Contrary quantum histories and contrary inferences

Marcelo Losada*

Universidad de Buenos Aires - CONICET
Ciudad Universitaria, 1428 Buenos Aires, Argentina.

Abstract

In the context of the quantum histories approach to quantum mechanics, we define a contrary relation between quantum histories, generalizing the standard definition of contrary quantum properties. Using this contrary relation, we study the possibility of inferring contrary quantum histories. We consider the quantum histories approach with three different consistency conditions: weak, medium and global consistency conditions. For the first two conditions, we show that, it is possible to infer contrary histories from different families of histories, in accordance with previous results. For global consistency condition, we prove that it is not possible to infer contrary histories.

Keywords: Consistent Histories; Decoherent Histories; contrary quantum histories; quantum foundations.

1. Introduction

According to the standard formulation of quantum mechanics, the states of a quantum system follow two types of time evolutions. When there is no measurement, evolutions are governed by the Schrödinger equation, leading to a continuous and deterministic dynamic. When a measurement is performed on the system, the state collapses onto one of the eigenstates of the measured observable in a discontinuous and non-deterministic process [1, 2]. This formulation has a remarkable predictive success, however, the distinction between

*Corresponding author
Email address: marcelolosada@yahoo.com (Marcelo Losada)

ordinary physical processes and measurement processes was considered conceptually unsatisfactory by some authors [3, 4, 5, 6, 7, 8]. On the one hand, since measuring devices are made of the same components as the remaining physical systems, it is expected that measurement processes are not essentially different from ordinary physical processes. On the other hand, since probability is defined only for measurements, it cannot be applied to closed systems which do not admit of being measured by an external apparatus.

These limitations become unavoidable in quantum cosmology. Since its study object is a closed system by definition, the notion of an external observer is completely inappropriate. In order to overcome these difficulties, it was proposed an approach to quantum mechanics based on the notion of quantum histories. In 1984, R. B. Griffiths presented the first version of his *Theory of Consistent Histories* [9]; some years later, he introduced some modifications [5, 10, 11]. R. Omnès also published a series of articles and two books in which he contributed to the development of the theory [6, 12, 13, 14]. Simultaneously, Murray Gell-Mann and James Hartle developed a similar formalism, called the *Decoherent Histories Interpretation* [7, 15, 16]. Although these proposals do not agree in every detail, their strong similarities justify to subsume all of them under the label *quantum histories approach*.

All versions of the quantum histories approach provide a formulation of quantum mechanics in which measurements are treated in the same way as other physical processes. Therefore, there is no additional collapse postulate, quantum dynamics is always described by the Schrödinger equation. Moreover, the notion of an external observer it is no longer necessary. As a result, the quantum histories approach is applicable to closed systems, and it provides a useful framework for studying quantum cosmology.

Since it is not possible to define a Kolmogorovian probability on the set of all quantum histories, it is necessary to restrict the probability definition to subsets of histories which constitute well-defined probabilistic sample spaces, called families of histories. These families must satisfy an additional condition, called the consistency condition. The two most relevant conditions that have been proposed are: the *weak consistency condition* (WCC) and the *medium consistency condition* (MCC) [7].

Since WCC and MCC depend on the initial state of the system, families of histories also depend on the initial state. This is an odd situation compared with what happens in standard quantum mechanics, where the sets of properties that can be considered simultaneously are independent of the state. In order to avoid the dependence with the initial state, R. Laura and L. Vanni proposed an alternative condition [17], which is equivalent to the WCC imposed for all states [18]. Moreover, it was proved that this condition can be applied for describing quantum measurements [19, 20], the decay process [21] and the double slit experiment with and without measurement instruments [22]. On the basis of these results, it is not difficult to show that the condition can be fruitfully applied to the most common experimental set-ups, which are, in one way or another, variations of the above cases. Throughout this paper, this condition will be called the *global consistency condition* (GCC).

An important weakness of the quantum histories approach is that there are too many consistent families of histories. A. Kent pointed out that the existence of multiple consistent families allows retrodictions of contrary properties from different consistent families [23]. In other words, he proved that, for a given initial data of a quantum system, it is possible to infer, with certainty, two inconsistent facts about the past, using two different consistent families of histories. This implies that, in general, in the quantum histories approach there is not a unique past for given present data [24].

Proponents of the quantum histories approach argue that this is not a real problem for the theory, because contrary retrodictions are obtained using incompatible families, which, accordingly to the single framework rule, cannot be considered simultaneously [25, 26]. If one accepts the single framework rule, explicit logical contradictions are not possible. However, this rule was criticized mainly for two reasons. On the one hand, for being an ad hoc solution imposed in order to avoid inconsistencies, and on the other hand, for implying that what is real depends on the choice of the consistent family of histories

[24, 27]. Therefore, some authors [23, 24, 28, 29] consider that the existence of contrary retrodictions is a serious failure of the quantum histories approach.

In [30], it was analyzed the problem of contrary retrodictions using the global consistency condition and it was proved, without appealing to the single framework rule, that it does not allow to retrodict contrary properties. In this paper, we generalize that result to a more general case of inferences of histories. We start defining a contrary relation between histories, and then we study the possibility of inferring contrary histories without appealing to the single framework rule.

The organization of this work is the following. In section 2, we summarize the main aspects of the quantum histories approach and we present the three consistency conditions that we will consider throughout this work: WCC, MCC and GCC. In section 3, we introduce the contrary relation between quantum histories and we discuss the possibility of contrary inferences of histories. First, we consider families of histories which satisfy the WCC and MCC, and we show that they admit inferences of contrary histories using different families of histories, in accordance with the result of A. Kent [23]. Second, we consider families which satisfy the GCC, and we prove, without appealing to the single framework rule, that it is not possible to infer contrary histories. This results is a natural generalization of the previous one obtained in [30]. In section 4 we present the conclusions, and finally, in the Appendix, we prove two lemmas that are used in Section 3.

2. Quantum histories approach

In this section, we introduce the main features of the quantum histories approach [5, 6, 7, 10, 14, 31]. First, we describe how quantum histories are represented. Second, we explain how probabilities are defined and why an additional consistency condition is necessary. Also, we introduce three consistency conditions which will be used in Section 3: weak, medium and global consistency conditions.

In standard quantum mechanics it is possible to represent properties of physical systems at a single time. Due to the orthocomplemented lattice structure of the set of properties, logical operations between them can be defined [32, 33]. However, since all properties of the system are considered at the same time, it is not possible to consider logical operations between properties at different times.

In order to be able to represent properties at different times, quantum histories formalism introduces the notion of history, which generalizes the notion of property. A product history is defined as a sequence of properties at different times, and a general history is defined as the result of logical operations between product histories. For example, if we consider n times, $t_1 < ... < t_n$, and properties p_i and q_i at each time t_i, then $\breve{p} = (p_1, ..., p_n)$ is the history in which, at each time t_i, the property p_i is the case, and $\breve{q} = (q_1, ..., q_n)$ is the history in which, at each time t_i, the property q_i is the case. From $\breve{p}$ and $\breve{q}$, we can obtain general histories which result from logical operations between them, such as the conjunction, the disjunction and the negation.

In standard quantum mechanics, properties can be represented by orthogonal projectors. If $\mathcal{H}$ is the Hilbert space of the system, the set of all properties is given by the set of all orthogonal projectors $P(\mathcal{H})$. The main idea of the quantum histories approach is to represent histories with orthogonal projectors defined on a bigger Hilbert space $\breve{\mathcal{H}} = \mathcal{H} \otimes \overset{n}{\dots} \otimes \mathcal{H}$, i.e., the Hilbert space obtained by doing the tensor product of n copies of the Hilbert space of the system. For example, an n-time product history $\breve{p} = (p_1, ..., p_n)$ is represented by the projector $\breve{P} = P_1 \otimes ... \otimes P_n$,

$$\breve{p} = (p_1, p_2, ..., p_n) \quad \longleftrightarrow \quad \breve{P} = P_1 \otimes ... \otimes P_n,$$

where each projector P_i represents the property p_i. In general, not every history is a product history. The disjunctions between product histories can be histories which cannot be represented by tensor products of projectors.

Since $\breve{\mathcal{H}}$ is also a Hilbert space, the set $P(\breve{\mathcal{H}})$, given by all projector operators on $\breve{\mathcal{H}}$, has the same structure that the lattice of properties of standard quantum mechanics $P(\mathcal{H})$, i.e., it is an orthocomplemented and non-distributive lattice

[33]. Therefore, using the logical operations of the lattice $P(\breve{\mathcal{H}})$, it is possible to define logical operations between quantum histories. Furthermore, the partial order relation of the lattice $P(\breve{\mathcal{H}})$ is defined in the following way:

$$\breve{P} \leq \breve{Q} \iff \breve{P}\mathcal{H} \subseteq \breve{Q}\mathcal{H}.$$

That means, $\breve{P}$ is less than $\breve{Q}$ if the range of $\breve{P}$ is included in the range of $\breve{Q}$.

In order to define Kolmogorovian probabilities for quantum histories, it is necessary to have a distributive lattice. Since the lattice $P(\breve{\mathcal{H}})$ is non-distributive, we need to choose a distributive sublattice. For this purpose, at each time t_i $(i = 1, ..., n)$, a projective decomposition of the identity of $\mathcal{H}$ must be selected, i.e., for each $i = 1, ..., n$, a set of orthogonal projectors $\{P_i^{k_i}\}_{k_i \in \sigma_i}$ (σ_i a set of index) which are mutually orthogonal and sum the identity of $\mathcal{H}$:

$$P_i^{k_i} P_i^{k_i'} = \delta_{k_i k_i'} P_i^{k_i}, \qquad \sum_{k_i \in \sigma_i} P_i^{k_i} = I,$$

where I is the identity operator of $\mathcal{H}$.

Then, we consider the product histories $\breve{P}^{\mathbf{k}}$, with $\mathbf{k} = (k_1, ..., k_n) \in \breve{\sigma} = \sigma_1 \times ... \times \sigma_n$, given by picking one projector $P_i^{k_i}$ at each time t_i:

$$\breve{P}^{\mathbf{k}} = P_1^{k_1} \otimes ... \otimes P_n^{k_n}, \qquad \mathbf{k} \in \breve{\sigma}.$$

It is easy to check that histories $\breve{P}^{\mathbf{k}}$ form a projective decomposition of the Hilbert space $\breve{\mathcal{H}}$:

$$\breve{P}^{\mathbf{k}} \breve{P}^{\mathbf{k}'} = \delta_{\mathbf{k}\mathbf{k}'} \, \breve{P}^{\mathbf{k}}, \qquad \textstyle\sum_{\mathbf{k}} \breve{P}^{\mathbf{k}} = \breve{I}, \qquad \mathbf{k}, \mathbf{k}' \in \breve{\sigma},$$

where $\breve{I}$ is the identity operator in $\breve{\mathcal{H}}$.

Finally, we consider the operators $\breve{P}^{\Lambda}$ $(\Lambda \subseteq \breve{\sigma})$, obtained by summing all histories $\breve{P}^{\mathbf{k}}$, with $\mathbf{k} \in \Lambda$, i.e.,

$$\breve{P}^{\Lambda} = \textstyle\sum_{\mathbf{k} \in \Lambda} \breve{P}^{\mathbf{k}}.$$

These operators are also orthogonal projectors, and they represent the quantum histories obtained by the disjunction of the histories $\breve{P}^{\mathbf{k}}$.

The set $\mathcal{F}$ of all histories $\breve{P}^{\Lambda}$ is a Boolean sublattice of $P(\breve{\mathcal{H}})$ and is called a *family of histories*,

$$\mathcal{F} = \left\{ \breve{P}^{\Lambda} \in P(\breve{\mathcal{H}}) \middle| \breve{P}^{\Lambda} = \sum_{\mathbf{k}\in\Lambda} \breve{P}^{\mathbf{k}}, \quad \Lambda \subseteq \breve{\sigma} \right\}.$$

The set of histories $\breve{P}^{\mathbf{k}}$ is a generator of the family of histories $\mathcal{F}$ and, since histories $\breve{P}^{\mathbf{k}}$ are the atoms of $\mathcal{F}$, they are called *atomic histories*. In any family of histories, the logical operations between two histories $\breve{P}$ and $\breve{Q}$ take the following form:

- Disjunction: $\breve{P} \vee \breve{Q} = \breve{P} + \breve{Q} - \breve{P}\breve{Q}$
- Conjunction: $\breve{P} \wedge \breve{Q} = \breve{P}\breve{Q}$
- Negation: $\breve{P}^{\perp} = \breve{I} - \breve{P}$

Since families of histories have a Boolean lattice structure, it is possible to define a probability on them. The probability definition of quantum histories is usually motivated from standard quantum mechanics [31, 34]. Given a product history represented by $\breve{P} = P_1 \otimes ... \otimes P_n$, the probability of measuring the sequence of properties P_1, ..., P_n, at times t_1, ..., t_n, respectively, is given by the expression

$$\mathrm{Pr}_{\rho_0}(P_1 \otimes ... \otimes P_n) = \mathrm{Tr}(P_{n,0}...P_{1,0}\rho_0 P_{1,0}...P_{n,0}), \tag{1}$$

where we have introduced the Heisenberg representation of the projector $P_i^{k_i}$, given by $P_{i,0} = U^{-1}(t_i, t_0) P_i U(t_i, t_0)$.

The main assumption of the quantum histories approach is the identification of the expression (1), not with the probability of measuring the history $P_1 \otimes ... \otimes P_n$, but with the probability of the occurrence of the history when the system is isolated and there is no measurements performed by external observers.

The probability defined above can be expressed in terms of an operator called *chain operator*. For each atomic history $\breve{P}^{\mathbf{k}} = P_1^{k_1} \otimes ... \otimes P_n^{k_n}$ of a family $\mathcal{F}$, its chain operator is defined as

$$C(\breve{P}^{\mathbf{k}}) = P_{1,0}^{k_1} P_{2,0}^{k_2} ... P_{n,0}^{k_n}, \tag{2}$$

with $P_{i,0}^{k_i}$ the Heisenberg representation of the projector $P_i^{k_i}$. Then, for a general history $\breve{P}^{\Lambda} = \sum_{\mathbf{k}\in\Lambda} \breve{P}^{\mathbf{k}}$, its chain operator is obtained by linear extension of the atomic case:

$$C(\breve{P}^{\Lambda}) = \sum_{\mathbf{k}\in\Lambda} C(\breve{P}^{\mathbf{k}}). \tag{3}$$

Finally, we define the probability of a general history $\breve{P}^{\Lambda}$ in the following way:

$$\mathrm{Pr}_{\rho_0}(\breve{P}^{\Lambda}) = \mathrm{Tr}\{C^{\dagger}(\breve{P}^{\Lambda})\rho_0 C(\breve{P}^{\Lambda})\}. \tag{4}$$

However, in general this definition does not satisfy the additivity axiom of probability theory. Given a family of histories $\mathcal{F}$, and two disjoint atomic histories $\breve{P}^{\mathbf{k}}$ and $\breve{P}^{\mathbf{k}'}$ (i.e. $\breve{P}^{\mathbf{k}}\breve{P}^{\mathbf{k}'} = 0$), the probability of the disjunction of the two histories is given by

$$\begin{aligned} &\mathrm{Pr}_{\rho_0}(\breve{P}^{\mathbf{k}} + \breve{P}^{\mathbf{k}'}) = \mathrm{Tr}\{C^{\dagger}(\breve{P}^{\mathbf{k}} + \breve{P}^{\mathbf{k}'})\rho_0 C(\breve{P}^{\mathbf{k}} + \breve{P}^{\mathbf{k}'})\} = \\ &= \mathrm{Pr}_{\rho_0}(\breve{P}^{\mathbf{k}}) + \mathrm{Pr}_{\rho_0}(\breve{P}^{\mathbf{k}'}) + 2\mathrm{Re}\left\{\mathrm{Tr}\left[C^{\dagger}(\breve{P}^{\mathbf{k}})\rho_0 C(\breve{P}^{\mathbf{k}'})\right]\right\}. \end{aligned} \tag{5}$$

We can see in equation (5) that the additivity condition does not hold in general. In order to obtain a well-defined probability, it is necessary to impose an additional condition to families of histories, such that the extra terms of expression (5) disappear. This extra condition is usually called the *consistency condition* and a family of histories which satisfies it is called a *consistent family of histories.*

Different consistency conditions have been proposed to eliminate the extra terms of equation (5). All of them are sufficient conditions to have well–defined probabilities, but not all of them are necessary conditions. The more restrictive conditions allow less consistent families of histories, which could be considered a weakness. However, they have some advantages when dealing with the classical limit problem or the inferences of contrary properties.

Usually, all these conditions are expressed in terms of what is known as the decoherence functional,

$$D_{\rho_0}\left(\breve{P}^{\mathbf{k}}, \breve{P}^{\mathbf{k}'}\right) = \mathrm{Tr}\left[C^{\dagger}(\breve{P}^{\mathbf{k}})\rho_0 C(\breve{P}^{\mathbf{k}'})\right].$$

The diagonal terms of the decoherence functional are the probabilities of the atomic histories, i.e.,

$$\mathrm{Pr}_{\rho_0}(\breve{P}^{\mathbf{k}}) = D_{\rho_0}\left(\breve{P}^{\mathbf{k}}, \breve{P}^{\mathbf{k}}\right).$$

In what follows we present three of the consistency conditions that have been proposed in the literature. Let $\mathcal{F}$ be a family of histories and $\breve{P}^{\mathbf{k}}$ its atomic histories.

- *Weak consistency condition (WCC):*

$$\mathrm{Re}\left\{D_{\rho_0}\left(\breve{P}^{\mathbf{k}}, \breve{P}^{\mathbf{k}'}\right)\right\} = 0, \qquad \forall\, \mathbf{k} \neq \mathbf{k}'. \tag{6}$$

  This condition was proposed by Gell-Mann and Hartle [7, 16], based on a similar condition presented by Griffiths [9] (for details, see note 4 of [16]). It is the necessary and sufficient condition in order to have well define probabilities. Therefore, it is the weakest of all possible conditions and it allows more consistent families than the others. A family $\mathcal{F}$ which satisfies the weak consistency condition will be called a *weakly consistent family of histories.*

  Although this condition is mathematically natural, some authors criticized it for being physically unsatisfactory. For example, Diosi showed in [35] that the weak consistency condition for two statistically independent subsystems does not imply the fulfillment of the same condition for the composite system. Also, he proved that weakly consistent families do not persist being weakly consistent if we alter the dynamics of the system, even when we expect them to persist. Other authors argued that it allows to many consistent families and it is not adequate for describing quasi–classical domains [16, 34, 36]. Stronger conditions were proposed in order to overcome some of the objections.

- *Medium consistency condition (MCC):*

$$D_{\rho_0}\left(\breve{P}^{\mathbf{k}}, \breve{P}^{\mathbf{k}'}\right) = 0, \qquad \forall\, \mathbf{k} \neq \mathbf{k}'. \tag{7}$$

This condition was proposed by Gell-Mann and Hartle [7] and it is stronger than the previous one. A family $\mathcal{F}$ which satisfies the medium consistency condition will be called a *medium consistent family of histories*. Although it is not a necessary condition, it is considered more useful to characterize quasi–classical domains and to study the emergence of classical world from quantum world [7, 16, 34]. Moreover, some authors [34] prefer this condition because they look favorably on conditions which reduce the number of families. They consider that less consistent families could help in solving the difficulties related with the prediction and retrodiction of contrary properties.

- *Global consistency condition (GCC):*

$$\mathrm{Re}\left\{D_{\rho_0}\left(\breve{P}^{\mathbf{k}}, \breve{P}^{\mathbf{k}'}\right)\right\} = 0, \quad \forall\, \mathbf{k} \neq \mathbf{k}', \quad \forall\, \rho_0. \tag{8}$$

This condition results from imposing the weak consistency condition (6) for all states of the system, and it is a necessary and sufficient condition for having consistent families independent of the state. A family $\mathcal{F}$ which satisfies the globally consistency condition will be called a *globally consistent family of histories.*

It is clear that GCC is stronger than WCC, but also it was proved in [18] that GCC is stronger than MCC. An equivalent condition to the GCC was first proposed by Laura and Vanni [17, 22], which consists in the commutation of all the orthogonal projectors in the Heisenberg representation:

$$[P_{i,0}^{k_i}, P_{j,0}^{k_j}] = 0, \quad \forall\, i,j, \ \forall\, k_i \in \sigma_i, \ \forall\, k_j \in \sigma_j. \tag{9}$$

In [18], it was proved that condition (9) is equivalent to the GCC, i.e., to the weak consistency condition imposed for all states. Also, in [18], it was argued that GCC is a natural requirement if we assume that families of histories have to be analog to complete sets of compatible properties of standard quantum mechanics, which are independent of the initial state.

Furthermore, it was proved that GCC has the advantage of not admitting contrary retrodictions [30].

In the next section we are going to generalize the results obtained in [30]. In the context of the quantum histories formalism, and for the three consistency conditions presented above, we will consider a more general case of contrary inferences.

3. Inferences of contrary quantum histories

A. Kent proved that in the quantum histories approach with the WCC or MCC the probabilistic retrodictions depend on the choice of the consistent family of histories. This freedom allows for a given initial data to infer with certainty two contrary properties of the past from different consistent families [23]. This result has no parallel in standard quantum theory, in which it is not possible to infer with certainty two contrary properties given an initial state. For these reasons, many authors [23, 24, 29] consider the existence of contrary inferences as a serious problem of the quantum histories approach.

In this section, we are going to analyze the existence of more general contrary inferences in the quantum histories approach. Throughout this analysis we will avoid appealing to the single framework rule. In this way, the conclusions obtained are not affected by the criticisms raised against this rule.

First, we define a contrary relation for quantum histories, generalizing the definition of contrary properties used in standard quantum mechanics [23]. Then, in Section 3.1, we show that weakly and medium consistent families allow inferences of contrary histories using different families of histories. To do that, it is enough to consider the example presented by J. B. Hartle [26] and to point out that the retrodiction of contrary properties is a particular case of inferences of contrary histories. Finally, in Section 3.2, we prove that it is not possible to infer contrary histories if we consider globally consistent families. This result generalizes a previous one presented in [30], in which it was proved that the commutation condition (9) does not allow retrodictions of contrary properties.

We begin defining the tools that will be necessary throughout this section: the contrary relation for histories, the notion of inference of histories and the definition of inference of contrary histories.

1. *Contrary relation of histories:*

   In standard quantum mechanics, two properties represented by orthogonal projectors P and Q are said to be contrary if they satisfy the relation $P \leq Q^{\perp} = I - Q$ [23]. This condition means that the image of projector P is included in the image of the orthogonal complement of projector Q. Since the set of quantum histories is also an orthocomplemented lattice, it has an order relation and an orthogonal complement. Therefore, the contrary relation between properties can be generalized to the set of histories in the following way. Given two histories represented by projectors $\breve{P}$ and $\breve{Q}$, we say that $\breve{P}$ is *contrary* to $\breve{Q}$ if they satisfy the following relation:

   $$\breve{P} \leq \breve{Q}^{\perp} = \breve{I} - \breve{Q}.$$

   This relation is symmetric: if $\breve{P} \leq \breve{I} - \breve{Q}$, then $\breve{Q} \leq \breve{I} - \breve{P}$. Then, we can simply say that $\breve{P}$ and $\breve{Q}$ are contrary histories.

   A useful characterization of contrary histories is the following (see [33], section 1.3): two histories represented by projectors $\breve{P}$ and $\breve{Q}$ are contrary if, and only if, their orthogonal projectors are orthogonal to each other, i.e.,

   $$\breve{P} \leq \breve{Q}^{\perp} \quad \Longleftrightarrow \quad \breve{P}\breve{Q} = \breve{Q}\breve{P} = 0.$$

2. *Inference of histories:*

   We consider an initial state ρ_0 and a consistent family of histories $\mathcal{F}$, which satisfies for ρ_0 any of the consistency conditions presented in Section 2. Given two histories of $\mathcal{F}$, represented by $\breve{P}$ and $\breve{Q}$, with $\Pr_{\rho_0}(\breve{Q}) \neq 0$, we define the conditional probability of $\breve{Q}$ given $\breve{P}$ in the following way

   $$\Pr_{\rho_0}(\breve{Q}|\breve{P}) = \frac{\Pr_{\rho_0}(\breve{Q}\breve{P})}{\Pr_{\rho_0}(\breve{Q})}. \tag{10}$$

Furthermore, we say that $\breve{Q}$ is *inferred* from $\breve{P}$ if the conditional probability of $\breve{Q}$ given $\breve{P}$ is equal to one, i.e.

$$\Pr_{\rho_0}(\breve{Q}|\breve{P}) = 1.$$

It should be noted that it is not possible to define the conditional probability between two histories which cannot be included in the same consistent family of histories.

3. *Inference of contrary histories:*

   The inference of contrary histories consists in the existence of two contrary histories, from different and incompatible consistent families, that can be inferred from a third history which belongs to both families of histories. More precisely, given an initial state ρ_0 and two consistent families of histories $\mathcal{F}_1$ and $\mathcal{F}_2$, we say that it is possible to infer contrary histories if there are three histories $\breve{P}$, $\breve{Q}$ and $\breve{R}$, such that $\breve{P}, \breve{R} \in \mathcal{F}_1$ and $\breve{Q}, \breve{R} \in \mathcal{F}_2$, satisfying the following conditions:

   - $\breve{P}$ and $\breve{Q}$ are contrary histories, i.e., $\breve{P}\breve{Q} = \breve{Q}\breve{P} = 0$.
   - $\breve{P}$ is inferred from $\breve{R}$, and $\breve{Q}$ is inferred from $\breve{R}$, i.e., $\Pr_{\rho_0}(\breve{P}|\breve{R}) = 1$ and $\Pr_{\rho_0}(\breve{Q}|\breve{R}) = 1$.

In what follows, we are going to study if it is possible to infer contrary histories, using the three consistency conditions previously defined.

3.1. Weak and medium consistency conditions

In this section, we show that consistent families of histories, which satisfy weak or medium consistency condition, admit inferences of contrary histories.

A particular case of contrary inference of histories was first presented by A. Kent in [23]. He proved that it is possible to retrodict with certainty two contrary properties from different consistent families which satisfy the WCC (6) or the MCC (7). Another example of retrodiction of contrary properties was developed by J. B. Hartle [26], based on the example of Kent. In what follows, we resume Hartle's example and we show that it is a particular case of inference of contrary histories.

Hartle considered a quantum system represented by a a three-dimensional Hilbert space. For simplicity, the Hamiltonian is chosen to be zero, i.e., $U(t,t') = I$. The state of the system at time t_0 is given by $\rho_0 = |\Psi\rangle\langle\Psi|$, with $|\Psi\rangle = \frac{1}{\sqrt{3}}(|A\rangle + |B\rangle + |C\rangle)$, where $|A\rangle$, $|B\rangle$ and $|C\rangle$ are three orthogonal and normalized vectors of the Hilbert space.

At time t_1, two projective decompositions of the identity are considered: $\{P_A, P_{\overline{A}}\}$ and $\{P_B, P_{\overline{B}}\}$, where $P_A = |A\rangle\langle A|$, $P_{\overline{A}} = I - P_A$, $P_B = |B\rangle\langle B|$ and $P_{\overline{B}} = I - P_B$. At time t_2, one projective decomposition of the identity is considered: $\{P_\Phi, P_{\overline{\Phi}}\}$, where $P_\Phi = |\Phi\rangle\langle\Phi|$ and $P_{\overline{\Phi}} = I - P_\Phi$, with $|\Phi\rangle = \frac{1}{\sqrt{3}}(|A\rangle + |B\rangle - |C\rangle)$. Two families of histories are defined, $\mathcal{F}_1$ and $\mathcal{F}_2$. $\mathcal{F}_1$ is the family generated by the following atomic histories

$$\breve{P}_{A\Phi} = P_A \otimes P_\Phi, \quad \breve{P}_{A\overline{\Phi}} = P_A \otimes P_{\overline{\Phi}},$$
$$\breve{P}_{\overline{A}\Phi} = P_{\overline{A}} \otimes P_\Phi, \quad \breve{P}_{\overline{A}\,\overline{\Phi}} = P_{\overline{A}} \otimes P_{\overline{\Phi}},$$

and $\mathcal{F}_2$ is the family generated by

$$\breve{P}_{B\Phi} = P_B \otimes P_\Phi, \quad \breve{P}_{B\overline{\Phi}} = P_B \otimes P_{\overline{\Phi}},$$
$$\breve{P}_{\overline{B}\Phi} = P_{\overline{B}} \otimes P_\Phi, \quad \breve{P}_{\overline{B}\,\overline{\Phi}} = P_{\overline{B}} \otimes P_{\overline{\Phi}}.$$

For the initial state $\rho_0 = |\Psi\rangle\langle\Psi|$, $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy weak and medium consistency conditions.

In order to show the existence of inferences of contrary histories, first we define the following contrary histories:

$$\breve{P}_{AI} = \breve{P}_{A\Phi} + \breve{P}_{A\overline{\Phi}} = P_A \otimes I,$$
$$\breve{P}_{BI} = \breve{P}_{B\Phi} + \breve{P}_{B\overline{\Phi}} = P_B \otimes I.$$

The history $\breve{P}_{AI}$ belongs to $\mathcal{F}_1$ and the history $\breve{P}_{BI}$ belongs to $\mathcal{F}_2$. To prove that they are contrary histories, it is enough to show

$$\breve{P}_{AI}\breve{P}_{BI} = (P_A \otimes I)(P_B \otimes I) = P_A P_B \otimes I = 0,$$

where in the last step we have used the orthogonality between vectors $|A\rangle$ and $|B\rangle$.

The next step is to show that there is a third history, which belongs to both families $\mathcal{F}_1$ and $\mathcal{F}_2$, from which it can be inferred $\breve{P}_{AI}$ and $\breve{P}_{BI}$. We defined the history $\breve{P}_{I\Phi}$ as

$$\breve{P}_{I\Phi} = I \otimes P_\Phi.$$

It is easy to see that $\breve{P}_{I\Phi}$ belongs to $\mathcal{F}_1$ and $\mathcal{F}_2$.

Finally, using family $\mathcal{F}_1$ and the initial state ρ_0, we can define the conditional probability of $\breve{P}_{AI}$ given $\breve{P}_{I\Phi}$; and using family $\mathcal{F}_2$ and the same initial state, we can define the conditional probability of $\breve{P}_{BI}$ given $\breve{P}_{I\Phi}$. From definitions (4) and (10), we obtain

$$\mathrm{Pr}_{\rho_0}(\breve{P}_{AI}|\breve{P}_{I\Phi}) = 1 \quad \text{and} \quad \mathrm{Pr}_{\rho_0}(\breve{P}_{BI}|\breve{P}_{I\Phi}) = 1.$$

Therefore, from history $\breve{P}_{I\Phi}$, we can infer contrary histories $\breve{P}_{AI}$ and $\breve{P}_{BI}$, using families $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively.

We have already mentioned that for some proponents of the quantum histories approach [25, 26] the retrodictions of contrary properties are not a problem, because each retrodiction is obtained using incompatible families of histories, which, accordingly to the single framework rule, cannot be simultaneously considered for describing a physical system. The solution appealing to the single framework rule can also be applied to the case of inferences of contrary histories. However, this rule was criticized by several authors [23, 24, 27, 28]. On the one hand, because it seems to be an ad hoc solution imposed in order to avoid inconsistencies, and on the other hand, for implying that what is real depends on the choice of the consistent family of histories. Therefore, looking for stronger consistency conditions which do not allow contrary inferences seems to be needed in order to have an adequate formulation of the quantum histories approach [23, 29].

In the next section, we are going to show that this problem disappear if we consider globally consistent families.

3.2. Global consistency condition

In a previous paper [30], it was proved that the quantum histories approach with the global consistency condition (8) does not admit retrodictions of contrary properties, which are a particular case of inference of contrary histories. In this section, we generalize that result. We will show that, if consistent families of histories satisfy the GCC, then it is not possible to infer contrary histories. More precisely, we are going to prove the following result.

Theorem:

There are no three histories $\breve{P}$, $\breve{Q}$ and $\breve{R}$, which satisfy the following conditions:

- There are two globally consistent families of histories $\mathcal{F}_1$ and $\mathcal{F}_2$, such that $\breve{P}, \breve{R} \in \mathcal{F}_1$ and $\breve{Q}, \breve{R} \in \mathcal{F}_2$.
- $\breve{P}$ and $\breve{Q}$ are contrary histories, i.e., $\breve{P}\breve{Q} = \breve{Q}\breve{P} = 0$.
- For some initial state ρ_0, $\breve{P}$ is inferred from $\breve{R}$, and $\breve{Q}$ is inferred from $\breve{R}$, i.e.,

$$\Pr_{\rho_0}(\breve{P}|\breve{R}) = 1, \quad \text{and} \quad \Pr_{\rho_0}(\breve{Q}|\breve{R}) = 1.$$

Proof:

We suppose that there are three histories $\breve{P}$, $\breve{Q}$ and $\breve{R}$, which satisfy the theorem conditions. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are globally consistent families of histories, then $C(\breve{P})$ and $C(\breve{Q})$ are orthogonal projectors (see Lemma 1 of the Appendix). Also, since $\breve{P}$ and $\breve{Q}$ are contrary properties, $C(\breve{P})C(\breve{Q}) = 0$ (see Lemma 2 of the Appendix).

The condition $C(\breve{P})C(\breve{Q}) = 0$ implies that $I - C(\breve{P}) - C(\breve{Q})$ is also an orthogonal projector and it is orthogonal to $C(\breve{P})$ and $C(\breve{Q})$. Therefore, $C(\breve{P})$, $C(\breve{Q})$ and $I - C(\breve{P}) - C(\breve{Q})$ form a projective decomposition of the identity of $\mathcal{H}$. Then, for all states ρ, the following condition holds:

$$\mathrm{Tr}\left\{\rho\left[C(\breve{P}) + C(\breve{Q})\right]\right\} + \mathrm{Tr}\left\{\rho\left[I - C(\breve{P}) - C(\breve{Q})\right]\right\} = 1.$$

Since the two terms are non-negative, for all states ρ we have:

$$\mathrm{Tr}\left[\rho C(\breve{P})\right] + \mathrm{Tr}\left[\rho C(\breve{Q})\right] \leq 1. \tag{11}$$

Furthermore, for each state ρ, the conditional probability of $\breve{P}$ given $\breve{R}$ is

$$\begin{aligned} \Pr_\rho(\breve{P}|\breve{R}) = \frac{\Pr_\rho(\breve{P}\breve{R})}{\Pr_\rho(\breve{R})} = \frac{\mathrm{Tr}\left[\rho C(\breve{P}\breve{R})\right]}{\mathrm{Tr}\left[\rho C(\breve{R})\right]} = \\ = \frac{\mathrm{Tr}\left[\rho C(\breve{P})C(\breve{R})\right]}{\mathrm{Tr}\left[\rho C(\breve{R})\right]} = \frac{\mathrm{Tr}\left[C(\breve{R})\rho C(\breve{R})C(\breve{P})\right]}{\mathrm{Tr}\left[C(\breve{R})\rho C(\breve{R})\right]}, \end{aligned} \tag{12}$$

where we have used the cyclic property of the trace and the properties proved in Lemma 1 of the Appendix: $C(\breve{P}\breve{R}) = C(\breve{P})C(\breve{R})$, $C(\breve{R})$ is an orthogonal projector and $[C(\breve{P}), C(\breve{R})] = 0$.

Analogously, for each state ρ, we can obtain the conditional probability of $\breve{Q}$ given $\breve{R}$,

$$\Pr_\rho(\breve{Q}|\breve{R}) = \frac{\mathrm{Tr}\left[C(\breve{R})\rho C(\breve{R})C(\breve{Q})\right]}{\mathrm{Tr}\left[C(\breve{R})\rho C(\breve{R})\right]}. \tag{13}$$

For each state ρ, we define another state $\rho^* = \frac{C(\breve{R})\rho C(\breve{R})}{\mathrm{Tr}\left\{C(\breve{R})\rho C(\breve{R})\right\}}$. It is easy to prove that ρ^* is a density operator. The conditional probabilities (12) and (13), in terms of ρ^*, take the following forms:

$$\Pr_\rho(\breve{P}|\breve{R}) = \mathrm{Tr}\left[\rho^* C(\breve{P})\right], \qquad \Pr_\rho(\breve{Q}|\breve{R}) = \mathrm{Tr}\left[\rho^* C(\breve{Q})\right].$$

Then, taken into account equation (11), we obtain

$$\Pr_\rho(\breve{P}|\breve{R}) + \Pr_\rho(\breve{Q}|\breve{R}) = \mathrm{Tr}\left\{\rho^* C(\breve{P})\right\} + \mathrm{Tr}\left\{\rho^* C(\breve{Q})\right\} \leq 1.$$

Therefore, both conditional probabilities cannot be equal to one. This result contradicts the previous assumptions, so we conclude that there are no three histories $\breve{P}$, $\breve{Q}$ and $\breve{R}$ satisfying the conditions of the theorem.■

We have proved that the global consistency condition does not allow inferences of contrary histories, even if we ignore the single framework rule. This result makes an important difference with the other two conditions, the WCC and the MCC, and it can be considered an interesting advantage of the GCC.

4. Conclusions

A. Kent proved that in the quantum histories approach, it is possible to retrodict with certainty two contrary properties from different consistent families which satisfy WCC and MCC [23]. In a previous paper [30], we analyzed that problem using the GCC, and we found that it does not allow to retrodict contrary properties. In this paper, we generalized this result to a more general case: contrary inferences of quantum histories.

First, in Section 2, we described the quantum histories approach, and we presented three different consistency conditions that can be imposed to families of histories: weak, medium and global consistency conditions. The first two are well-known conditions, which were studied in several works. The last condition results from imposing the WCC for all states of the system, and it is a necessary and sufficient condition for having consistent families independent of the initial state [18]. An equivalent condition to the GCC was first proposed by Laura and Vanni [17, 22], and the equivalence was proved in [18].

In Section 3, we defined a contrary relation between histories, which generalizes the definition of contrary properties used in the discussions of contrary retrodictions. This definition is based on the orthocomplemented lattice structure of the set of quantum histories. Then, we used the contrary relation to study the possibility of inferences of contrary histories. We considered the quantum histories approach with the previously mentioned consistency conditions.

In Section 3.1, we showed that families of histories which satisfy the WCC and MCC allow inferences of contrary histories. To do that, we considered the example presented by J. B. Hartle [26] and we pointed out that a retrodiction of contrary properties is a particular case of inference of contrary histories. Finally, in Section 3.2, we considered globally consistent families and we proved a general result: families of histories which satisfy the GCC do not admit inferences of contrary histories.

This result points out an interesting advantage of the GCC compared with the other two conditions. However, since this condition is more restrictive than the others, it is still necessary to continue studying the possibility that some relevant families of histories are not admitted with this condition. We do not have a definite answer to this question, but some successful results in that direction where obtained in [20, 21, 22]. On the basis of those results, we consider that the GCC condition can be fruitfully applied to the most common experimental set-ups, which are, in one way or another, variations of the above cases. Furthermore, the quantum histories approach based on weak and medium consistency conditions has been target of several further criticisms, beyond the problem of contrary inferences [24]. Therefore, it seems reasonable to explore alternative consistency conditions which avoid those criticisms.

Appendix A.

Lemma 1: Given a globally consistent family of histories $\mathcal{F}$, the following statements are true:

1. If $\breve{P} \in \mathcal{F}$, then $C(\breve{P})$ is an orthogonal projector.
2. If $\breve{P}, \breve{Q} \in \mathcal{F}$, then $C(\breve{P}\breve{Q}) = C(\breve{P})C(\breve{Q}) = C(\breve{Q})C(\breve{P})$.

Proof:

Let $\breve{P}^{\mathbf{k}}$ be the atomic histories which generate the family $\mathcal{F}$, where

$$\breve{P}^{\mathbf{k}} = P_1^{k_1} \otimes ... \otimes P_n^{k_n}, \qquad \mathbf{k} \in \sigma_1 \times ... \times \sigma_n = \breve{\sigma}, \tag{A.1}$$

and with each $\{P_i^{k_i}\}_{k_i \in \sigma_i}$ a projective decomposition of the identity of $\mathcal{H}$. Since $\mathcal{F}$ is a globally consistent family, it must satisfy the equivalent condition to the GCC (9), i.e., for $i, j = 1, ... n$,

$$[P_{i,0}^{k_i}, P_{j,0}^{k_j}] = 0, \quad \forall\, k_i \in \sigma_i, \forall\, k_j \in \sigma_j. \tag{A.2}$$

Given two histories $\breve{P}, \breve{Q} \in \mathcal{F}$, we can express them as sums of the atomic histories $\breve{P}^{\mathbf{k}}$,

$$\breve{P} = \sum_{\mathbf{k} \in \Delta_1 \subseteq \breve{\sigma}} \breve{P}^{\mathbf{k}}, \qquad \breve{Q} = \sum_{\mathbf{k}' \in \Delta_2 \subseteq \breve{\sigma}} \breve{P}^{\mathbf{k}'}.$$

According to expressions (2) and (3), the chain operators of $\breve{P}$ and $\breve{Q}$ are given by

$$C(\breve{P}) = \sum_{\mathbf{k}\,\in\,\Delta_1} C(\breve{P}^{\mathbf{k}}) = \sum_{\mathbf{k}\,\in\,\Delta_1} P_{1,0}^{k_1}...P_{n,0}^{k_n},$$
$$C(\breve{Q}) = \sum_{\mathbf{k}'\,\in\,\Delta_2} C(\breve{P}^{\mathbf{k}'}) = \sum_{\mathbf{k}'\,\in\,\Delta_2} P_{1,0}^{k_1'}...P_{n,0}^{k_n'}.$$

1. Each product $P_{1,0}^{k_1}...P_{n,0}^{k_n}$ is an orthogonal projector, because a product of commuting orthogonal projectors is an orthogonal projector. Also, a sum of orthogonal projectors, which are orthogonal to each other, is also an orthogonal projector.
   Therefore,
   $$C(\breve{P}) = \sum_{\mathbf{k}\,\in\,\Delta_1} P_{1,0}^{k_1}...P_{n,0}^{k_n}$$
   is an orthogonal projector.
2. The product of $C(\breve{P})$ and $C(\breve{Q})$ is given by
   $$\begin{aligned} C(\breve{P})C(\breve{Q}) &= \sum_{\mathbf{k}\,\in\,\Delta_1,\,\mathbf{k}'\,\in\,\Delta_2} P_{1,0}^{k_1}...P_{n,0}^{k_n}\,P_{1,0}^{k_1'}...P_{n,0}^{k_n'} = \\ &= \sum_{\mathbf{k}\,\in\,\Delta_1,\,\mathbf{k}'\,\in\,\Delta_2} P_{1,0}^{k_1}...P_{n,0}^{k_n}\delta_{k_1,k_1'}...\delta_{k_n,k_n'} = \\ &= \sum_{\mathbf{k}\,\in\,\Delta_1\,\cap\,\Delta_2} P_{1,0}^{k_1}...P_{n,0}^{k_n} = \\ &= \sum_{\mathbf{k}\,\in\,\Delta_1\,\cap\,\Delta_2} C(\breve{P}^{\mathbf{k}}), \end{aligned} \tag{A.3}$$
   where we have used the commutation condition (A.2). From (A.3), we conclude that $C(\breve{P})C(\breve{Q}) = C(\breve{Q})C(\breve{P})$.
   Furthermore, the product of $\breve{P}$ and $\breve{Q}$ is given by
   $$\begin{aligned} \breve{P}\breve{Q} &= \sum_{\mathbf{k}\,\in\,\Delta_1,\,\mathbf{k}'\,\in\,\Delta_2} \breve{P}^{\mathbf{k}}\breve{P}^{\mathbf{k}'} = \\ &= \sum_{\mathbf{k}\,\in\,\Delta_1,\,\mathbf{k}'\,\in\,\Delta_2} \breve{P}^{\mathbf{k}}\delta_{\mathbf{k},\mathbf{k}'} = \sum_{\mathbf{k}\,\in\,\Delta_1\cap\Delta_2} \breve{P}^{\mathbf{k}}. \end{aligned}$$
   Then, $C(\breve{P}\breve{Q}) = \sum_{\mathbf{k}\,\in\,\Delta_1\cap\Delta_2} C(\breve{P}^{\mathbf{k}})$, and comparing with equation (A.3), we conclude that $C(\breve{P}\breve{Q}) = C(\breve{P})C(\breve{Q})$. Therefore, $C(\breve{P}\breve{Q}) = C(\breve{P})C(\breve{Q}) = C(\breve{Q})C(\breve{P})$.■

Lemma 2:

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two globally consistent families of histories, and let $\breve{P} \in \mathcal{F}_1$ and $\breve{Q} \in \mathcal{F}_2$ be two contrary histories. Then, $C(\breve{P})C(\breve{Q}) = 0$.

Proof:

Let $\{\breve{P}^{\mathbf{k}}\}_{\mathbf{k}\in\breve{\sigma}}$ and $\{\breve{Q}^{\mathbf{l}}\}_{\mathbf{l}\in\breve{\sigma}'}$ be the atomic histories which generate the families $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, where

$$\breve{P}^{\mathbf{k}} = P_1^{k_1} \otimes ... \otimes P_n^{k_n}, \qquad \mathbf{k} \in \sigma_1 \times ... \times \sigma_n = \breve{\sigma},$$
$$\breve{Q}^{\mathbf{l}} = Q_1^{l_1} \otimes ... \otimes Q_n^{l_n}, \qquad \mathbf{l} \in \sigma'_1 \times ... \times \sigma'_n = \breve{\sigma}',$$

and with each $\{P_i^{k_i}\}_{k_i\in\sigma_i}$ and each $\{Q_i^{l_i}\}_{l_i\in\sigma'_i}$ a projective decomposition of the identity of $\mathcal{H}$.

Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are globally consistent families, they must satisfy the equivalent condition to the GCC (9), i.e., for $i, j = 1, ... n$,

$$[P_{i,0}^{k_i}, P_{j,0}^{k_j}] = 0, \qquad \forall\, k_i \in \sigma_i, \forall\, k_j \in \sigma_j, \tag{A.4}$$
$$[Q_{i,0}^{l_i}, Q_{j,0}^{l_j}] = 0, \qquad \forall\, l_i \in \sigma'_i, \forall\, l_j \in \sigma'_j. \tag{A.5}$$

We can express $\breve{P}$ and $\breve{Q}$ as sums of the atomic histories $\breve{P}^{\mathbf{k}}$ and $\breve{Q}^{\mathbf{l}}$, respectively,

$$\breve{P} = \sum_{\mathbf{k}\,\in\,\Delta\,\subseteq\,\breve{\sigma}} \breve{P}^{\mathbf{k}}, \qquad \breve{Q} = \sum_{\mathbf{l}\,\in\,\Delta'\,\subseteq\,\breve{\sigma}'} \breve{Q}^{\mathbf{l}}.$$

The product of the chain operators $C(\breve{P})$ and $C(\breve{Q})$ is

$$C(\breve{P})C(\breve{Q}) = \sum_{\mathbf{k}\,\in\,\Delta}\sum_{\mathbf{l}\,\in\,\Delta'} C(\breve{P}^{\mathbf{k}})C(\breve{Q}^{\mathbf{l}}). \tag{A.6}$$

Since $\breve{P}$ and $\breve{Q}$ are contrary properties, $\breve{P}\breve{Q} = \sum_{\mathbf{k}\,\in\,\Delta}\sum_{\mathbf{l}\,\in\,\Delta'} \breve{P}^{\mathbf{k}}\breve{Q}^{\mathbf{l}} = 0$. Then, for all $\mathbf{k} \in \Delta$ and for all $\mathbf{l} \in \Delta'$, we have $\breve{P}^{\mathbf{k}}\breve{Q}^{\mathbf{l}} = 0$.

The condition $\breve{P}^{\mathbf{k}}\breve{Q}^{\mathbf{l}} = P_1^{k_1}Q_1^{l_1} \otimes ... \otimes P_n^{k_n}Q_n^{l_n} = 0$ implies that, for some $1 \leq i \leq n$, $P_i^{k_i}Q_i^{l_i} = 0$, and also $P_{i,0}^{k_i}Q_{i,0}^{l_i} = 0$.

Moreover, it is easy to see that $C(\breve{P}^{\mathbf{k}}) = C(\breve{P}^{\mathbf{k}})P_{i,0}^{k_i}$ and $C(\breve{Q}^{\mathbf{l}}) = Q_{i,0}^{l_i}C(\breve{Q}^{\mathbf{l}})$:

$$\begin{aligned} C(\breve{P}^{\mathbf{k}}) &= P_{1,0}^{k_1} ... P_{i,0}^{k_i} ... P_{n,0}^{k_n} = P_{1,0}^{k_1} ... P_{i,0}^{k_i} P_{i,0}^{k_i} ... P_{n,0}^{k_n} = \\ &= P_{1,0}^{k_1} ... P_{n,0}^{k_n} P_{i,0}^{k_i} = C(\breve{P}^{\mathbf{k}})P_{i,0}^{k_i}, \\ C(\breve{Q}^{\mathbf{l}}) &= Q_{1,0}^{l_1} ... Q_{i,0}^{l_i} ... Q_{n,0}^{l_n} = Q_{1,0}^{l_1} ... Q_{i,0}^{l_i} Q_{i,0}^{l_i} ... Q_{n,0}^{l_n} \\ &= Q_{i,0}^{l_i} Q_{1,0}^{l_1} ... Q_{n,0}^{l_n} = Q_{i,0}^{l_i} C(\breve{Q}^{\mathbf{l}}), \end{aligned}$$

where we have used the chain operator definition (2), the relations $P_{i,0}^{k_i}P_{i,0}^{k_i} = P_{i,0}^{k_i}$ and $Q_{i,0}^{l_i}Q_{i,0}^{l_i} = Q_{i,0}^{l_i}$, and the commutation relations (A.4) and (A.5). Therefore,

$$C(\breve{P}^{\mathbf{k}})C(\breve{Q}^{\mathbf{l}}) = C(\breve{P}^{\mathbf{k}})P_{i,0}^{k_i}Q_{i,0}^{l_i}C(\breve{Q}^{\mathbf{l}}) = 0, \tag{A.7}$$

where we have used the result $P_{i0}^{k_i}Q_{i0}^{l_i} = 0$. Finally, replacing expression (A.7) in (A.6), we obtain $C(\breve{P})C(\breve{Q}) = 0$.■

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