THE LIMIT CASE OF THE CESÀRO-α CONVERGENCE OF THE ERGODIC AVERAGES AND THE ERGODIC HILBERT TRANSFORM

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Abstract. Recently, M. D. Sarrión and the authors gave a sufficient condition on invertible Lamperti operators on L^p which guarantees the convergence in the Cesàro-α sense of the ergodic averages and the ergodic Hilbert transform for all $f \in L^p$ with $p > \frac{1}{1+\alpha}$ and $-1 < \alpha \leq 0$. The result does not hold for the space $L^{\frac{1}{1+\alpha}}$. In this paper we give a positive result for the smaller Lorentz space $L_{\frac{1}{1+\alpha},1}$.

1. Introduction.

Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space. In [14], R. Sato studied the almost everywhere and the L^p-norm convergence of the ergodic averages $R_n f = \frac{1}{n}\sum_{k=0}^{n-1} T^k f$ and the ergodic partial sums $H_n f = \sum_{k=1}^{n} \frac{T^k f - T^{-k} f}{k}$ corresponding to the ergodic Hilbert transform. This study was done for invertible Lamperti operators on $L^p(\mu)$, $1 < p < \infty$, (see [9] for the definition and the properties) such that the linear modulus $|T|$ of T satisfies the following norm condition: $\sup_{n\geq 0} \left\| \frac{1}{2n+1} \sum_{k=-n}^{n} |T|^k \right\|_p < \infty$.

The convergence of $\{R_n f\}$ is the Cesàro-1 $(C, 1)$ convergence of the sequence $\{T^n f\}$ and the convergence of the partial sums $\{H_n f\}$ is the $(C, 0)$ summability of the ergodic Hilbert transform, i.e., of the series $\sum_{k=1}^{\infty} \frac{T^k f - T^{-k} f}{k}$. The general aim of this paper is to continue the study of the Cesàro-α convergence of $\{T^n f\}$ with $0 < \alpha < 1$ and $\{H_n f\}$ with $-1 < \alpha < 0$, which was initiated in [3], [4] and [8].

Recently the authors and M. D. Sarrión studied in [2], in the setting of Lamperti operators (see also [10]), the convergence in the Cesàro-α sense (see [17] or [5]) of the sequences $\{T^n f\}$ and $\{H_n f\}$. In fact, for $-1 < \alpha < 0$, it was studied the $(C, 1+\alpha)$ convergence of $\{T^n f\}$ and the (C, α) summability of the ergodic Hilbert transform, i.e., the limits of the following sequences

$$R_{n,1+\alpha} f = \frac{1}{A_n^{1+\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha} T^k f \quad \text{and} \quad H_{n,\alpha} f = \frac{1}{A_n^{\alpha}} \sum_{k=1}^{n+1} A_{n+1-k}^{\alpha} \left(\frac{T^k f - T^{-k} f}{k} \right)$$

where the Cesàro numbers A_n^α are defined by $A_n^\alpha = \frac{(\alpha+1)\cdots(\alpha+n)}{n!}$ and $A_0^\alpha = 1$. More precisely, in [2] it was proved the following theorem.

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Theorem 1.1 [2]**.** *Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space, $-1 < \alpha < 0$, $\frac{1}{1+\alpha} < p < \infty$ and T an invertible Lamperti operator on $L^p(\mu)$ such that the linear modulus $|T|$ of T satisfies*

$$\sup_{n\geq 0} \left\| \frac{1}{2n+1} \sum_{k=-n}^{n} |T|_{1+\alpha}^{k} \right\|_{p(1+\alpha)} < \infty, \tag{1.1}$$

where T_β is the linear operator given by $T_\beta f = [T(f^\beta)]^{1/\beta}$ for $f \geq 0$. Then, for all $f \in L^p(\mu)$, there exist the limits

$$\lim_{n\to\infty} R_{n,1+\alpha} f \quad \text{and} \quad \lim_{n\to\infty} H_{n,\alpha} f$$

almost everywhere and in the strong operator topology.

The study of the averages $R_{n,1+\alpha}f$ in Theorem 1.1 is a simple modification of the corresponding result proved in [10] in the setting of the positive linear operators with positive inverse. It is clear from the proofs of Theorem 1.1 in [2] or Theorem 3.1 in [10] that for the convergence of the averages $R_{n,1+\alpha}f$ we actually need a weaker condition than the assumption (1.1).

Theorem 1.2 ([10] **or** [2]**).** *Let $(X, \mathcal{F}, \mu)$, α and p be as in Theorem 1.1. Let T be an invertible Lamperti operator on $L^p(\mu)$ such that*

$$\sup_{n\geq 0} \left\| \frac{1}{n+1} \sum_{k=0}^{n} |T|_{1+\alpha}^{k} \right\|_{p(1+\alpha)} < \infty. \tag{1.2}$$

Then, for all $f \in L^p(\mu)$, there exist the limits $\lim_{n\to\infty} R_{n,1+\alpha}f$ almost everywhere and in the strong operator topology.

Theorems 1.1 and 1.2 do not hold in the limit case $p = \frac{1}{1+\alpha}$. Y Deniel proved in [4] this fact for the averages $R_{n,1+\alpha}$ and the corresponding result for $H_{n,\alpha}$ was proved in [2].

On the other hand, in [12], R. Sato studied the limit case $p = 1$ of his result, mentioned at the beginning of the introduction. He studied the ergodic Hilbert transform but it is worth noting that the result also holds for the averages $R_n f$. The theorem proved by Sato is the following.

Theorem 1.3 [12]**.** *Let T be an invertible Lamperti operator on $L^1(\mu)$ such that*

$$\sup_{n\in\mathbb{Z}} ||T^n||_1 < \infty \tag{1.3}$$

and

$$\sup_{n\in\mathbb{Z}} ||T^n||_\infty < \infty. \tag{1.4}$$

Then the limit $\lim_{n\to\infty} H_n f$ exists almost everywhere for all $f \in L^1(\mu)$.

The goal of this paper is to give a positive result of Theorems 1.1 and 1.2 in the the limit case $p = \frac{1}{1+\alpha}$. In both cases we shall obtain almost everywhere convergence for functions f in the Lorentz space $L_{\frac{1}{1+\alpha},1}(\mu) = \{f : ||f||_{\frac{1}{1+\alpha},1;\mu} = \int_0^\infty \left(\lambda_f(t)\right)^{1+\alpha} dt < \infty\}$, where $\lambda_f(t) = \mu(\{x : |f(x)| > t\})$ is the distribution function of f. More precisely, we prove the following theorems.

Theorem 1.4. *Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space, $-1 < \alpha \leq 0$ and T an invertible Lamperti operator on $L^{\frac{1}{1+\alpha}}(\mu)$ such that*

$$\sup_{n\geq 0} \left\| \frac{1}{n+1} \sum_{k=0}^{n} |T|_{1+\alpha}^{k} \right\|_1 = M_1 < \infty \tag{1.5}$$

and

$$\sup_{n\in\mathbb{Z}} ||T^n||_\infty = M_\infty < \infty. \tag{1.6}$$

Then the limit $\lim_{n\to\infty} R_{n,1+\alpha} f$ *exists almost everywhere, for any f in $L_{\frac{1}{1+\alpha},1}(\mu)$.*

Theorem 1.5. *Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space, $-1 < \alpha \leq 0$ and T an invertible Lamperti operator on $L^{\frac{1}{1+\alpha}}(\mu)$ such that*

$$\sup_{n\geq 0} \left\| \frac{1}{2n+1} \sum_{k=-n}^{n} |T|_{1+\alpha}^{k} \right\|_1 = M_2 < \infty \tag{1.7}$$

and

$$\sup_{n\in\mathbb{Z}} ||T^n||_\infty = M_\infty < \infty. \tag{1.6}$$

Then the limit $\lim_{n\to\infty} H_{n,\alpha} f$ *exists almost everywhere, for any f in $L_{\frac{1}{1+\alpha},1}(\mu)$.*

Notice that, Theorem 1.4 generalizes Corollary 1 in [3]. We have also that Theorem 1.3 is obtained from Theorem 1.5 when $\alpha = 0$, since condition (1.3) implies condition (1.7).

Throughout this paper α will be a number such that $-1 < \alpha \leq 0$ and the letter C will mean a positive constant non necessarily the same at each ocurrence.

2. Preliminary results.

This section is devoted to establish some results and several properties of the Lamperti operators that we shall need in the proof of the theorems.

Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space, $-1 < \alpha \leq 0$ and T an invertible Lamperti operator on $L^{\frac{1}{1+\alpha}}(\mu)$. T is called a Lamperti operator on $L^{\frac{1}{1+\alpha}}(\mu)$ if T is a bounded linear operator on $L^{\frac{1}{1+\alpha}}(\mu)$ such that T separates supports. It follows [9] that there exists a positive linear operator Φ on the space of measurable functions, induced by a σ-endomorphism of the σ-algebra $\mathcal{F}$, also denoted by Φ, such that

$$\Phi(\chi_E) = \chi_{\Phi(E)},$$

and a measurable function h on X, with $0 < |h(x)| < \infty$ a.e. on X, such that

$$Tf(x) = h(x)\Phi f(x), \qquad \text{for all } f \in L^{\frac{1}{1+\alpha}}(\mu).$$

The operator Φ verifies that $\Phi 1 = 1$ and $\Phi(|f|^r) = |\Phi(f)|^r$, for any positive r. On the other hand, since T^{-1} is also an invertible Lamperti operator, it is clear that if $h_1(x) = h(x)$ then the powers of T and T^{-1} are of the form

$$T^j f(x) = h_j(x)\Phi^j f(x), \quad \text{for all } j \in \mathbb{Z},$$

where the functions h_j verify also that $0 < |h_j(x)| < \infty$ a.e. on X and $h_{j+k} = h_j \Phi^j(h_k)$, for all $j, k \in \mathbb{Z}$. It is easy to see that

$$||T^j||_\infty = ||h_j||_\infty, \quad \text{for all } j \in \mathbb{Z}. \tag{2.1}$$

Other important property is the existence of a sequence of positive measurable functions $\{w_j\}$ such that $w_j \Phi^j(w_{-j}) = 1$ and

$$\int \left|T^j(f)\right|^{\frac{1}{1+\alpha}} w_j \, d\mu = \int |f|^{\frac{1}{1+\alpha}} \, d\mu. \tag{2.2}$$

For such an operator T, there exists a positive linear operator $|T|$ on $L^{\frac{1}{1+\alpha}}(\mu)$ such that $|Tf| = |T||f|$, for every $f \in L^{\frac{1}{1+\alpha}}(\mu)$. $|T|$ is also an invertible Lamperti operator, called the linear modulus of T. We can easily see that $|T|f = |h|\Phi f$ and for each integer j, $|T|^j f = |h_j|\Phi^j f$. Observe that $|T|^j_{1+\alpha} f = |h_j|^{\frac{1}{1+\alpha}} \Phi^j(f)$.

From (2.1) and the relation $h_j \Phi^j(h_{-j}) = 1$ we can prove easily the following proposition that we will use in the proof of the theorems.

Proposition 2.1. *Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space, $-1 < \alpha \leq 0$ and T an invertible Lamperti operator on $L^{\frac{1}{1+\alpha}}(\mu)$ such that T verifies (1.6). Then, for all $j \in \mathbb{Z}$ and almost every $x \in X$*

$$\frac{1}{M_\infty} \leq |h_j(x)| \leq M_\infty.$$

A key fact in the proof of the theorems is the relation between the assumptions on the operator and the classes of weights $A_1(\mathbb{Z})$ and $A_1^+(\mathbb{Z})$.

Definition 2.2. *Let v be a nonnegative function on the integers.*
(i) $v \in A_1(\mathbb{Z})$ if there exists a constant C such that

$$\sup_{n \geq 0} \frac{1}{2n+1} \sum_{k=-n}^{n} v(i+k) \leq C v(i), \quad \textit{for all } i \in \mathbb{Z}.$$

(ii) $v \in A_1^+(\mathbb{Z})$ if there exists a constant C such that

$$\sup_{n \geq 0} \frac{1}{n+1} \sum_{k=0}^{n} v(i-k) \leq C v(i), \qquad \textit{for all } i \in \mathbb{Z}.$$

The next proposition establishes the relations between the assumptions (1.5) and (1.7) and the classes $A_1(\mathbb{Z})$ and $A_1^+(\mathbb{Z})$.

Proposition 2.3. *Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space, $-1 < \alpha \leq 0$ and T an invertible Lamperti operator on $L^{\frac{1}{1+\alpha}}(\mu)$.*
(i) If T verifies (1.5) then, for almost every $x \in X$, the functions $w_x(i) = w_i(x) \in A_1^+(\mathbb{Z})$, with a constant independent of x.
(ii) If T verifies (1.7) then, for almost every $x \in X$, the functions $w_x(i) = w_i(x) \in A_1(\mathbb{Z})$, with a constant independent of x.

Proof. (i) By the assumption (1.5) applied to a function $f = |T|^i_{1+\alpha}(g)$ with $g \geq 0$, $g \in L^1(\mu)$ and $i \in \mathbb{Z}$, we have

$$\frac{1}{n+1}\sum_{k=0}^{n}\int_X \left[|T|^{i+k}\left(g^{1+\alpha}\right)\right]^{\frac{1}{1+\alpha}} d\mu \leq M_1 \int_X \left[|T|^{i}\left(g^{1+\alpha}\right)\right]^{\frac{1}{1+\alpha}} d\mu,$$

for all $n \in \mathbb{N}$. On the other hand, the property (2.2) of T for $f \geq 0$ can be written as $\int \left[|T|^j(f)\right]^{\frac{1}{1+\alpha}} d\mu = \int f^{\frac{1}{1+\alpha}} w_{-j}\, d\mu$ for all $j \in \mathbb{Z}$. Then, applying this equality in both integrals of the above inequality we have

$$\int_X g(x)\left[\frac{1}{n+1}\sum_{k=0}^{n} w_{-i-k}(x)\right] d\mu(x) \leq M_1 \int_X g(x) w_{-i}(x)\, d\mu(x),$$

for all $g \geq 0$, $g \in L^1(\mu)$ and all $n \in \mathbb{N}$. Now (i) follows immediately with constant M_1.

The proof of (ii) is similar. Thus, we omit it.

In the proofs of the theorems we shall use weighted inequalities for the discrete maximal Hilbert transform

$$h^* a(i) = \sup_{n\geq 1}\left|\sum_{k=1}^{n}\frac{a(i+k)-a(i-k)}{k}\right|$$

and the maximal operator $m^+_{1+\alpha}$ associated with the Cesàro averages of functions defined on the integers

$$m^+_{1+\alpha}a(i) = \sup_{n\geq 0}\frac{1}{A_n^{1+\alpha}}\sum_{k=0}^{n} A^{\alpha}_{n-k}|a(i+k)|.$$

For the operator h^* we have the following result (see [6]).

Theorem 2.4 [6]**.** *Let v be a positive measurable function on the integers. Then, $v \in A_1(\mathbb{Z})$, if and only if, there exists a constant C such that*

$$\sum_{\{i:h^*a(i)>\lambda\}} v(i) \leq \frac{C}{\lambda}\sum_{i=-\infty}^{\infty}|a(i)|v(i)$$

for all $\lambda > 0$ and all functions a on $\mathbb{Z}$.

On the other hand, for the operator $m^+_{1+\alpha}$, we have the following result which is Theorem 2.7 proved in [11] applied in our setting (see Proposition 2.4, Example 2.5 (3) and Remark 2.8 in [11]).

Theorem 2.5 [11]**.** *Let $-1 < \alpha \leq 0$ and v be a positive function on the integers. Then, the following statements are equivalent.*

(i) The operator $m^+_{1+\alpha}$ applies the space $L_{\frac{1}{1+\alpha},1}(v)$ into the space $L_{\frac{1}{1+\alpha},\infty}(v)$, that is, there exists a constant C such that

$$\sum_{\{i:m^+_{1+\alpha}a(i)>\lambda\}} v(i) \leq \frac{C}{\lambda^{\frac{1}{1+\alpha}}}\left(\int_0^\infty\left(\sum_{\{i:a(i)>t\}} v(i)\right)^{1+\alpha} dt\right)^{1/1+\alpha}, \tag{2.3}$$

for all functions a on $\mathbb{Z}$.

(ii) *There exists a constant C such that*

$$\left\|\chi_{[r,s]}\right\|_{\frac{1}{1+\alpha},1;v}\left\|\frac{A^{\alpha}_{k-\cdot}}{v(\cdot)}\chi_{[s,k]}\right\|_{-\frac{1}{\alpha},\infty;v}\leq CA^{1+\alpha}_{k-r},$$

for all integer numbers r, s and k with $r\leq s\leq k$, i.e., there exists a constant C such that

$$\sup_{r\leq s\leq k}\left(A^{1+\alpha}_{k-r}\right)^{-1}\left(\sum_{j=r}^{s}v(j)\right)^{1+\alpha}\sup_{t>0}t\left(\sum_{\{j\in[s,k]:\ \frac{A^{\alpha}_{k-j}}{v(j)}>t\}}v(j)\right)^{-\alpha}<\infty. \tag{2.4}$$

An important result that we shall need in the proof of Theorem 1.4 is the equivalence between condition (2.4) and the class of weights A_1^+.

Lemma 2.6. *Let v be a positive measurable function on the integers. Then, $v\in A_1^+(\mathbb{Z})$, if and only if, v verifies (2.4).*

Proof. We start proving that $A_1^+(\mathbb{Z})$ implies (2.4). For fixed t, r, s and k, let $E=\{j : j\leq k,\ \frac{A^{\alpha}_{k-j}}{v(j)}>t\}$ and $E_{s,k}=E\cap\{s,\dots,k\}$. Let ℓ be the number of elements of $E_{s,k}$. We may assume that $\ell>0$. Since v satisfies $A_1^+(\mathbb{Z})$ then we have for all $i\in E_{s,k}$

$$\sum_{j=r}^{s}v(j)\leq C(k-r+1)v(i)\leq C(k-r+1)\frac{A^{\alpha}_{k-i}}{t}.$$

Therefore

$$\sum_{j=r}^{s}v(j)\leq\frac{C(k-r+1)}{t\ell}\sum_{i\in E_{s,k}}A^{\alpha}_{k-i}. \tag{2.5}$$

By (2.5) and the definition of $E_{s,k}$ we get that

$$\begin{aligned}\left(A^{1+\alpha}_{k-r}\right)^{-1}\left(\sum_{j=r}^{s}v(j)\right)^{1+\alpha}t\left(\sum_{i\in E_{s,k}}v(i)\right)^{-\alpha}&\leq\\ \leq\left(A^{1+\alpha}_{k-r}\right)^{-1}\left[\frac{C(k-r+1)}{t\ell}\sum_{i\in E_{s,k}}A^{\alpha}_{k-i}\right]^{1+\alpha}&t\left[\frac{1}{t}\sum_{i\in E_{s,k}}A^{\alpha}_{k-i}\right]^{-\alpha}\\ =\frac{C(k-r+1)^{1+\alpha}}{A^{1+\alpha}_{k-r}}\,\frac{\sum_{i\in E_{s,k}}A^{\alpha}_{k-i}}{\ell^{1+\alpha}}.&\end{aligned}$$

Observe that $\sum_{i\in E_{s,k}}A^{\alpha}_{k-i}\leq\sum_{k=0}^{\ell-1}A^{\alpha}_{k}$ since the possible terms to add are $A^{\alpha}_0,\dots A^{\alpha}_{k-s}$ and we know [17] that $A^{\alpha}_0\geq A^{\alpha}_1\geq\cdots\geq A^{\alpha}_{k-s}$. Now, keeping in mind that $\sum_{k=0}^{\ell-1}A^{\alpha}_k=A^{1+\alpha}_{\ell-1}$ and $A^{1+\alpha}_n$ is essentially equivalent to $n^{1+\alpha}$ [17], the right hand-side of the above inequality is less than or equal to a constant C. The proof of the first implication is finished.

Now, we shall prove that (2.4) implies $A_1^+(\mathbb{Z})$. Let r and s be such that $r \le s$. Choosing $k = s$ in (2.4) we have

$$\left(A_{s-r}^{1+\alpha}\right)^{-1}\left(\sum_{j=r}^{s} v(j)\right)^{1+\alpha}\frac{1}{v(s)}\left[v(s)\right]^{-\alpha} \le C.$$

Since $A_{s-r}^{1+\alpha}$ is essentially equivalent to $(s-r+1)^{1+\alpha}$ we have that

$$\frac{1}{s-r+1}\sum_{j=r}^{s} v(j) \le Cv(s),$$

which means that v satisfies $A_1^+(\mathbb{Z})$.

Remark 2.7. For $\alpha = 0$, the operator $m_{1+\alpha}^+$ is the Hardy-Littlewood maximal operator on $\mathbb{Z}$ and the equivalence between the statement $v \in A_1^+(\mathbb{Z})$ and the weak type (1,1) inequality for m_1^+ with respect to v was previously proved in [1] (see also [15]).

3. Proofs of Theorems 1.4 and 1.5

In order to prove Theorem 1.4 we need to study the maximal operator $M_{1+\alpha,T}f = \sup_{n\ge 0}|R_{n,1+\alpha}f|$. We shall prove that $M_{1+\alpha,T}$ applies $L_{\frac{1}{1+\alpha},1}(\mu)$ into weak-$L^{\frac{1}{1+\alpha}}(\mu)$. It suffices to prove this fact for positive operators, since $M_{1+\alpha,T}f \le M_{1+\alpha,|T|}|f|$.

Theorem 3.1. *Let $(X, \mathcal{F}, \mu)$, α and T be as in Theorem 1.4. Assume also that T is positive. (Therefore $|T| = T$). Then, there exists a constant C such that*

$$\mu(\{x:\ M_{1+\alpha,T}f(x) > \lambda\}) \le \frac{C}{\lambda^{\frac{1}{1+\alpha}}}||f||_{\frac{1}{1+\alpha},1;\mu}^{1/1+\alpha},$$

for all $\lambda > 0$ and all $f \in L_{\frac{1}{1+\alpha},1}(\mu)$.

Proof. Let f be a nonnegative function and let $L \in \mathbb{N}$, $L > 0$. Let us define

$$M_{1+\alpha}^L f = M_{1+\alpha,T}^L f = \sup_{0\le n\le L} R_{n,1+\alpha}f.$$

Now, given $N \in \mathbb{N}$, by the properties of the Lamperti operator T and Proposition 2.1 we get that

$$\begin{aligned}\mu(\{x:\ M_{1+\alpha}^L f(x) > \lambda\}) &= \frac{1}{N+1}\sum_{i=0}^{N}\int_X \chi_{\{M_{1+\alpha}^L f>\lambda\}}(x)\,d\mu(x)\\ &= \frac{1}{N+1}\sum_{i=0}^{N}\int_X \left[T^i\left(\chi_{\{M_{1+\alpha}^L f>\lambda\}}\right)(x)\right]^{\frac{1}{1+\alpha}} w_i(x)\,d\mu(x)\\ &\le \frac{M_\infty^{\frac{1}{1+\alpha}}}{N+1}\int_X \sum_{\{0\le i\le N:\ x\in\Phi^i(\{M_{1+\alpha}^L f>\lambda\})\}} w_i(x)\,d\mu(x).\end{aligned} \tag{3.1}$$

Now, applying again Proposition 2.1 and the properties of T we have

$$\begin{aligned}\chi_{\Phi^i(\{M^L_{1+\alpha}f>\lambda\})}(x) &\le M_\infty T^i\left(\chi_{\{M^L_{1+\alpha}f>\lambda\}}\right)(x)\\ &\le \frac{M_\infty}{\lambda}T^i\left(\left[M^L_{1+\alpha}f\right]\chi_{\{M^L_{1+\alpha}f>\lambda\}}\right)(x)\\ &\le \frac{M_\infty}{\lambda}T^i\left(M^L_{1+\alpha}f\right)(x).\end{aligned}\tag{3.2}$$

On the other hand, by the definition of $M^L_{1+\alpha}$, there exist pairwise disjoint measurable subsets of X, E_0, E_1, ... , E_L, such that $M^L_{1+\alpha}f = \sum_{j=0}^L \chi_{E_j}R_{j,1+\alpha}f$. Since T separates supports, we have, for every i with $0 \le i \le N$ and all $x \in X$, that

$$\begin{aligned}T^i(M^L_{1+\alpha}f)(x) &= \sum_{j=0}^L T^i(\chi_{E_j}R_{j,1+\alpha}f)(x) = \sum_{j=0}^L \chi_{\Phi^i(E_j)}(x)T^i\left(R_{j,1+\alpha}f\right)(x)\\ &\le \sum_{j=0}^L \chi_{\Phi^i(E_j)}(x)M^L_{1+\alpha}(T^if)(x) \le M^L_{1+\alpha}(T^if)(x)\\ &\le m^+_{1+\alpha}(g_x\chi_{[0,N+L]})(i)\end{aligned}\tag{3.3}$$

where $g_x(j) = T^jf(x)$. Now, putting together (3.2) and (3.3) we obtain that $\{0 \le i \le N:\ x \in \Phi^i(\{M^L_{1+\alpha}f > \lambda\})\} \subset \{i:\ m^+_{1+\alpha}(g_x\chi_{[0,N+L]})(i) > \frac{\lambda}{M_\infty}\}$. Then, by (3.1) we have

$$\mu(\{x:\ M^L_{1+\alpha}f(x) > \lambda\}) \le \frac{M_\infty^{\frac{1}{1+\alpha}}}{N+1}\int_X \sum_{\{i:\ m^+_{1+\alpha}(g_x\chi_{[0,N+L]})(i)>\frac{\lambda}{M_\infty}\}} w_i(x)\,d\mu(x).\tag{3.4}$$

If $\alpha = 0$, by Proposition 2.3 (i), Lemma 2.6, Theorem 2.5 and the property (2.2) we obtain that

$$\begin{aligned}\mu(\{x:\ M^L_1f(x) > \lambda\}) &\le \frac{CM^2_\infty}{\lambda(N+1)}\sum_{i=0}^{N+L}\int_X T^if(x)w_i(x)\,d\mu\\ &= \frac{CM^2_\infty(N+L+1)}{\lambda(N+1)}||f||_1,\end{aligned}$$

and letting N and then L tend to ∞ the proof is completed in this case.

On the other hand, if $-1 < \alpha < 0$, then $L_{\frac{1}{1+\alpha},\infty;\mu}$ is a Banach space, since $\frac{1}{1+\alpha} > 1$. Therefore applying Theorem 3.13 in [16, p.195], which also holds for the sublinear operator $M_{1+\alpha,T}$, it suffices to prove the theorem for characteristic functions, i.e, to prove that

$$\mu(\{x : M_{1+\alpha}\chi_E(x) > \lambda\}) \le \frac{C}{\lambda^{\frac{1}{1+\alpha}}}\int_X \chi_E\,d\mu$$

for all $\lambda > 0$ and all measurable sets E. Taking $f = \chi_E$ in (3.4), applying Proposition 2.3 (i), Lemma 2.6 and Theorem 2.5 we get that

$$\mu(\{x : M^L_{1+\alpha}\chi_E(x) > \lambda\}) \le$$

$$\frac{CM_\infty^{\frac{2}{1+\alpha}}}{(N+1)\lambda^{\frac{1}{1+\alpha}}}\int_X\left[\int_0^\infty\left(\sum_{\{0\le i\le N+L:\ g_x(i)>t\}} w_i(x)\right)^{1+\alpha}dt\right]^{\frac{1}{1+\alpha}}d\mu(x).$$

Now, since the integral on t is zero if $t > M_\infty$, by the Hölder inequality and the properties of the Lamperti operators we have that

$$\begin{aligned}
\mu(\{x:\ M^L_{1+\alpha}\chi_E(x) > \lambda\}) &\le \frac{CM_\infty^{\frac{2-\alpha}{1+\alpha}}}{(N+1)\lambda^{\frac{1}{1+\alpha}}}\int_X\int_0^{M_\infty}\left(\sum_{\{0\le i\le N+L:\ g_x(i)>t\}} w_i(x)\right)dt\,d\mu(x)\\
&= \frac{CM_\infty^{\frac{2-\alpha}{1+\alpha}}}{(N+1)\lambda^{\frac{1}{1+\alpha}}}\sum_{i=0}^{N+L}\int_X T^i(\chi_E)(x)w_i(x)\,d\mu(x)\\
&= \frac{CM_\infty^{\frac{2-\alpha}{1+\alpha}}}{(N+1)\lambda^{\frac{1}{1+\alpha}}}\sum_{i=0}^{N+L}\int_X \left[T^i(\chi_E)(x)\right]^{\frac{1}{1+\alpha}}(h_i(x))^{1-\frac{1}{1+\alpha}}w_i(x)\,d\mu(x)\\
&\le \frac{CM_\infty^{\frac{2(1-\alpha)}{1+\alpha}}(N+L+1)}{(N+1)\lambda^{\frac{1}{1+\alpha}}}\mu(E).
\end{aligned}$$

Letting N and then L tend to ∞ we are done.

Proof of Theorem 1.4. If T verifies the hypothesis of Theorem 1.4 then it is easy to see that T is an invertible Lamperti operator on $L^p(\mu)$ with $\frac{1}{1+\alpha} < p < \infty$ and verifies condition (1.2) in Theorem 1.2. In fact,

$$\begin{aligned}
\left\|\frac{1}{n+1}\sum_{k=0}^n |T|^k_{1+\alpha}(f)\right\|_{p(1+\alpha)} &\le \left\|\frac{1}{n+1}\sum_{k=0}^n |h_k|^p\Phi^k\left(|f|^{p(1+\alpha)}\right)\right\|_1^{\frac{1}{p(1+\alpha)}}\\
&\le M_\infty^{\frac{p(1+\alpha)-1}{p(1+\alpha)^2}}\left\|\frac{1}{n+1}\sum_{k=0}^n |T|^k_{1+\alpha}\left(|f|^{p(1+\alpha)}\right)\right\|_1^{\frac{1}{p(1+\alpha)}}\\
&\le M_\infty^{\frac{p(1+\alpha)-1}{p(1+\alpha)^2}} M_1^{\frac{1}{p(1+\alpha)}}||f||_{p(1+\alpha)}.
\end{aligned}$$

So that, by Theorem 1.2 there exists the limit $\lim_{n\to\infty} R_{n,1+\alpha}f$ for all $f\in L^p(\mu)\cap L_{\frac{1}{1+\alpha},1}(\mu)$ for $\frac{1}{1+\alpha} < p < \infty$. Therefore, Theorem 1.4 follows from Theorem 3.1 and the Banach's Principle, since $f\in L^p(\mu)\cap L_{\frac{1}{1+\alpha},1}(\mu)$ is a dense subset of $L_{\frac{1}{1+\alpha},1}(\mu)$.

Now, we shall prove Theorem 1.5. Acting as in the proof of Theorem 1.4, but using Theorem 1.1, we see that in order to prove Theorem 1.5 we only need to establish the boundednees of the maximal operator $H^*_\alpha f = \sup_{n\ge 1}|H_{n,\alpha}f|$.

In [2], it was proved the pointwise estimate

$$H^*_\alpha f(x) \le C\left[H^*f(x) + M_{1+\alpha,|T|}(|f|)(x) + M_{1+\alpha,|T|^{-1}}(|f|)(x)\right],$$

where $H^*f = H^*_0 f$ is the maximal ergodic Hilbert transform. This result was proved for an invertible Lamperti operator on $L^p(\mu)$ with $\frac{1}{1+\alpha} < p < \infty$ and $-1 < \alpha \le 0$, but it is easy to see that also holds for invertible Lamperti operators on $L^{\frac{1}{1+\alpha}}(\mu)$ (see the proof of Lemma 2.1 in [2]). In the following theorem we shall prove that the maximal operator H^* is of weak type $(\frac{1}{1+\alpha},\frac{1}{1+\alpha})$. This fact together with Theorem 3.1 gives the desired boundedness of H^*_α:

$$\mu(\{x:\ H^*_\alpha f(x) > \lambda\}) \le \frac{C}{\lambda^{\frac{1}{1+\alpha}}}||f||^{1/1+\alpha}_{\frac{1}{1+\alpha},1;\mu},$$

for all $\lambda > 0$ and all $f\in L_{\frac{1}{1+\alpha},1}(\mu)$.

Theorem 3.2. *Let $(X,\mathcal{F},\mu)$, α and T be as in Theorem 1.5. Then, there exists a constant C such that*

$$\mu(\{x:\ H^*f(x)>\lambda\})\leq \frac{C}{\lambda^{\frac{1}{1+\alpha}}}||f||_{\frac{1}{1+\alpha}}^{1/1+\alpha},$$

for all $\lambda>0$ and all $f\in L^{\frac{1}{1+\alpha}}(\mu)$.

Proof. If $-1<\alpha<0$, the result follows from a Sato's result (see Lemma in [13]); actually Sato proved the strong type $(\frac{1}{1+\alpha},\frac{1}{1+\alpha})$ inequality for H^*.

Now we sketch the proof for $\alpha=0$. As in the proof of Theorem 3.1, we write $H_L^*f=\sup_{1\leq n\leq L}|H_nf|=\sum_{j=1}^L|H_jf|\chi_{E_j}$, for $L\in\mathbb{N}$, $L>0$, where the sets E_j are pairwise disjoint measurable subsets of X. Then, we obtain as in (3.1) with $\alpha=0$ that

$$\mu(\{x:\ H_L^*f(x)>\lambda\})\leq\frac{M_\infty}{N+1}\int_X\sum_{\{0\leq i\leq N:\ x\in\Phi^i(\{H_L^*f>\lambda\})\}}w_i(x)\,d\mu(x)$$

On the other hand,

$$\begin{aligned}\frac{\lambda}{M_\infty}\chi_{\Phi^i(\{H_L^*f>\lambda\})}(x)&\leq|T|^i(H_L^*f)(x)\\&\leq H_L^*(T^if)(x)\leq h^*(g_x\chi_{[-L,N+L]})(i)\end{aligned}$$

for all integers i, with $0\leq i\leq N$ and where h^* is the discrete maximal Hilbert transform. The proof finishes as in the proof of Theorem 3.1 with $\alpha=0$ but using now Proposition 2.3 (ii) and Theorem 2.4.

4. Final remarks

Remark 4.1. Theorem 1.4 does not hold if we omit the assumption (1.6). In order to see this, we consider a positive invertible isometry S on $L^1([0,1])$ such that S^n1/n does not converge a.e. to zero (the existence of S is guaranteed in the proof of Theorem 3 in [7]). Since S is a positive invertible isometry then S is a Lamperti operator and therefore it is of the form $Sf=h\Phi f$. Then $Tf=h^{1+\alpha}\Phi f$ is a Lamperti operator which is an isometry on $L^{\frac{1}{1+\alpha}}(dx)$ and it satisfies (1.5). We shall see by contradiction that the conclusion of Theorem 1.4 does not hold. In fact, if there exists the limit $\lim_{n\to\infty}R_{n,1+\alpha}f$ a.e. for all $f\in L_{\frac{1}{1+\alpha},1}(dx)$ then, argueing as in Theorem 1.22 in [17], $\lim_{n\to\infty}\frac{T^nf}{n^{1+\alpha}}=0$ a.e. for all $f\in L_{\frac{1}{1+\alpha},1}(dx)$ (this was pointed out and shown to us by M. D. Sarrión). In particular, for $f=1$, we have $0=\lim_{n\to\infty}\frac{T^n1}{n^{1+\alpha}}=\lim_{n\to\infty}(\frac{S^n1}{n})^{1+\alpha}$ a.e. which is a contradiction.

Remark 4.2. In [13], Sato proves that Theorem 1.5 does not hold in the case $\alpha=0$ if we omit (1.6).

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