \tag{11}$$

since, being $R_G(x,u)$ continuous at $x = y$,

$$\begin{aligned}&\underset{y\to x}{\text{Sch-lim}}\,\mathrm{tr}\left(\gamma^\mu R_G(x,u)\ e^{i\int_x^y A.dz}\right)\\ &\quad= \underset{y\to x}{\text{Sch-lim}}\,\mathrm{tr}\,(\gamma^\mu R_G(x,u))\\ &\quad= \lim_{u\to 0}\mathrm{tr}\,(\gamma^\mu R_G(x,u)) = \mathrm{tr}\left(\gamma^\mu \underset{s=-1}{\mathrm{FP}}\ K_s(x,x)\right).\end{aligned} \tag{12}$$

Now, we shall see how this works in $n = 2$, 3 and 4. By computing the $G_{-n+1+\ell}\ (x,u)$'s we shall be able to establish when (11) holds and so, when both methods yield the same regularized currents.

It will be enough for our purposes to consider a flat coordinate patch. In Cartesian coordinates

$$\not{D} = \gamma^\mu D_\mu = \gamma^\mu(i\partial_\mu + A_\mu), \tag{13}$$

where the algebra of the γ-matrices is

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\ \delta^{\mu\nu}. \tag{14}$$

Its symbol, $\sigma(\not{D}; x, \xi)$, is

$$\sigma(\not{D}; x, \xi) = -\ \not{\xi} - \not{A}(x). \tag{15}$$

The symbol of the resolvent, $\sigma((\not{D} - \lambda)^{-1}; x, \xi)$, has an asymptotic expansion $\sum_\ell \widetilde{C}_{-1-\ell}(x, \xi, \lambda)$, where $\widetilde{C}_{-1-\ell}(x, \xi, \lambda)$ is homogeneous in ξ and λ of degree $-1 - \ell$ [2]. Then

$$(\not{D} - \lambda)^{-1}\ \varphi(x) \sim \frac{1}{(2\pi)^{n/2}} \int \sum_\ell \widetilde{C}_{-1-\ell}(x, \xi, \lambda)\ \mathrm{e}^{i\xi.x}\ \widehat{\varphi}(\xi)\ \mathrm{d}\xi. \tag{16}$$

Applying $\not{D} - \lambda$ to Equation (3) we get recursive equations for determining the $\widetilde{C}_{-1-\ell}(x, \xi, \lambda)$'s:

$$\begin{aligned}&-(\not{\xi} + \lambda)\ \widetilde{C}_{-1}(x, \xi, \lambda) = 1,\\ &\not{D}_x\widetilde{C}_{-1-\ell}(x, \xi, \lambda)\ -(\not{\xi} + \lambda)\ \widetilde{C}_{-1-\ell-1}(x, \xi, \lambda) = 0.\end{aligned} \tag{17}$$

Owing to the particular features of the Dirac operator, the standard symbolic cal-

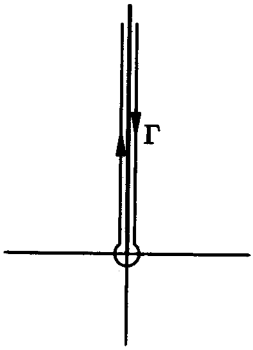

Figure 1. The Γ curve in the λ-plane.

culus [2] simplifies remarkably in our case. In fact, the solution of (17) can be written in a very concise form:

$$\widetilde{C}_{-1-\ell}(x,\xi,\lambda) = -\frac{(\not{\xi}-\lambda)}{\xi^2-\lambda^2}\left[\not{D}_x\,\frac{(\not{\xi}-\lambda)}{\xi^2-\lambda^2}\right]^{\ell}. \tag{18}$$

Now, from Equation (1),

$$\begin{aligned} &H_{-n-s+\ell}(x,u)\\ &=\frac{1}{(2\pi)^n}\int \widetilde{c}_{s-\ell}(x,\xi)\;\mathrm{e}^{i\xi.u}\;\mathrm{d}\lambda\;\mathrm{d}\xi\\ &=\frac{i}{(2\pi)^{n+1}}\int\int_{\Gamma}\widetilde{C}_{-1-\ell}(x,\xi,\lambda)\;\lambda^s\;\mathrm{e}^{i\xi.u}\;\mathrm{d}\lambda\;\mathrm{d}\xi, \end{aligned} \tag{19}$$

where the contour Γ can be chosen as shown in Figure 1. Therefore,

$$\begin{aligned} &H_{-n-s+\ell}(x,u)\\ &=\frac{-i}{(2\pi)^{n+1}}\int\int_{\Gamma}\frac{(\not{\xi}-\lambda)}{(\xi^2-\lambda^2)^{\ell+1}}\;[\not{D}_x\,(\not{\xi}-\lambda)]^{\ell}\;\lambda^s\;\mathrm{e}^{i\xi.u}\;\mathrm{d}\lambda\;\mathrm{d}\xi\\ &=\frac{-i}{(2\pi)^{n+1}}\int\int_{\Gamma}\frac{(-i\,\not{\partial}_u-\lambda)}{(\xi^2-\lambda^2)^{\ell+1}}\;[\not{D}_x\,(-i\,\not{\partial}_u-\lambda)]^{\ell}\;\lambda^s\;\mathrm{e}^{i\xi.u}\;\mathrm{d}\lambda\;\mathrm{d}\xi. \end{aligned} \tag{20}$$

Taking into account that, for any polynomial $P(\lambda)$,

$$\begin{aligned} &\frac{i}{2\pi}\int_{\Gamma}\frac{\lambda^s\;P(\lambda)}{(\xi^2-\lambda^2)^{\ell+1}}\;\mathrm{d}\lambda\\ &=\frac{i}{2\pi}\left\{\int_{\infty}^{0}\frac{(z\;e^{i\frac{\pi}{2}})^s\;P(iz)}{(\xi^2+z^2)^{\ell+1}}\;i\,\mathrm{d}z+\int_0^{\infty}\frac{(z\;\mathrm{e}^{-i\frac{3\pi}{2}})^s\;P(iz)}{(\xi^2+z^2)^{\ell+1}}\;i\,\mathrm{d}z\right\}\\ &=\frac{i}{\pi}\;\mathrm{e}^{-i\frac{\pi}{2}s}\;\sin(\pi s)\;P(-\partial_a)\left[\int_0^{\infty}\frac{z^s\;\mathrm{e}^{-iaz}}{(\xi^2+z^2)^{\ell+1}}\;\mathrm{d}z\right]_{a=0}, \end{aligned} \tag{21}$$

we can write

$$\begin{aligned} &H_{-n-s+\ell}(x,u) \\ &\quad = \frac{-i}{\pi}\ \mathrm{e}^{-\frac{i\pi}{2}s}\ \sin(\pi s)(-i\ \partial\!\!\!/_u + \partial_a)\ [\ D\!\!\!\!/_x\ (-i\ \partial\!\!\!/_u + \partial_a)]^{\ell}\times \\ &\qquad \times \sum_{k=0}^{\ell+1}\frac{(-ia)^k}{k!}\int_0^\infty z^{s+k}\frac{1}{(2\pi)^n}\int\frac{1}{(\xi^2+z^2)^{\ell+1}}\ \mathrm{e}^{i\xi.u}\ \mathrm{d}\xi\ \mathrm{d}z\Big|_{a=0}. \end{aligned} \tag{22}$$

Now, the integrals in (22) can be performed using the known identities

$$\frac{1}{(2\pi)^n}\int(\xi^2+z^2)^s\ \mathrm{e}^{i\xi.u}\ \mathrm{d}\xi = \frac{2^{1+s}}{(2\pi)^{\frac{n}{2}}}\ \frac{1}{\Gamma(-s)}\ \left(\frac{z}{u}\right)^{\frac{n}{2}+s}\cdot\mathbf{K}_{\frac{n}{2}+s}(zu), \tag{23}$$

where $\mathbf{K}_\mu$ is a Bessel function (see, for instance, [8]) and

$$\int_0^\infty z^\mu\ \mathbf{K}_\nu(zu)\ \mathrm{d}z = 2^{\mu-1}\ u^{-\mu-1}\ \Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right) \tag{24}$$

(see, for example, [7]).

Finally, we get the following expression for $H_{-n-s+\ell}(x,u)$:

$$\begin{aligned} &H_{-n-s+\ell}(x,u) \\ &\quad = \frac{-i\ 2^{s-2\ell-2}}{\pi^{\frac{n}{2}+1}\ \ell!}\mathrm{e}^{-\frac{i\pi}{2}s}\ \sin(\pi s)\times \\ &\qquad \times(-i\ \partial\!\!\!/_u + \partial_a)\ [\ D\!\!\!\!/_x\ (-i\ \partial\!\!\!/_u + \partial_a)]^{\ell}\sum_{k=0}^{\ell+1}\frac{(-ia)^k}{k!}\times \\ &\qquad \times\Gamma\left(\frac{1+s+k}{2}\right)\Gamma\left(\frac{s+k+n-1-2\ell}{2}\right)\ u^{-s-n+2\ell+1-k}\Big|_{a=0}. \end{aligned} \tag{25}$$

The first four $H_{-n-s+\ell}(x,u)$ terms, obtained from (25) after a straightforward but tedious computation involving γ-matrices's algebra and derivatives, are shown in Table I. There, as usual, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -i(D_\mu A_\nu - D_\nu A_\mu)$. It is worth noticing that the first terms of the exponential

$$\mathrm{e}^{-i\int_x^y A.\mathrm{d}z} = 1 + i(u.A) - \frac{(u.D)(u.A)}{2!} - i\frac{(u.D)(u.D)(u.A)}{3!} + \ldots \tag{26}$$

start to appear as an overall factor in the sum of the expansion (4) for $K_s(x,y)$.

Now, we shall compute the sum in expression (11) in order to see whether both methods coincide or not. Taking into account that

$$G_{-n+1+\ell}\ (x,u) = \lim_{s\to-1}\ H_{-n-s+\ell}\ (x,u) \quad \text{for } \ell < N$$

Table I. The first four $H_{-n-s+\ell}(x, u)$.

$$H_{-n-s}(x,u) = \frac{2^{s-1}}{\pi^{\frac{n}{2}+1}} \, e^{-i\frac{\pi}{2}s} \sin(\pi s) \times$$
$$\times \left[\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{n+s+1}{2}\right) u^{-n-s-1} \, \not{u} - \Gamma\left(\frac{2+s}{2}\right)\Gamma\left(\frac{n+s}{2}\right) u^{-n-s} \right]$$

$$H_{-n-s+1}(x,u) = \frac{2^{s-1}}{\pi^{\frac{n}{2}+1}} \, e^{-i\frac{\pi}{2}s} \sin(\pi s) \times$$
$$\times \left[\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{n+s+1}{2}\right) u^{-n-s-1} \, \not{u} - \Gamma\left(\frac{2+s}{2}\right)\Gamma\left(\frac{n+s}{2}\right) u^{-n-s} \right] i(u.A)$$

$$H_{-n-s+2}(x,u) = \frac{2^{s-1}}{\pi^{\frac{n}{2}+1}} \, e^{-i\frac{\pi}{2}s} \sin(\pi s) \times$$
$$\times \left\{ \left[\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{n+s+1}{2}\right) u^{-n-s-1} \, \not{u} - \Gamma\left(\frac{2+s}{2}\right)\Gamma\left(\frac{n+s}{2}\right) u^{-n-s} \right] \left(-\frac{(u.D)(u.A)}{2!} \right) + \right.$$
$$\left. + \frac{i}{8}\left[\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{n+s-1}{2}\right) u^{-n-s+1} \, u_\rho \gamma^\mu \gamma^\rho \gamma^\nu + \Gamma\left(\frac{2+s}{2}\right)\Gamma\left(\frac{n+s-2}{2}\right) u^{-n-s+2} \, \gamma^\mu \gamma^\nu \right] F_{\mu\nu} \right\}$$

$$H_{-n-s+3}(x,u) = \frac{2^{s-1}}{\pi^{\frac{n}{2}+1}} \, e^{-i\frac{\pi}{2}s} \sin(\pi s) \times$$
$$\times \left\{ \left[\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{n+s+1}{2}\right) u^{-n-s-1} \, \not{u} - \Gamma\left(\frac{2+s}{2}\right)\Gamma\left(\frac{n+s}{2}\right) u^{-n-s} \right] \left(-i\frac{(u.D)(u.D)(u.A)}{3!} \right) + \right.$$
$$+ \frac{i}{8}\left[\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{n+s-1}{2}\right) u^{-n-s+1} \, u_\rho \gamma^\mu \gamma^\rho \gamma^\nu + \Gamma\left(\frac{2+s}{2}\right)\Gamma\left(\frac{n+s-2}{2}\right) u^{-n-s+2} \, \gamma^\mu \gamma^\nu \right] F_{\mu\nu} \, i(u.A) +$$
$$+ \frac{1}{24}\left[\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{n+s-1}{2}\right) u^{-n-s+1} \left(-\tfrac{3}{2} u_\rho u^\sigma \gamma^\mu \gamma^\rho \gamma^\nu \partial_\sigma F_{\mu\nu} - u^\mu u_\rho \gamma^\rho \partial^\nu F_{\mu\nu} + u^\mu u^\nu \gamma^\rho \partial_\nu F_{\mu\rho} \right) + \right.$$
$$+ \Gamma\left(\frac{2+s}{2}\right)\Gamma\left(\frac{n+s-2}{2}\right) u^{-n-s+2} \left(-\tfrac{3}{2} u^\mu \gamma^\nu \gamma^\rho \partial_\mu F_{\nu\rho} + u^\mu \partial^\nu F_{\mu\nu} \right) +$$
$$\left. \left. + \Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{n+s-3}{2}\right) u^{-n-s+3} \, \gamma^\nu \partial^\mu F_{\mu\nu} \right] \right\}$$

and

$$G_{-n+1+N}(x,u) = \underset{s=-1}{\mathrm{FP}} \; H_{-n-s+N}(x,u),$$

from Table I, we get the following relations.

For $n = 2$, we have

$$\sum_{\ell=0}^{1} G_{-2+1+\ell}(x,u) \, e^{i\int_x^y A.\mathrm{d}z} = -\frac{i}{2\pi}\frac{\not{u}}{u^2}(1 + o(u^2)), \tag{27}$$

so it is clear that (11) holds in this case.

For $n = 3$, we get

$$\sum_{\ell=0}^{2} G_{-3+1+\ell}(x,u)\, e^{i\int_x^y A.\mathrm{d}z} = -\frac{i}{4\pi}\frac{\not{u}}{u^3}(1+\mathrm{o}(u^3)) + \frac{1}{16\pi}\left[\frac{u_\rho}{u}\gamma^\mu\gamma^\rho\gamma^\nu + \gamma^\mu\gamma^\nu\right]F_{\mu\nu}, \tag{28}$$

and so

$$\operatorname*{Sch-lim}_{y\to x} \mathrm{tr}\left(\gamma^\mu \sum_{\ell=0}^{2} G_{-3+1+\ell}(x,u)\, e^{i\int_x^y A.\mathrm{d}z}\right) = \frac{1}{16\pi}\mathrm{tr}[\gamma^\mu\gamma^\rho\gamma^\nu]F_{\rho\nu}, \tag{29}$$

which does or does not vanish depending on the γ's representation (it does not vanish if the 2×2 Pauli matrices are chosen).

Finally, we consider $n = 4$. In this case, a pole is present in $H_{-4-s+3}(x, u)$ at $s = -1$. After computing the finite part in order to get $G_{-4+1+3}(x, u)$ we have

$$\begin{aligned}\sum_{\ell=0}^{3} G_{-4+1+\ell}(x,u)\, e^{i\int_x^y A.\mathrm{d}z} &= -\frac{i}{2\pi^2}\frac{\not{u}}{u^4}(1+\mathrm{o}(u^4)) + \frac{1}{16\pi^2}\frac{u_\rho}{u^2}\gamma^\mu\gamma^\rho\gamma^\nu F_{\mu\nu}(1+\mathrm{o}(u^2)) - \\ &- \frac{i}{48\pi^2}\frac{u_\rho u^\sigma}{u^2}(-\tfrac{3}{2}\gamma^\mu\gamma^\rho\gamma^\nu\partial_\sigma F_{\mu\nu} - \gamma^\rho\partial^\mu F_{\sigma\mu} + \gamma^\mu\partial^\rho F_{\sigma\mu}) - \\ &- \frac{i}{24\pi^2}(\ln 2 - \ln u - \frac{i\pi}{2} + \Gamma'(1))\gamma^\nu\partial^\mu F_{\mu\nu},\end{aligned} \tag{30}$$

which, in general, clearly yields a nonzero result for expression (11).

So, we see that although Schwinger and complex powers methods are both gauge-invariant, they only coincide for the two-dimensional case. In 3 dimensions, the coincidence depends on the representation chosen for the γ-matrices, while for $n = 4$ they in general disagree.

Had we considered the general case, additional terms depending on the Riemannian curvature would have appeared. Nevertheless, those terms coud not counterbalance the computed $F_{\mu\nu}$-depending terms which produced the difference between both methods.

References

1. Schwinger, J.: *Phys. Rev.* **128** (1962), 2425.
2. Seeley, R. T.: *Amer. J. Math.* **91** (1969), 889.
3. Hawking, S. W.: *Comm. Math. Phys.* **55** (1977), 133.
4. Gamboa Saraví, R. E., Muschietti, M. A., Schaposnik, F. A. and Solomin, J. E.: *J. Math. Phys.* **26** (1985), 2045.

5. Kontsevich, M. and Vishik, S.: Determinants of elliptic pseudo-differential operators (1994), hep-th/9404046.
6. Shubin, M. A.: *Pseudodifferential Operators and Spectral Theory*, Springer-Verlag, Berlin, 1987.
7. Gradshteyn, I. S. and Ryzhik, I. M.: *Table of Integrals, Series, and Products*, Academic Press, New York, 1980.
8. Gel'fand, I. M. and Shilov, G. E.: *Generalized Functions*, Academic Press, New York, 1964.