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On the search for weighted inequalities for operators related to the Ornstein-Uhlenbeck semigroup⋆

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Abstract. In this paper, for each given p, $1 < p < \infty$, we characterize the weights v for which the centered maximal function with respect to the gaussian measure and the Ornstein-Uhlenbeck maximal operator are well defined for every function in $L^p(vd\gamma)$ and their means converge almost everywhere. In doing so, we find that this condition is also necessary and sufficient for the existence of a weight u such that the operators are bounded from $L^p(vd\gamma)$ into $L^p(ud\gamma)$. We approach the poblem by proving some vector valued inequalities. As a byproduct we obtain the strong type $(1, 1)$ for the "global" part of the centered maximal function.

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1 Introduction

In this work we obtain some weighted inequalities for operators related to the gaussian harmonic analysis, namely those associated to the study of the Ornstein-Uhlenbeck semigroup.

Let γ be the gaussian measure in $\mathbb{R}^n$, that is $d\gamma(z) = e^{-|z|^2}dz$. The initial question we pose is the following: given a fixed operator T, bounded in $L^p(d\gamma)$ for some p, $1 < p < \infty$, find necessary and sufficient conditions for a weight v in order to have $Tf(x) < \infty a.e.x$ for every $f \in L^p(vd\gamma)$. However, it turns out that this question is equivalent to the problem: given a fixed operator T, bounded in $L^p(d\gamma)$ for some p, $1 < p < \infty$, find sufficient and necessary conditions for a weight v in order to exist a non trivial weight u such that T becomes bounded from $L^p(vd\gamma)$ into $L^p(ud\gamma)$.

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This last question, nowadays classical in the context of Euclidean Harmonic Analysis, was first formulated by B. Muckenhoupt and solved for particular operators by several authors, see [GC,R] and the references there. Latter on, J.L. Rubio de Francia realized that this type of results could be obtained from appropiate vector valued inequalities for the operator under consideration.

In this paper, we take Rubio de Francia's approach, so we start the study of the centered maximal function with respect to the gaussian measure, by deriving a kind of Fefferman-Stein inequality (see Theorem 2.10).

To achieve this result, as it is usual in gaussian harmonic analysis, we decompose the operator in its “local" and “global" parts. For the first, we use that it essentially behaves as a singular integral and we apply the theory of vector valued Calderón-Zygmund operators. As for the global part, we get a bound in terms of a positive integral operator which we prove it is of strong type $(1, 1)$ (see Theorem 2.7). This behaviour came as a surprise to us, since the same procedure, when applied to other related gaussian operators like Ornstein-Uhlenbeck maximal opeartor, first and second order Riesz transforms and multipliers, always leads to positive integrals operators which are, as far as we know, just of weak type $(1, 1)$, see [P] [MPS] [FGS] [GMST1] [GMST2].

In this way we found the precise condition on the weight v, namely $v^{-\frac{1}{p-1}} \in L^1(d\gamma)$, which solves the problems stated above. This condition is also the natural necessary and sufficient condition for uniform boundedness and almost everywhere convergence of the gaussian means of functions in $L^p(v d\gamma)$. All these results are contained in Theorem 2.12.

In section 3 we also give some outlines of how to achieve similar results for others operators, like the Ornstein-Ulhenbeck maximal operator, Riesz Transforms and $g-$functions.

Next we introduce some notation and definitions and we state several known results which will be used often in the sequel.

Given B_1, B_2 Banach spaces, let $d\mu$ denote either the Lebesgue or the Gauss measure on $\mathbb{R}^n$ and N_t denote the region $\{(x, y) : |x - y| < \frac{t}{1+|x|+|y|}\}$. We shall consider a linear operator T defined in L^∞_{0,B_1}, the space of B_1- valued, compactly supported and essentially bounded functions, into the space of B_2-valued and strongly measurable functions on $\mathbb{R}^n$, satisfying the following assumptions:

1. T extends to a bounded operator either from $L^q_{B_1}(d\mu)$ into $L^q_{B_2}(d\mu)$ for some q, $1 < q < \infty$, or from $L^1_{B_1}(d\mu)$ into weak-$L^1_{B_2}(d\mu)$.
2. There exists a $\mathcal{L}(B_1, B_2)-$valued measurable function K, defined on the complement of the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$, such that for every function f in L^∞_{0,B_1}

$$Tf(x) = \int K(x, y)\, f(y) dy,$$

for all x outside the support of f; where the kernel K satisfies the estimates

$$\|K(x,y)\| \leq \frac{C}{|x-y|^n}, \qquad \|\partial_x K(x,y)\| + \|\partial_y K(x,y)\| \leq \frac{C}{|x-y|^{n+1}},$$

for all (x, y) in the local region N_2, $x \neq y$.

Following [GMST2] we introduce some definitions. For an operator T as above, given φ a smooth function on $\mathbb{R}^n \times \mathbb{R}^n$ such that $\varphi(x, y) = 1$ if $(x, y) \in N_1$, $\varphi(x, y) = 0$ for $(x, y) \notin N_2$ and

$$|\partial_x \varphi(x,y)| + |\partial_y \varphi(x,y)| \leq C\,|x-y|^{-1} \text{ if } x \neq y, \tag{1.1}$$

we define the **global** and the **local** parts of the operator T by

$$T_{glob} f(x) = \int K(x,y)(1-\varphi(x,y)) f(y) dy,$$
$$T_{loc} f(x) = Tf(x) - T_{glob} f(x).$$

Definition 1.2 *We shall say that an operator T defined on L^{∞}_{0,B_1} into the space of B_2-valued strongly measurable functions is* **local** *if its kernel is supported in N_2.*

We shall use the following results, which can be found in [GMST2].

Proposition 1.3 *If the operator T satisfies assumptions 1 and 2 as above, then the operator T_{loc} is bounded from $L^p_{B_1}(d\gamma)$ into $L^p_{B_2}(d\gamma)$ and from $L^p_{B_1}(dx)$ into $L^p_{B_2}(dx)$, for $1 < p < \infty$. Moreover T_{loc} is bounded from $L^1_{B_1}(d\mu)$ into weak-$L^1_{B_2}(d\mu)$, both, with respect to the Lebesgue and the Gauss measure.*

Proposition 1.4 *If S is a local operator, then the weak type $(1, 1)$ for Lebesgue and Gauss measures are equivalent.*

We shall also need the following theorem due to Rubio de Francia, see [GC,R] p. 554.

Theorem 1.5 *Let (X, μ) be a measure space, G a Banach space and T a sublinear operator from G into $L^s(X)$, which satisfies for some $s < p$, the folllowing inequality*

$$\left\| \left(\sum_j |Tf_j|^p \right)^{1/p} \right\|_{L^s(X)} \leq C_{p,s} \left(\sum_j \|f_j\|_G^p \right)^{1/p}$$

where $C_{p,s}$ is a constant depending on p and s. Then there exists a positive function u such that $u^{-1} \in L^{\frac{s}{p-s}}(X)$ and

$$\left(\int_X |Tf(x)|^p\, u(x) d\mu(x) \right)^{1/p} \leq \|f\|_G$$

We derive a simple consequence of Rubio de Francia theorem, since this will be the useful statement for us.

Corollary 1.6 *Let T be a sublinear operator such that*

$$\gamma\left\{x : \left(\sum_j |Tf_j(x)|^p\right)^{1/p} > \lambda\right\} \leq \frac{C}{\lambda}\int_{\mathbb{R}^n}\left(\sum_j |f_j(x)|^p\right)^{1/p} d\gamma(x)$$

then, for any v such that $\int_{\mathbb{R}^n} v^{-\frac{1}{p-1}}(x)d\gamma(x) < \infty$ and $s < p$, there exists a positive function u such that $u^{-1} \in L^{\frac{s}{p-s}}(X)$, and

$$\int_{\mathbb{R}^n} |Tf|^p u(x)d\gamma(x) \leq \int_{\mathbb{R}^n} |f|^p v(x)d\gamma(x)$$

Proof. Since $\gamma(\mathbb{R}^n)$ is finite and $s < p$, we have by using Kolmogorov's inequality that

$$\begin{aligned}
\left\|\left(\sum_j |Tf_j|^p\right)^{1/p}\right\|_{L^s(d\gamma)} &\leq C_s \sup_{\lambda>0} \lambda\gamma\left(\left\{x : \left(\sum_j \left|Tf_j(x)\right|^p\right)^{1/p} > \lambda\right\}\right) \\
&\leq C_s \int_{\mathbb{R}^n}\left(\sum_j \left|f_j(x)\right|^p\right)^{1/p} d\gamma(x) \\
&\leq C_s\left(\int_{\mathbb{R}^n}\sum_j \left|f_j(x)\right|^p v(x)d\gamma(x)\right)^{1/p} \\
&\quad \times\left(\int_{\mathbb{R}^n} v^{-\frac{1}{p-1}}(x)d\gamma(x)\right)^{1/p'} \\
&\leq C_s\left(\int_{\mathbb{R}^n}\sum_j \left|f_j(x)\right|^p v(x)d\gamma(x)\right)^{1/p} \\
&= C_s\left(\sum_j \|f_j\|^p_{L^p(vd\gamma)}\right)^{1/p}.
\end{aligned}$$

Therefore we are in the hypothesis of Theorem 1.5 with $G = L^p(vd\gamma)$, and the corollary follows. □

2 The centered maximal function

Given the centered Hardy-Littlewood maximal operator with respect to the gaussian measure γ, that is

$$M_\gamma f(x) = \sup_{r>0} |A_r f(x)| = \sup_{r>0} \left| \frac{1}{\gamma(B(x,r))} \int_{\mathbb{R}^n} f(y) d\gamma(y) \right|,$$

we consider the operators $M_{\gamma,1}$ and $M_{\gamma,2}$ defined by

$$M_{\gamma,1} f(x) = \sup_{r>0} \left| \frac{1}{\gamma(B(x,r))} \int_{B(x,r)} \chi_{\left\{|x-y|\leq \min\left(\frac{1}{2},\frac{1}{2|x|}\right)\right\}}(y) f(y) d\gamma(y) \right|$$

and

$$M_{\gamma,2} f(x) = \sup_{r>0} \left| \frac{1}{\gamma(B(x,r))} \int_{B(x,r)} \chi_{\{|x-y|>\min\left(\frac{1}{2},\frac{1}{2|x|}\right)\}}(y) f(y) d\gamma(y) \right|.$$

For the latter operator we have the following result

Theorem 2.7 *There exists a positive kernel $P(x,y)$ such that the operator defined by*
$Pf(x) = \int P(x,y) f(y) dy$ *satisfies*

$$M_{\gamma,2} f(x) \leq Pf(x) \text{ a.e.} x,$$

and P is of strong type $(1,1)$ with respect to the gaussian measure. Moreover, the kernel is supported in the region $|x-y| \geq \min(\frac{1}{2}, \frac{1}{2|x|})$, and the following estimates hold

$$\begin{aligned} P(x,y) &\leq C|y|^n e^{|x|^2-|y|^2-\frac{2\delta}{3}|y||x-y|} \quad \text{if} \quad |x| \geq 1 \quad \text{and} \quad |x-y| \leq \frac{|x|}{2} \\ &\leq C|x|^2 e^{|x|^2-|y|^2-|x|\varepsilon} \quad \text{if} \quad |x| \geq 1 \quad \text{and} \quad |x-y| \geq \frac{|x|}{2} \\ &\leq C e^{-|y|^2} \quad \text{if} \quad |x| \leq 1, \end{aligned}$$

for some positive ε and δ.

Proof. Let us denote by L the set $L = \{(x,y) : |x-y| \leq \min(\frac{1}{2}, \frac{1}{2|x|})\}$, then

$$M_{\gamma,2} f(x) = \sup_{r>0} \left| \frac{1}{\gamma(B(x,r))} \int_{B(x,r)} \chi_{L^c}(x,y) f(y) d\gamma(y)| \right|,$$

It is clear that this operator is bounded by the integral operator with kernel $H(x,y)$ given by

$$H(x,y) = \sup_{r>0} \frac{1}{\gamma(B(x,r))} \chi_{L^c}(x,y) \chi_{B(x,r)}(y) e^{-|y|^2}$$

Also, it is easy to see that

$$H(x, y) \le P(x, y) = \frac{1}{\gamma(B(x, |x - y|))} \chi_{L^c}(x, y) e^{-|y|^2}$$

Now we shall make some calculations, for $n \ge 3$, to estimate by below the Gauss measure of a general ball $B(x, R)$. Since the gaussian measure is rotation invariant, we can assume that $x = |x|e_n$, and then we have

$$\begin{aligned}
\int_{\{|z-x|<R\}} e^{-|z|^2} dz &= \int_{\{|u|<R\}} e^{-|u+x|^2} du = \int_{\{|u|<R\}} e^{-|u|^2-|x|^2-2<u,x>} du \\
&= \int_0^{2\pi} \int_0^{\pi} \dots \int_0^{\pi} \int_0^{R} e^{-r^2-|x|^2-2r\cos\varphi_1|x|} r^{n-1} \sin^{n-2}\varphi_1 \\
&\quad \dots \sin\varphi_{n-2} d\varphi_1 \dots d\varphi_{n-1} dr \\
&\ge e^{-|x|^2} \int_0^{R} \int_0^{2\pi} \int_0^{\pi} \dots \int_0^{\pi} \int_0^{\pi/2} e^{-r^2} e^{2r|x|\cos\varphi_1} r^{n-1} \sin^{n-2}\varphi_1 \\
&\quad \dots \sin\varphi_{n-2} d\varphi_1 \dots \varphi_{n-1} dr \\
&\ge C_{n,\varphi_0} e^{-|x|^2} e^{-R^2} \int_0^{R} \int_{\varphi_0}^{\pi/2} e^{2r|x|\cos\varphi_1} r \sin\varphi_1 d\varphi_1 r^{n-2} dr \\
&= C_{n,\varphi_0} \frac{e^{-|x|^2} e^{-R^2}}{|x|} \int_0^{R} \int_0^{2r|x|\cos\varphi_0} e^u du r^{n-2} dr \\
&\ge C_{n,\varphi_0} \frac{e^{-|x|^2} e^{-R^2} R^{n-2}}{|x|} \int_{R/2}^{R} \int_0^{2r|x|\cos\varphi_0} e^u du dr,
\end{aligned}$$

where φ_0, $0 < \varphi_0 < \pi/2$ denotes a fixed angle and we have done the change of variables $u = 2r|x|\cos\varphi_1$. If we now assume that

$$R|x|\cos\varphi_0 \ge \beta > 0, \tag{2.8}$$

the double integral above can be estimated by below by

$$\frac{(1 - e^{-\beta})^2}{2|x|\cos\varphi_0} e^{2R|x|\cos\varphi_0}.$$

Therefore in this case, we get the estimate

$$\gamma(B(x, R)) \ge C_{n,\varphi_0,\beta} \frac{e^{-|x|^2} e^{-R^2} R^{n-2}}{|x|^2} e^{2R|x|\cos\varphi_0} \tag{2.9}$$

for $n \ge 3$. It is easy to check that for $n = 2$ we get the same type of estimate, while for $n = 1$ we obtain

$$\gamma(B(x, R)) \ge C_{n,\varphi_0,\beta} \frac{e^{-x^2} e^{-R^2}}{|x|} e^{2R|x|}.$$

In order to estimate the kernel, we consider first the case when $\min(\frac{1}{2}, \frac{1}{2|x|}) = \frac{1}{2|x|}$, that is $|x| \geq 1$. Then for $(x, y) \in L^c$, the condition 2.8 is satisfied for $R = |x - y|$, so we may apply estimate 2.9 with $R = |x - y|$. Then we have

$$P(x, y) \leq C_{n,\varphi_0,\beta} \frac{e^{|x|^2} e^{|x-y|^2} |x|^2 e^{-2|x-y||x| \cos\varphi_0}}{|x - y|^{n-2}} e^{-|y|^2}$$
$$\leq C_{n,\varphi_0,\beta} e^{|x|^2} e^{|x-y|^2} |x|^n e^{-2|x-y||x| \cos\varphi_0} e^{-|y|^2},$$

where we have used $|x - y| > \frac{1}{2|x|}$. We observe that even the estimate 2.9 for the case $n = 1$ was sligthly different we can also arrive to the last inequality.

To further estimate the kernel we distinguish two cases.

First suppose $|x - y| \leq \frac{|x|}{2}$, which implies $\frac{|x|}{2} \leq |y| \leq \frac{3|x|}{2}$. Therefore, under these assumptions, we have

$$|x|^n e^{|x-y|^2} e^{-2|x-y||x| \cos\varphi_0} = C|y|^n e^{|x-y|(|x-y|-2|x| \cos\varphi_0)}$$
$$\leq C|y|^n e^{-|x-y||x|(2\cos\varphi_0 - \frac{1}{2})} \leq C|y|^n e^{-\frac{2}{3}\delta|x-y||y|},$$

where δ is some positive number, provided we take φ_0 small enough.

Second, let us assume $|x - y| > \frac{|x|}{2}$. This, together with $|x| \geq 1$, gives $|x - y| \geq \frac{1}{2}$ and hence

$$\gamma\left(B(x, |x - y|)\right) \geq \gamma\left(B\left(x, \frac{1}{2}\right)\right) \geq C_{n,\varphi_0,\beta} \frac{e^{-|x|^2} e^{|x| \cos\varphi_0}}{|x|^2}.$$

Consequently

$$P(x, y) \leq C_{n,\varphi_0,\beta} e^{|x|^2} e^{-|y|^2} e^{-|x| \cos\varphi_0} |x|^2.$$

We point out that for $n = 1$ this estimate also holds since $|x| \geq 1$.

Finally if $\min(\frac{1}{2}, \frac{1}{2|x|}) = \frac{1}{2}$, we have $|x| \leq 1$. In this case $B(x, \frac{1}{2}) \subset B(x, |x - y|)$ and therefore

$$\gamma(B(x, |x - y|)) \geq \int_{|x-z|<\frac{1}{2}} e^{-|z|^2} dz \geq e^{-4} |B(x, 1)| = C$$

This ends the proof of the estimates stated in the theorem. Let us see now that the operator P is of strong type $(1, 1)$.

$$\int\int P(x, y)|f(y)| dy d\gamma(x) \leq C \int |f(y)| e^{-|y|^2} \int \left(\int |y|^n e^{-\frac{2}{3}\delta|y||x-y|} dx\right) dy$$
$$+ C \int |f(y)| e^{-|y|^2} \left(\int |x|^2 e^{-|x|\varepsilon} dx\right) dy$$
$$+ C \int |f(y)| e^{-|y|^2} \left(\int e^{-|x|^2} dx\right) dy$$
$$\leq C \|f\|_{L^1(d\gamma)}.$$

where in the innner integral of the first term we have perfomed the change of variables $u = (x - y)|y|$. □

Now we are going to combine the estimate just proved for $M_{\gamma,2}$ with the results for local operators stated in Proposition 1.4 in order to get a vector valued inequality for the centered maximal operator M_γ.

Theorem 2.10 *Given* $1 < p < \infty$, *the following inequality holds*

$$\gamma\left(\left\{x : \left(\sum_j |M_\gamma f_j(x)|^p\right)^{1/p} > \lambda\right\}\right) \leq \frac{C}{\lambda}\int_{\mathbb{R}^n}\left(\sum_j |f_j(x)|^p\right)^{1/p} d\gamma(x).$$

Proof. It is clear that

$$M_{\gamma,1}f(x) = \sup_{0<r\leq\min\left(\frac{1}{2},\frac{1}{2|x|}\right)}\left|\frac{1}{\gamma(B(x,r))}\int_{B(x,r)} f(y)d\gamma(y)\right|.$$

Moreover if $z \in B(x,r), r \leq \min(\frac{1}{2}, \frac{1}{2|x|})$, we get that there exists a constant C such that $C^{-1}e^{-|z|^2} \leq e^{-|x|^2} \leq Ce^{-|z|^2}$. To see this obseve that $e^{-|z|^2} = e^{-|z-x|^2}e^{-2<z-x,x>}e^{-|x|^2}$, and that $| < z - x, x > | \leq \frac{1}{2}$, and $|z - x| \leq r \leq \frac{1}{2}$. Then we get $M_{\gamma,1}f(x) \leq CMf(x)$, where M is the Hardy-Littlewood operator with respect to the Lebesgue measure. Then by the well known result for the operator M, we get that $M_{\gamma,1}$ satisfies

$$\left|\left\{x : \left(\sum_j |M_{\gamma,1}f_j(x)|^p\right)^{1/p} > \lambda\right\}\right| \leq \frac{C}{\lambda}\int_{\mathbb{R}^n}\left(\sum_j |f_j(x)|^p\right)^{1/p} dx. \tag{2.11}$$

If now we consider the $l^\infty-$valued version of the operator $M_{\gamma,1}$ given by

$$Vf(x) = \left\{\frac{1}{\gamma(B(x,r))}\int_{B(x,r)} \chi_{\left\{(x,y):|x-y|\leq\min\left(\frac{1}{2},\frac{1}{2|x|}\right)\right\}}(y)f(y)d\gamma(y)\right\}_r,$$

inequality 2.11 says that V is bounded from $L^1_{l^p}(dx)$ into weak-$L^1_{l^p(l^\infty)}(dx)$. We notice that V is a local operator since it is easy to check that $\{(x, y) : |x - y| \leq \min(\frac{1}{2}, \frac{1}{2|x|})\} \subset N_2$. Therefore we can apply Proposition 1.4 to get that V is bounded from $L^1_{l^p}(d\gamma)$ into weak-$L^1_{l^p(l^\infty)}(d\gamma)$, but this, in turn, implies that $M_{\gamma,1}$ satisfies the desired inequality.

On the other hand it is well known that a positive linear operator bounded from $L^1(d\mu)$ into $L^1(d\mu)$ with $d\mu$ any measure, has a vector valued bounded extension from $L^1_B(d\mu)$ into $L^1_B(d\mu)$ for any Banach space B, see [RT]. Applying Theorem 2.7 and this remark to the operator P, to the Banach space $B = l^p$ and to the gaussian measure, we have that

$$\int \left(\sum_j |M_{\gamma,2} f_j(x)|^p \right)^{1/p} d\gamma(x) \leq \int \left(\sum_j |P f_j(x)|^p \right)^{1/p} d\gamma(x)$$
$$\leq C \int \left(\sum_j |f_j(x)|^p \right)^{1/p} d\gamma(x),$$

giving the result. □

The Theorem we just proved allows us to use the connection between vector-valued estimates and weighted L^p inequalities, as stated in Corollary 1.6

Theorem 2.12 *Given $1 < p < \infty$ and v a positive measurable function, the following conditions are equivalent*

(i) *For every $f \in L^p(v d\gamma)$, $\lim_{r\to 0} A_r f(x) = f(x)$ a.e. x, where $A_r f(x) = \frac{1}{\gamma(B(x,r))} \int_{B(x,r)} f(y) d\gamma(y)$.*

(ii) *For every $f \in L^p(v d\gamma)$, $M_\gamma f(x) < \infty$ a.e. x.*

(iii) *There exists a positive measurable function u and a constant C such that for every $f \in L^p(v d\gamma)$ and all $\lambda > 0$, we have*

$$\int_{\{x\in\mathbb{R}^n : M_\gamma f(x) > \lambda\}} u(x) d\gamma(x) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p v(x) d\gamma(x).$$

(iv) *There exists a positive measurable function u and a constant C such that for every $f \in L^p(v d\gamma)$ we have*

$$\int_{\mathbb{R}^n} M_\gamma f(x)^p u(x) d\gamma(x) \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x) d\gamma(x).$$

(v) *There exists a positive measurable function u and a constant C such that for every $f \in L^p(v d\gamma)$ and all $\lambda > 0$, we have*

$$\sup_{r>0} \int_{\{x\in\mathbb{R}^n : |A_r f(x)| > \lambda\}} u(x) d\gamma(x) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p v(x) d\gamma(x).$$

(vi) *There exists a positive measurable function u and a constant C such that for every $f \in L^p(v d\gamma)$ we have*

$$\sup_{r>0} \int_{\mathbb{R}^n} |A_r f(x)|^p u(x) d\gamma(x) \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x) d\gamma(x).$$

(vii) $v^{-\frac{1}{p-1}} \in L^1(d\gamma)$.

Moreover the weight u whose existence is guaranteed in (iii),(iv),(v) or (vi) satisfies $\|u^{-1}\|_{L^{\frac{s}{p-s}}(d\gamma)} < \infty$ *for every* $0 < s < 1$.

Proof. We shall prove this theorem as follows:

(vii) ⇒ (iv) ⇒ (vi) ⇒ (v) ⇒ (vii);

(iv) ⇒ (iii) ⇒ (i) ⇒ (ii) ⇒ (iii) ⇒ (v).

We first observe that implications (iv) ⇒ (vi), (vi) ⇒ (v), (iv) ⇒ (iii), (i) ⇒ (ii) and (iii) ⇒ (v) are obvious. Since $\lim_{r\to\infty} A_r f(x) = f(x)$ a.e.x. for every $f \in L^p(vd\gamma) \cap L^1(d\gamma)$ which is a dense subset of $L^p(vd\gamma)$, the Banach Principle says that (iii) ⇒ (i). In order to see that (v) ⇒ (vii) we first observe that there is a positive radius S such that $0 < \int_{B(0,S)} u(y)d\gamma(y)$. Now for any $R > S$ we have $A_{2R}(\chi_{B(0,R)}|f|)(y) \geq (\gamma(\mathbb{R}^n))^{-1} \int |\chi_{B(0,R)}(z)f(z)|d\gamma(z)$, $y \in B(0,S)$. Applying the hypothesis, we have

$$0 < \int_{B(0,S)} u(y)d\gamma(y) \leq \int_{\{y: A_{2R}(\chi_{B(0,R)}|f|)(y) \geq (\gamma(\mathbb{R}^n))^{-1}\int |\chi_{B(0,R)} f|d\gamma\}} u(y)d\gamma(y)$$
$$\leq \frac{C}{\left(\int \left|\chi_{B(0,R)} f\right| d\gamma\right)^p} \int \left|\chi_{B(0,R)}(y)f(y)\right|^p v(y)d\gamma(y).$$

Hence we get $(\int \left|\chi_{B(0,R)}(y)f(y)\right| d\gamma(y))^p \leq C \int |\chi_{B(0,R)} f(y)|^p v(y)\, d\gamma(y)$, for every $R > S$, in particular this implies that $(\int |f(y)|d\gamma(y))^p \leq C \int |f(y)|^p v(y)d\gamma(y)$. If we set $f = gv^{-1/p}$ the last inequality can be written as $(\int |g(y)|v^{-1/p}(y)d\gamma(y))^p \leq C \int |g(y)|^p d\gamma(y)$, and this implies that $v^{-1/p} \in L^{p'}(d\gamma)$ which is (vii).

Next we show that (vii) ⇒ (iv)

By Theorem 2.10, the operator M_γ satisfies the hypotheses of Corollary 1.6, therefore there exists a weight u such that

$$\int_{\mathbb{R}^n} \left|M_\gamma f(x)\right|^p u(x)d\gamma(x) \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x)d\gamma(x).$$

and $\|u^{-1}\|_{L^{\frac{s}{p-s}}(d\gamma)} < \infty$ for every $0 < s < 1$.

Now, we shall prove (ii) ⇒ (iii). If $1 < p \leq 2$ we can use Nikishin's Theorem, see VI.1.4 and VI.2.7 in [GC,R] obtaining (iii) for this range of p. If $2 < p$ we write $w = v^{\frac{1}{p-1}}$ and we have

$$L^2(wd\gamma) \subset L^p(vd\gamma) + L^1(d\gamma),$$

since any positive function $f \in L^2(wd\gamma)$ can be decomposed as $f = g+h$, with $g(x) = f(x)$ if $f(x) \leq v^{-\frac{1}{p-1}}(x)$ and 0 otherwise. By hypothesis $M_\gamma g(x) < \infty$, moreover as $h \in L^1(d\gamma)$ we also have $M_\gamma h(x) < \infty$ therefore $M_\gamma f(x) < \infty$.

Applying Nikishin's theorem again with $p = 2$ and the weight w, there exists a positive function u such that (iii) holds with $p = 2$ and w. Since we already proved that (iii) $\Rightarrow$ (v) and (v) $\Rightarrow$ (vii) we can conclude that the weight w satisfies $\int w^{-1}(x)d\gamma(x) < \infty$ and that means $\int v^{-\frac{1}{p-1}}d\gamma(x) < \infty$. $\square$

3 Weighted inequalities for related operators

What we have done for the maximal operator can also be carried out for other operators. The crucial step in our development has been the vector-valued inequality in Theorem 2.10. We remind that its proof relies on an appropiate analysis of the "local" and "global" parts of the operator, allowing suitable vector-valued extensions. Now, as an example, we work out this program for the Ornstein-Uhlenbeck maximal operator, namely

$$O^* f(x) = \sup_{0\le r<1} |O_r f(x)| = \sup_{0\le r<1} \left| \int_{\mathbb{R}^n} \mathcal{M}_r(x, y) f(y) dy \right|,$$

where

$$\mathcal{M}_r(x, y) = \pi^{-n/2} \left(1 - r^2\right)^{-n/2} \exp\left(-\frac{|rx - y|^2}{1 - r^2}\right).$$

Theorem 3.13 *Given* $1 < p < \infty$, *the following inequality holds*

$$\gamma\left(\left\{x : \left(\sum_j |O^* f_j(x)|^p\right)^{1/p} > \lambda\right\}\right)$$

$$\le \frac{C}{\lambda} \int_{\mathbb{R}^n} \left(\sum_j |f_j(x)|^p\right)^{1/p} d\gamma(x). \tag{3.14}$$

Proof. Given the operator O^*, we consider its vector valued version given by

$$Wf(x) = \left\{ \int_{\mathbb{R}^n} \mathcal{M}_r(x, y) f(y) dy \right\}_r.$$

Since O^* is of weak type $(1, 1)$ with respect to the Gaussian measure, see [P], we have that W is bounded from $L^1(d\gamma)$ into weak$-L^1_{l^\infty}(d\gamma)$ and so it is W_{loc}. By proposition 1.4 the operator W_{loc} is bounded from $L^1(dx)$ into weak$-L^1_{l^\infty}(dx)$. Moreover the kernel satisfies assumption 2; in particular it is a Calderón-Zygmund operator and consequently bounded from $L^1_{l^p}(dx)$ into weak-$L^1_{l^p(l^\infty)}(dx)$, see [RRT]. Applying again Proposition 1.4 we get that W_{loc} is bounded from $L^1_{l^p}(d\gamma)$ into weak-$L^1_{l^p(l^\infty)}(d\gamma)$. On the other hand

$$\left\| W_{glob} f(x) \right\|_{l^\infty} \le Qf(x)$$
$$(3.15) \qquad = \int_{\mathbb{R}^n} \sup_{0 \le r < 1} |(1 - \varphi(x, y))\mathcal{M}_r(x, y) f(y)| \, dy.$$

Since this operator Q is positive and of weak type $(1, 1)$ with respect to the gaussian measure, see [P], it has a bounded extension from $L^1_{l^p}(d\gamma)$ into weak-$L^1_{l^p}(d\gamma)$. It follows from 3.15 that W_{glob} is bounded from $L^1_{l^p}(d\gamma)$ into weak-$L^1_{l^p(l^\infty)}(d\gamma)$. This ends the proof by observing that $O^* f(x) = \|Wf(x)\|_{l^\infty}$. □

Now we are in position to state a parallel result to Theorem 2.12 for the Ornstein-Uhlenbeck maximal operator.

Theorem 3.16 *Given* $1 < p < \infty$ *and* v *a positive measurable function, all the conditions (i) to (vii) of the Theorem 2.12 are equivalent if we change the operators* A_r, M_γ *by the operators* O_r, O^*, *and* $\lim_{r\to 0} A_r$ *by* $\lim_{r\to 1} O_r$.

Moreover the weight u *whose existence is guaranteed satisfies* $\|u^{-1}\|_{L^{\frac{s}{p-s}}(d\gamma)} < \infty$ *for every* $0 < s < 1$.

Proof. The proof follows the lines of that of Theorem 2.12. We shall only point out where some differences appear. First, we use Theorem 3.13 in order to prove (vii) ⇒ (iv). Observe that $O_0 f(x) \ge C \int f(y)\gamma(y)dy$ for every $x \in \mathbb{R}^n$ and then we can repeat the argument of (v) ⇒ (vii). On the other hand it is well known, see [M], that $\lim_{r\to 1} O_r f(x) = f(x)$ for every $x \in \mathbb{R}^n$ and f a continuous function with compact support, therefore the Banach Principle can be applied as in the case of the A_r means. □

Remark 3.17 We point out that inequality 3.13 holds true for the first and second order Riesz Transforms and also for some suitable $g-$functions. As in the case of the Ornstein-Ulhenbeck operator the proofs rely again in the analysis of the local and global parts. As in the previous cases it can be shown that their local parts are essentially Calderón-Zygmund operators (possibly with vector valued kernels). On the other hand, the global parts are known to be controlled by positive linear operators of weak type $(1, 1)$, see [MPS] [GMST1] [FGS]. In this way both parts can be extended boundedly to l^p-valued functions.

As a consequence we obtain, for example, that for weights v satisfying $v^{-\frac{1}{p-1}} \in L^1(d\gamma), 1 < p < \infty$, there exists a weight u such that any of the above operators is bounded from $L^p(vd\gamma)$ into $L^p(ud\gamma)$.

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