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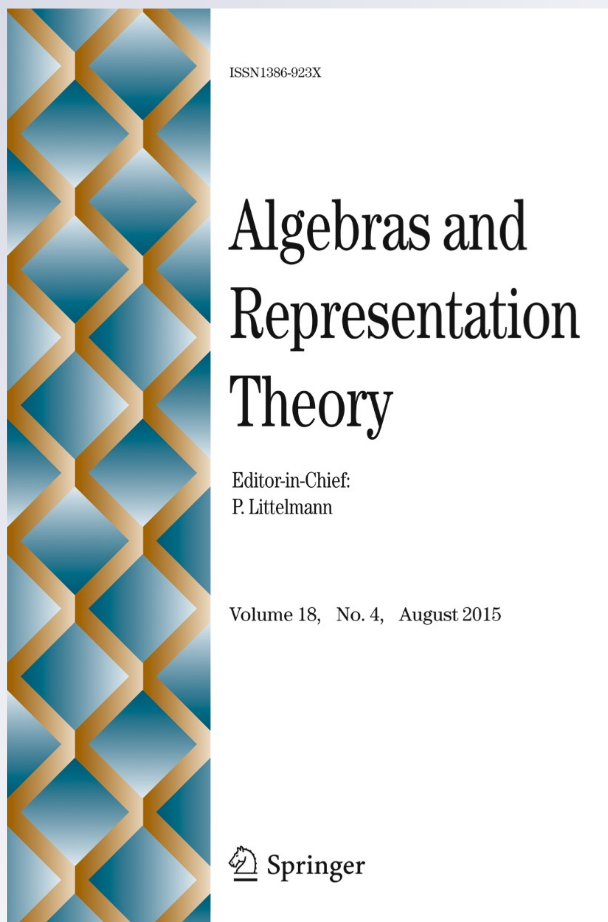

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Differentials for Lie Algebras

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Abstract We develop a theory of relative Kähler differentials for Lie algebras. The main result is that the functor of relative differentials is representable, and that the universal object which represents it behaves properly with respect to étale base change. We illustrate how our construction yields a detailed analysis of the structure of derivations of multiloop algebras which is needed for the construction of Extended Affine Lie Algebras.

Keywords Relative Kähler differentials of Lie algebras · Twisted form · Centroid · Faithfully flat descent

Mathematics Subject Classification (2010) Primary 17B67; Secondary 17B01 · 12G05 · 20G10

1 Introduction

Let $\mathfrak{g}$ be a Lie algebra over a field k (which we will assume for simplicity in this Introduction to be algebraically closed and of characteristic 0). The k-derivations of $\mathfrak{g}$ with values in a $\mathfrak{g}$-module M are used to define the first Lie algebra cohomology. More importantly, the derivations with values in $\mathfrak{g}$ itself are an infinitesimal approximation of the automorphism

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group of $\mathfrak{g}$, hence give information about both, $\mathfrak{g}$ and its automorphisms. This infinitesimal approximation applies, mutatis mutandi, if k is an arbitrary commutative ring.

Many Lie algebras arising in important applications are not just Lie algebras over k, but over some k-algebra R. An example of paramount importance which arises in infinite dimensional Lie theory are the twisted forms of the R-Lie algebra $\mathfrak{g} \otimes_k R$, where R is a Laurent polynomial ring. Affine Kac-Moody theory falls within this framework. Assume L is such a Lie algebra. It seems a natural idea, developed and analyzed by Neher and the second author in [13], to connect the R-derivations and the k-derivations in a meaningful way. Since every R-derivation is a k-derivation, the interesting problem here is to describe those k-derivations that are not R-derivations (see Remark below). The main result of [13] in this context is that, given an $R-L$-module M, there exists an exact sequence

$$0 \to \mathrm{Der}_R(L, M) \to \mathrm{Der}_k(L, M) \to \mathrm{Der}_k\big(R, C_R(M)\big) \to 0 \tag{1}$$

where $C_R(M)$ is an R-module constructed naturally out of the Lie algebra L and M. In many interesting cases this sequence is split as a sequence of R-modules.

There is a very good reason for why the understanding of the k-derivations of L is important: The most interesting Lie algebras are constructed by taking a central extension of L and adding to this a suitable space of derivations. This is already the case for the affine Kac-Moody Lie algebras, and their natural generalizations, namely Extended Affine Lie Algebras (or EALAs, for short). EALAs where first conceived by physicists. A rigorous definition of these algebras and some of their important invariants and properties was given in [1]. One of these invariants is the centreless core of an EALA (often called the bottom algebra; these are the algebras L that we had in mind in the previous paragraph). E. Neher conceived a beautiful theory in which a set of axioms is given for the Lie algebras that can constitute the centreless core of an EALA, and a precise procedure of how to build any EALA by adding to a bottom algebra L a centre and a space of derivations. This is the "modern" approach to EALAS, and will be explained in some detail below. See [9–11], and [12].

Another promising application of our work is to the classification of differential superconformal algebras. The motivation comes from the theory of differential conformal superalgebras developed in [7] which was used to explain the family of $N = 2, 3, 4$ superconformal Lie algebras described in [15]. The classification of differential superconformal algebras using non-abelian cohomology methods (torsors. See [3]) would require, out of first principles, a complete functorial understanding of the space of derivations of certain algebras given by étale descent. The methods that we develop in the present paper are perfectly suited for this purpose.

Many of our results can be "interpreted" in the language of [13] applied to the special case of the universal derivation $d\colon L \to \Omega_{R,L/k}$ we will define. There are, however, two distinct advantages to our approach here. First, it allows to treat *all* derivations at once, regardless of the specific R-module in question. Second, for perfect Lie algebras, it provides for a way of sheafification, that is, the module of differentials gives rise to a quasi-coherent sheaf defined for any scheme over k (see Remark 5.5 below).

For k-algebras, derivations and Kähler differentials behave nicely under étale ring extension. The exact sequence in Eq. 1 is established first for Lie algebras of the form $\mathfrak{g} \otimes_k R$, and then by descent for twisted forms. Since the twisted forms we are interested in are split by étale extensions, the present approach to the study of derivations of Lie algebras seems to be, not only philosophically correct, but also useful.

Remark The "relative" R/k setting mentioned above is crucial. The motivation comes from infinite dimensional Lie theory (e.g. affine or extended affine Kac-Moody Lie algebras). This are infinite dimensional Lie algebras L over k, and understanding the structure of derivations of these algebras with coefficients in certain modules (e.g. L^*) is central to the theory. The ring R appears naturally (as the centroid of L). By descent considerations one has perfect understanding about the structure of the derivations of L as a Lie algebra over R. The full picture is completed by describing the "defect" between R and k-derivations. This is the approach that we shall take.

Note that since every L-module (over R) is also a module for the universal enveloping algebra $\mathrm{U}_R(L)$ of L over R, and since there is a well-developed theory of differentials for associative algebras, it is tempting to try and connect this theory for the enveloping algebra to the derivations of L. However, it is unclear to us, how this could work, mainly because a derivation of L with values in some module M is *not* a derivation of $\mathrm{U}_R(L)$ with values in M in any natural way. While the theory of differentials for associative algebras will play a role, it is not connected to the one of the enveloping algebras.

2 Preliminaries

In this section we introduce the setup that we will be working with. Throughout k will denote a (commutative, unital) ring; later on, we will require k to be a field. We assume that k is fixed throughout our discussions. It is referred to as the base ring. A k-algebra will be a commutative unital ring R together with a fixed ring homomorphism $k \to R$. If R is a k-algebra, an *R-Lie algebra*, or Lie algebra over R, is an R-module together with an R-bilinear product $[\ ,\ ]$ such that $[x, x] = 0$ for all x and for which the Jacobi identity is satisfied. If L is an R-Lie algebra, it is of course also a Lie algebra over k, and an *L-module* is a k-module M with compatible L-action. To be precise, the action $(x\ ,\ m) \mapsto xm$ is given by a k-linear map $L \otimes_k M \to M$ such that $xym - yxm = [x, y]m$ for all $x, y \in L$ and $m \in M$. If M is also an R-module, then M is an *$R - L$-module* if in addition, $x(rm) = (rx)m = r(xm)$ for all $r \in R$, $x \in L$, $m \in M$, i.e., the action map factors through $L \otimes_R M$. If M, N are $R - L$-modules, $\mathrm{Hom}_{R-L}(M, N)$ denotes the set of L-equivariant R-module maps. This has a natural R-module structure given by $(rf)(m) = rf(m)$. If $R = k$, we simply write $\mathrm{Hom}_L(M, N)$.

Remark Of course an $R - L$-module is nothing but a (left-) $\mathrm{U}_R(L)$-module where $\mathrm{U}_R(L)$ denotes the universal enveloping algebra of the R-Lie algebra L. The $R - L$-equivariant maps between such modules then become simply the $\mathrm{U}_R(L)$-linear maps.

2.1 Derivations

Let L be a Lie algebra over k, and M an L-module. A *k-derivation* of L with values in M is a k-linear map $\delta\colon L \to M$ satisfying

$$\delta([x, y]) = x\delta(y) - y\delta(x).$$

If L is an R-Lie algebra and M is an $R - L$-module, it makes sense to talk about R-derivations, that is, derivations that are also R-linear. We will write $\mathrm{Der}_R(L, M)$ for the space of R-derivations with values in M.

Below we will also encounter the derivations of R with values in a bimodule M. Here a *bimodule* of R is always a bimodule where k acts centrally, i.e., it is just a left $R \otimes_k R$-module. A k-derivation of R with values in such a bimodule is a k-linear function $\delta\colon R \to M$ with the property that $\delta(rs) = r\delta(s) + \delta(r)s$. We write $\mathrm{Der}_k^b(R, M)$ for the set of all such derivations. It is well known that the functor $M \rightsquigarrow \mathrm{Der}_k^b(R, M)$ on the category of R-bimodules is represented by the ideal of the diagonal in $R \otimes_k R$, that is, the kernel $\Omega^b_{R/k}$ of the multiplication map $R \otimes_k R \to R$. Being an ideal, it is an R-bimodule in our sense. Any R-module M is naturally an R-bimodule if we put $(r, s)m = rsm$.[1] Then any R-bimodule map $\Omega^b_{R/k} \to M$ factors through $\Omega^b_{R/k}/\left(\Omega^b_{R/k}\right)^2 \simeq \Omega^b_{R/k} \otimes_{R\otimes_k R} R$, which is canonically isomorphic to the module $\Omega_{R/k}$ of Kähler differentials of R over k, and, restricted to the category of R-modules, $\mathrm{Der}_k^b(R, \cdot)$ is represented by $\Omega_{R/k}$.

It is clear that $\mathrm{Der}_k(L, M)$ is an R-module via $(r\delta)(x) = r\delta(x)$. The assignment $M \rightsquigarrow \mathrm{Der}_k(L, M)$ is clearly a functor from the category of $R - L$-modules to the category of R-modules. As mentioned in the introduction, our main goal is to show that this functor is representable, and to describe the representing $R - L$-module. Note that L is a Lie algebra over R and M is an $R - L$ module (this is the relative set up mentioned in the Introduction). The notation $\mathrm{Der}_k(L, M)$ hides R, and we at times use $\mathrm{Der}_{R/k}(L, M)$ instead to emphasize the presence of R.

Definition 2.1 Let L be an R-Lie algebra. A *module of R/k-differentials* for L is an $R - L$ module $\Omega_{R,L/k}$ representing the functor $\mathrm{Der}_{R/k}(L, \cdot)$.

By Yoneda's Lemma, $\Omega_{R,L/k}$ is unique up to canonical isomorphism, if it exists. Similarly, the identity map in $\mathrm{Hom}_{R-L}(\Omega_{R,L/k}, \Omega_{R,L/k})$ corresponds to a derivation $d_{R,L,k}\colon L \to \Omega_{R,L/k}$ which we will refer to as the *universal derivation*. It is characterized by the fact that any derivation $\delta\colon L \to M$ corresponds to a unique $R - L$-homomorphism $\sigma_\delta\colon \Omega_{R,L/k} \to M$ such that $\sigma_\delta \circ d_{R,L,k} = \delta$.

2.2 Change of Rings

An *extension* of R is a homomorphism $R \to S$ of k-algebras. If M is an R-module, and $R \to S$ an extension, we often write M_S for the S-module $M \otimes_R S$. Since S/R will be fixed within a given discussion, this notation will not lead to confusion. The tensor product with respect to our base ring k will be denoted by $\otimes_k$ to avoid any possible misunderstanding.

Suppose L is an R-Lie algebra, and M is an $R - L$-module. If $R \to S$ is any extension, then $M_S = M \otimes_R S$ is canonically an $S - L_S$-module. If L is an S-Lie algebra and $R \to S$ is an extension, then L will be viewed as an R-Lie algebra in the natural way. Note, however, that in this case if M is an $R - L$-module, then M_S is not, in general, an $S - L$-module. Indeed, $xm \otimes s$ may not equal $(sx)m \otimes 1$ for example. To remedy this, we define

$$T(M, S) = (M \otimes_R S)/U$$

where U is the S-submodule generated by all $(sx)m \otimes t - xm \otimes st$ for $s, t \in S$, $m \in M$, and $x \in L$. We can generalize this picture by replacing S by an arbitrary $S - L$-module N. In this case $(M \otimes_R N)/U$ is an $S - L$-module if U is the R-submodule generated by all $(sx)m \otimes n - xm \otimes sn$, where $s \in S$, $x \in L$, $m \in M$, and $n \in N$. If we denote $(M \otimes_R N)/U$

[1] Or $(r \otimes s)m = rsm$ in the tensor product interpretation of our bimodules.

by $T(M, N)$, then $T(-, N)$ is a functor from the category of $R-L$-modules to the category of $S-L$-modules (playing a similar role as $\otimes_R N$ does on the category of R-modules).

Let A be an S-module that is also an L-module (but not necessarily an $S-L$-module). Given an $S-L$-module B, we still write $\mathrm{Hom}_{S-L}(A, B)$ for the set of S- and L-equivariant maps from A to B.

Lemma 2.2 *On the category of $S-L$-modules, $T(M, N)$ represents the functor $P \mapsto \mathrm{Hom}_{S-L}(M \otimes_R N, P)$.*

Proof The assignment in question is a functor from the category of $S-L$-modules to the category of S-modules. Since every $S-L$ map $M \otimes_R N \to P$ factors uniquely through $T(M, N)$ the Lemma is clear. $\square$

The main application of this concept in our context is the case when $S = R \otimes_k R$. This S is an extension of R in two obvious ways. Clearly any R-Lie algebra L is also an $R \otimes_k R$-Lie algebra, if we give L the canonical bimodule structure.

We will be mostly interested in (twisted) forms of Lie algebras over k.

Definition 2.3 Ler R be a k-algebra which we assume is fixed in our discussion. Let L and L' be Lie algebras over R. We say that L is a form of L' if $L \otimes_R S \simeq L' \otimes_R S$ as S-Lie algebras for some faithfully flat an0d finitely presented extension S/R. The given S is said to *split* L, and we also say that L is an S/R-form of L'.

Finally, if $\mathfrak{g}$ is a Lie algebra over k, we say (by a harmless abuse of language) that L is a form of $\mathfrak{g}$ if L is a form of $\mathfrak{g} \otimes_k R$.

Arguably the most important examples are forms of loop algebras, which are a major building block of interesting families of infinite dimensional Lie algebras: here k is typically $\mathbb{C}$, but can be any algebraically closed field of characteristic zero, $\mathfrak{g}$ is a (finite dimensional) simple Lie algebra over k, and $R = k\left[t_1^{\pm 1}, \cdots, t_n^{\pm 1}\right]$ is a Laurent polynomials ring or the field $k((t_1)) \cdots ((t_n))$ of iterated Laurent series over k. In these cases, any form of $\mathfrak{g} \otimes_k R$ is split by an étale (in fact, Galois-) extension of R,[2] which will make the discussion of étale base change below relevant. Details about the "torsor" approach to these algebras can be found in [4, 6] and [14]. The case $n = 1$ appears in the theory of affine Kac-Moody-Lie algebras, while the general case in the theory of Extended Affine Lie Algebras (EALAs).

Remark 2.4 The purpose of this remark is to point out a useful "finiteness" property of forms that has not been recorded in the literature Let $\mathfrak{g}$, R and k be as above. Assume that $\mathfrak{g}$ is finite dimensional, and consider the algebraic group $\mathbf{G} := \mathbf{Aut}(\mathfrak{g})$. The forms of $\mathfrak{g} \otimes_k R$ are classified by the pointed set $H^1_{\mathrm{fppf}}(R, \mathbf{G}_R)$. Since $\mathbf{G}$ is smooth, we may replace the fppf topology by the étale topology.

The claim is the following. If L is a form of $\mathfrak{g} \otimes_k R$, then there exists a Noetherian subring R' of R containing k, and a Lie algebra L' over R' such that $L \simeq L' \otimes_{R'} R$. The Lie algebra L' is a form of $\mathfrak{g} \otimes_k R'$. Furthermore, L' is unique in the sense that if L'' is an R''-Lie algebra with similar properties, then there exists a Noetherian subring R''' of R containing both R' and R'' such that $L' \otimes_{R'} R'''$ and $L'' \otimes_{R''} R'''$ are isomorphic Lie algebras over R'''.

[2] For Laurent polynomials, this follows for the Isotriviality Theorem of [5].

To see this write R as a direct limit of Noetherian subrings $(R_\lambda)_{\lambda\in\Lambda}$ where Λ is a partially ordered set and each R_λ contains k. There is no loss of generality in assuming that Λ has a minimal element 0 and that $R_0 = k$. For $\lambda \in \Lambda$ consider the R_λ-group scheme $\mathbf{G}_\lambda = \mathbf{G} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R_\lambda)$. By a theorem of Grothendieck-Margaux (see Theorem 2.1 of [8]) the natural map

$$\varinjlim_{\lambda\in\Lambda} H^1(R_\lambda, \mathbf{G}_\lambda) \to H^1(R, \mathbf{G}_R)$$

is bijective. The claim follows easily from this fact.

We finish the present remark by pointing out that the same consideration holds if $\mathfrak{g}$ is an *arbitrary* finite dimensional algebra over k.

3 The Construction

The goal of this section is to show the existence of the $R - L$-module $\Omega_{R,L/k}$ and gain preliminary insights into its structure.

Recall that the universal enveloping algebra of an R-Lie algebra L is defined as

$$\mathrm{U}_R(L) = T_R(L)/\langle x \otimes y - y \otimes x - [x, y] | x, y \in L\rangle$$

where $T_R(L) = \bigoplus_p L^{\otimes p}$ is the tensor algebra of L over R. We will write $\mathfrak{M}_{L,R}$ for the *augmentation ideal* of $\mathrm{U}_R(L)$, namely the ideal of $\mathrm{U}_R(L)$ generated by the image of L in $\mathrm{U}_R(L)$.[3] We have the canonical map $\mathrm{U}_R(L) \to R$ having the augmentation ideal as kernel. Let $d_{L,R} \colon L \to \mathrm{U}_R(L)$ be the canonical map $x \mapsto x$ (this last x is of course an abuse of notation. In all the cases we are interested the map is injective, so that the abuse of notation is harmless). It is an R-derivation into the $R - L$-module $\mathfrak{M}_{L,R}$, where L acts on $\mathfrak{M}_{L,R}$ by left-multiplication.

Lemma 3.1 *Let $\delta \colon L \to M$ be an R-derivation where M is some $R - L$-module. Then there is a unique $R - L$-linear map $\varphi_\delta \colon \mathfrak{M}_{L,R} \to M$, such that $\delta = \varphi_\delta \circ d_{L,R}$.*

Proof This is straightforward: first let $T_R^+(L) = \bigoplus_{p>0} L^{\otimes p}$, and define $\varphi' \colon T_R^+(L) \to M$ to be the unique k-linear map satisfying

$$\varphi'(x_1 \otimes x_2 \otimes \cdots \otimes x_p) = x_1 x_2 \cdots x_{p-1} \delta(x_p).$$

The elements of the ideal defining $\mathrm{U}_R(L)$ are finite sums of elements of the form

$$u = f(x \otimes y - y \otimes x - [x, y])g$$

where f and g are pure tensors. If g is not in R, then $\varphi'(u) = 0$ by the defining property of a Lie algebra action on M. Otherwise, g may be absorbed into f, and

$$\varphi'(f(x \otimes y - y \otimes x - [x, y]) = fx\delta(y) - fy\delta(x) - f\delta([x, y]) = 0$$

because of the defining property of a derivation. It follows that φ' factors through $\mathfrak{M}_{L,R}$, with the latter viewed naturally as a quotient of $T_R^+(L)$. The resulting map $\varphi = \varphi_\delta$ is as desired. Any $R - L$-linear map from $\mathfrak{M}_{L,R}$ to M is also $\mathrm{U}_R(L)$-linear. Since as an $\mathrm{U}_R(L)$-module, $\mathfrak{M}_{L,R}$ is generated by $d_{L,R}(L)$, the map φ is necessarily unique. $\square$

[3]Note that $\mathfrak{M}_{L,k}$ is also meaningful. The base ring will always be made explicit.

Corollary 3.2 *The functor $M \mapsto \mathrm{Der}_R(L, M)$ from the category of $R - L$-modules to the category of R-modules is represented by $\mathfrak{M}_{L,R}$.*

Proof This is now clear: any $R - L$-linear map $\varphi\colon \mathfrak{M}_{L,R} \to M$, when composed with $d_{L,R}\colon L \to \mathfrak{M}_{L,R}$ results in a derivation. □

Of course, the structure of $\mathfrak{M}_{L,R}$ is not necessarily much simpler than the structure of L itself, so for example if $R = k$, this may not be that helpful information. As we will see however, if R and k are different, and L is "reasonable" as an R-Lie algebra, then this approach may give information, mainly on those derivations that are k- but not R-linear. The relevance of this point comes from infinite dimensional Lie theory as already lauded to. More precisely. EALAs, of which the affine algebras are examples, are of the form

$$L \oplus Z \oplus D.$$

The Lie algebra L is a multi loop algebra[4], so that its centroid R is a Laurent polynomial ring, and L is a form of $\mathfrak{g} \otimes_k R$ for some unique $\mathfrak{g}$. $L \oplus Z$ is a central extension of L (the universal one in the affine case), while D is a Lie algebra of derivations (one dimensional in the affine case) of L with values on L^*. Understanding the Lie algebras D that give place to EALA structures is quite delicate. A detailed and conceptual knowledge of the relevant derivations is essential.[5] It is here that the "relative" set up is crucial. One needs to exploit that L has a natural R-Lie algebra structure which splits after an étale extension S/R. Perhaps it is pertinent to draw an analogy with the usual Kähler differentials to put the present work in perspective. Our intention is not to create a Lie analogue of $\Omega_{R/k}$, but rather of a concept of relative Kähler differentials which for lack of better notation can be thought of as $\Omega_{S/R/k}$. This is the object that is of use in infinite dimensional Lie theory.

Proposition 3.3 *Let L be an R-Lie algebra. Then the functor from the category of $R - L$-modules to the category of R-modules given by $M \mapsto \mathrm{Der}_{R,k}(L, M)$ is represented by $T(\mathfrak{M}_{L,k}, R)$.*

Proof We know that on $k - L$-modules, $M \mapsto \mathrm{Der}_k(L, M)$ is represented by $\mathfrak{M}_{L,k}$. The rest of the assertion is just the universal property of $T(\mathfrak{M}_{L,k}, R)$ according to Lemma 2.2. To be precise, if $\varphi\colon \mathfrak{M}_{L,k} \to M$ is any $k - L$-map, then we have seen it induces a unique map $(\mathfrak{M}_{L,k} \otimes_k R)/U = T(\mathfrak{M}_{L,k}, R) \to M$. Conversely any such map gives rise to a derivation when composed with the natural map $L \to T(\mathfrak{M}_{L,k}, R)$ coming from $x \mapsto d_{L,k}(x) \otimes 1$. □

It follows from our definition of relative Lie differentials that that $\Omega_{R,L/k} = T(\mathfrak{M}_{L,k}, R)$, and that the universal derivation $d_{R,L,k}\colon L \to \Omega_{R,L/k}$ is given by $x \mapsto d_{L,k}(x) \otimes 1$. Note that $\Omega_{R,L/k}$ is generated as an $R - L$-module by the image of L. While this follows from the construction we gave above, it is also a immediate consequence of the universal property because the submodule generated by L would also be a representing module.

[4] Assuming a technical condition on the EALA which is irrelevant to the present discussion.

[5] The reader is referred again to Neher's work for a beautiful description of this construction.

If $R \to S$ is any ring extension, and L is an S-algebra, the map $d_{S,L,R}\colon L \to \Omega_{S,L/R}$ is a k-derivation into an $S-L$-module and hence induces a map $\Omega_{S,L/k} \to \Omega_{S,L/R}$. As both modules are generated by the image of L, it is a surjection and we obtain an exact sequence

$$0 \to \Gamma_{S,L/k,R} \to \Omega_{S,L/k} \to \Omega_{S,L/R} \to 0.$$

For us the most interesting case is when $S = R$. We then simply write $\Gamma_{R,L/k}$ instead of $\Gamma_{R,L/k,R}$, and we get the *fundamental exact sequence*

$$0 \to \Gamma_{R,L/k} \to \Omega_{R,L/k} \to \Omega_{R,L/R} \to 0. \tag{2}$$

We will show that this sequence is split as a sequence of $R-L$ in the cases that interest us, which is the analogue for differentials of Proposition 3.3 in [13]. The $R-L$-module $\Gamma_{R,L/k}$ is called the *defect module of* R/k *derivations of* L.

By Corollary 3.2 we know that $\Omega_{R,L/R}$ is isomorphic to $\mathfrak{M}_{L,R}$. It is unrealistic to expect being able to say much about $\mathfrak{M}_{L,R}$ in general. As already mentioned, the case which is of most interest to us is when L is a form of some Lie algebra $\mathfrak{g}$ over k, and in this setting $\mathfrak{M}_{L,R}$ may be understood by understanding $\mathfrak{M}_{\mathfrak{g},k}$

4 The Structure of $\Gamma_{R,L/k}$

In this section, we will describe $\Gamma_{R,L/k}$ in terms of the Kähler differentials of R over k, at least for "reasonable"Lie algebras.

To begin with, we note the following.

Lemma 4.1 *$\Gamma_{R,L/k}$ is the $R-L$-submodule of $\Omega_{R,L/k}$ generated by all*

$$d_{R,L,k}\, rx - r d_{R,L,k}\, x \qquad (x \in L, r \in R). \tag{3}$$

Proof Let M be the $R-L$-submodule generated by all the elements in Eq. 3, and consider $\Omega = \Omega_{R,L/k}/M$. Note that $M \subset \Gamma_{R,L/k}$. Let $\bar{d}\colon L \to \Omega$ be the map induced by the universal derivation $d_{R,L,k}$. It is clear that $\bar{d}$ is a k-derivation. However, by the structure of M, it is also an R-derivation. Hence there exists a map $\sigma\colon \Omega_{R,L/R} \to \Omega$ such that $\sigma \circ d_{R,L,R} = \bar{d}$. Moreover σ is a section of the canonical projection $\Omega \to \Omega_{R,L/R}$ since σ maps $d_{R,L,R}x$ to $\bar{d}(x)$ for all $x \in L$. For the same reason, it is surjective, and thus σ is an isomorphism. But that means $M = \Gamma_{R,L/k}$. □

The following lemma is as special case of Proposition 2.1 in [13]. We include a short proof for the reader's convenience.

Lemma 4.2 *Let $\delta\colon L \to M$ be any k-derivation into an $R-L$-module. Then for each $r \in R$, the map $L \to M$ defined by $x \mapsto \delta(rx) - r\delta(x)$ is L-equivariant.*

Proof Let $x, y \in L$. Then

$$\begin{aligned}\delta(r[y,x]) - r\delta([y,x]) &= \delta([y,rx]) - r\delta([y,x])\\ &= y\delta(rx) - (rx)\delta(y) - r(y\delta(x)) + rx\delta(y)\\ &= y(\delta(rx) - r\delta(x)).\end{aligned}$$

The above allows us to obtain a stronger version of Lemma 4.1. □

Corollary 4.3 *$\Gamma_{R,L/k}$ is generated as an R-module by the elements of the form $d_{R,L,k}\,rx - rd_{R,L,k}\,x \quad (x \in L, r \in R)$.*

Proof We know that these elements generate $\Gamma_{R,L/k}$ as an $R-L$-module. By the previous lemma, the R-module span of these elements is L-stable. The corollary follows. □

We will write $C_R(M)$ for the set of $R-L$ linear maps $L \to M$, i.e., $C_R(M) = \mathrm{Hom}_{R-L}(L, M)$. For any $R-L$-module M, $C_k(M)$ is naturally an R-bimodule, with $(r,s)\varphi = r\varphi s$. A k-derivation $R \to C_k(M)$ is then a k-linear map $\delta\colon R \to C_k(M)$ such that

$$\delta(rs) = r\delta(s) + \delta(r)s.$$

For any R-bimodule M we write $\mathrm{Der}_k^b(R, M)$ for the set of such derivations. The most general observation we can make about $\Gamma_{R,L/k}$ is the following:

Proposition 4.4 *There is a canonical injective natural transformation*

$$\mathrm{Hom}_{R-L}(\Gamma_{R,L/k}, \cdot) \to \mathrm{Der}_k^b(R, C_k(\cdot))$$

on the category of covariant functors from the category of $R-L$-modules to the category of R-modules.

Proof Let M be an $R-L$-module, and $\sigma \in \mathrm{Hom}_{R-L}(\Gamma_{R,L/k}, M)$. We define $\delta\colon R \to C_k(M)$ as follows:

$$\delta(r)(x) = \sigma(d_{R,L,k}(rx) - rd_{R,L,k}(x)).$$

By Lemma 4.2, $\delta(r) \in C_k(M)$, so it remains to verify that δ is a derivation. For simplicity we write d instead of $d_{R,L,k}$.

$$\begin{aligned}\delta(rs)(x) &= \sigma(d(rsx) - rsd(x)) = \sigma(d(rsx) - rd(sx) + rd(sx) - rsdx)\\ &= \delta(r)(sx) + r\delta(s)(x).\end{aligned}$$

If $\sigma \neq 0$, then not all $\delta(r)$ can be equal to zero, as the elements $drx - rdx$ generate $\Gamma_{R,L/k}$ we conclude that $\delta \neq 0$.

To verify that $\sigma \mapsto \delta$ is functorial in M is straightforward. □

By definition of $\Omega^b_{R/k}$ we have $\mathrm{Der}_k^b(R, C_k(M)) = \mathrm{Hom}_{(R,R)}(\Omega^b_{R/k}, C_k(M))$.

If A, B are R-bimodules, i.e. $S = R \otimes_k R$-modules, we write $A \boxtimes_R B$ for the tensor product of the right R-module A and the left R-module B. It is again an S-module, acting as $(r,s)a \otimes b = ra \otimes sb = ras \otimes b$. If we give L the canonical R-bimodule structure, it becomes a Lie algebra over $S = R \otimes_k R$. S is an extension of R in two obvious ways, so let us fix $R \to S$ as $r \mapsto 1 \otimes r$.[6] Then for any S-module A, with trivial L-action, $T(L, A) = (A \boxtimes_R L)/U$ where U is the S-module generated by all

$$a \otimes [(rxs), y] - ras \otimes [x, y] \tag{4}$$

with $r, s \in R$, $x, y \in L$, and $a \in A$ (which is just the k-span of all these elements). We think of $T(L, A)$ as an R-module by means of the left R-action on A. For any $R-L$-module M, then, $\mathrm{Hom}_L(L, M)$ is an R-bimodule, and so $\mathrm{Hom}_{(R,R)}(A, \mathrm{Hom}_L(L, M))$ is

[6]Using the map $r \mapsto r \otimes 1$ leads to an equivalent theory.

an R-bimodule as well. Finally, if $\mathrm{Hom}_{R-L}\big(T(L,A),M\big)$ is an R-bimodule as well, if we define the right action by means of $(\varphi r)(a \otimes x) = \varphi(ar \otimes x)$.

Lemma 4.5 *There is a canonical isomorphism of R-bimodules*

$$\mathrm{Hom}_{(R,R)}(A, \mathrm{Hom}_L(L,M)) \to \mathrm{Hom}_{R-L}(T(L,A),M).$$

Proof Let $\varphi \in \mathrm{Hom}_{(R,R)}(A, \mathrm{Hom}_L(L,M))$. We then define $\beta\colon A \times L \to M$ as

$$\beta(a,x) = \varphi(a)(x).$$

Clearly, β is a k-bilinear map. It also satisfies $\beta(ar,x) = \beta(a,rx)$ by the definition of the right action of R on $\mathrm{Hom}_L(L,M)$ and the fact that φ is an (R,R)-linear map:

$$\beta(ar,x) = \varphi(ar)(x) = \varphi(a)(rx) = \beta(a,rx).$$

We obtain a k-linear map $\sigma\colon A \boxtimes_R L \to M$. In fact, σ is R-linear:

$$\sigma(r(a\otimes x)) = \sigma(ra\otimes x) = \varphi(ra)(x) = r\varphi(a)(x) = r\sigma(a\otimes x).$$

Moreover, σ maps any element of the form Eq. 4 to 0:

$$\begin{aligned}\sigma(a\otimes[(rxs)y]) &= \varphi(a)([(rxs)y]) = (rx)\varphi(a)(sy) = x(r\varphi(a)(sy))\\ &= x\varphi(ras)(y) = \varphi(ras)([x,y]) = \sigma(ras\otimes[x,y]).\end{aligned}$$

Thus, σ factors through an R-linear map $\sigma_\varphi\colon T(L,A) \to M$. It is also L-equivariant by construction, and hence $R-L$-linear.

Conversely, suppose $\sigma\colon T(L,A) \to M$ is an $R-L$-linear map. The projection $A\boxtimes_R L \to M$ then gives rise to a k-linear map $A\boxtimes_R L \to M$, also denoted σ. For $a \in A$, let $\varphi_\sigma(a)\colon L \to M$ be defined as

$$\varphi_\sigma(a)(x) = \sigma(a\otimes x).$$

$\varphi_\sigma(a)$ is k-linear, and L-equivariant:

$$\varphi_\sigma(a)([x,y]) = \sigma(a\otimes[x,y]) = x\sigma(a\otimes y) = x\varphi_\sigma(a)(y).$$

Thus, $\varphi_\sigma(a) \in \mathrm{Hom}_L(L,M)$. We obtain a k-linear map $\varphi_\sigma : A \to \mathrm{Hom}_L(L,M)$. It is also (R,R)-linear, since for each $x \in L$

$$\varphi_\sigma(ras)(x) = \sigma(ras\otimes x) = r\sigma(a\otimes sx) = r\varphi_\sigma(a)(sx) = ((r,s)\varphi_\sigma(a))(x).$$

Now $\sigma_{\varphi_\sigma} = \sigma$ and $\varphi_{\sigma_\varphi} = \varphi$. □

It is clear that the isomorphism of the lemma is functorial on M. The natural transformation of Proposition 4.4 then translates into a natural transformation

$$\mathrm{Hom}_{R-L}(\Gamma_{R,L/k},\cdot) \to \mathrm{Hom}_{(R,R)}(\Omega^b_{R/k}, C_k(\cdot)) \simeq \mathrm{Hom}_{R-L}(T(L,\Omega^b_{R/k}),\cdot)$$

which of course is the same as an $R-L$-equivariant map $T(L,\Omega^b_{R/k}) \to \Gamma_{R,L/k}$. Chasing diagrams we see that this map sends the equivalence class of

$$dr\otimes x \text{ to } d_{R,L,k}(rx) - rd_{R,L,k}(x). \tag{5}$$

This shows that the induced map $T(L,\Omega^b_{R/k}) \to \Gamma_{R,L/k}$ is always surjective.

Remark Of course one could use Formula (5) to define the map $T(L,\Omega^b_{R/k}) \to \Gamma_{R,L/k}$ directly. However, it seems that the effort in then proving that it is a well-defined map is about the same as the more conceptual detour we took.

Proposition 4.6 *Suppose the natural transformation of Proposition 4.4 is an isomorphism. Then*

$$\Gamma_{R,L/k} \simeq T(L, \Omega^b_{R/k}).$$

Proof This is clear by Yoneda considerations. □

Suppose that L is flat as an R-module.[7] Then after applying $\boxtimes_R L$ (on the right) the exact sequence

$$0 \to \Omega^b_{R/k} \to R \otimes_k R \to R \to 0$$

becomes

$$0 \to \Omega^b_{R/k} \boxtimes_R L \to R \otimes_k L \to L \to 0$$

where the right hand map is simply scalar multiplication. So $\Omega^b_{R/k} \boxtimes_R L$ is the kernel of the k-linear map $R \otimes_k L \to L$ satisfying $r \otimes x \mapsto rx$. As a submodule of $R \otimes_k L$ this kernel is spanned (as a left R-module, or even as an abelian group) by all elements of the form

$$r \otimes x - 1 \otimes rx.$$

By definition $T(L, \Omega^b_{R/k})$ is the quotient of that submodule by the submodule spanned by all

$$r \otimes [(sx)y] - 1 \otimes r[(sx)y] - sr \otimes [x, y] + s \otimes r[x, y]$$

(to see this use (4) above). In general we do not know whether one can say more, either about the structure of $T(L, \Omega^b_{R/k})$ or of $\Gamma_{R,L/k}$. By Corollary 4.3, $\Gamma_{R,L/k}$ is contained in the R-module M generated by the image $d_{R,L/k}L$ of L in $\Omega_{R,L/k}$. We now attempt to describe M, but let us now add the assumption that L admits a faithful $R - L$-module. Equivalently, the canonical map $L \to \mathrm{U}_R(L)$ is injective. This is for example true if L is a finitely generated projective R-module.[8] So $d_{R,L,R}$, hence also $d_{R,L,k}$, are injective.

$M \subset \Omega_{R,L/k}$ is in general not L-stable. However, much as it happens for $\mathrm{U}_k(L)$, there is a second action (by commutator) which makes M an L-module (however, not an $R - L$-module, as we shall see). Essentially, the idea is to define the action of x on $d_{R,L,k}y$ by x as $d_{R,L,k}[x, y]$. Now M is the image of the map $\varphi\colon R \otimes_k L \to \Omega_{R,L/k}$ defined by $r \otimes x \mapsto r d_{R,L,k}x$. $R \otimes_k L$ carries the natural L-action, so to get an action on M, all that is needed is to show that the kernel of φ is L-stable. So let

$$\varphi\left(\sum_i r_i \otimes x_i\right) = 0.$$

Composing with the projection to $\Omega_{R,L/R}$, we see that this means $\sum_i r_i x_i$ is in the kernel of $d_{R,L,R}$. By our assumption this kernel is trivial, and therefore $\sum_i r_i x_i = 0$. Now for a given $y \in L$,

$$\varphi\left(\sum_i r_i \otimes [yx_i]\right) = \sum_i r_i d_{R,L,k}[yx_i] = y\left(\sum_i r_i d_{R,L,k}x_i\right) - \left(\sum_i r_i x_i\right) d_{R,L,k}y = 0.$$

Thus, the kernel of φ is L-stable, and we obtain an L-action on M. Moreover, note that the kernel of φ is contained in K, the kernel of the multiplication $R \otimes_k L \to L$. As an L-module, $R \otimes_k L$ decomposes as $K \oplus L$, where we identify L with $1 \otimes_k L$. Similarly, M

[7] Which is the case for all the Lie algebras we are interested in.

[8] The map is injective if L is free by [2] I §2.7 Cor. 2. The finitely generated projective case follows by faithfully flat descent since the construction of the enveloping algebra commutes with base change.

decomposes as $M = \Gamma_{R,L/k} \oplus d_{R,L,k}L$. In both cases the summand isomorphic to L is only a k- but not an R-submodule. The map $R \otimes_k L \to M$ factors through $T\left(L, \Omega^b_{R/k}\right) \oplus L$.

From the foregoing discussion we have.

Proposition 4.7 *Suppose L is flat as an R-module, and that $d_{R,L,R}$ is injective. Then the R-module M generated by $d_{R,L,k}L$ is isomorphic to $\Gamma_{R,L/k} \oplus L$ as an L-module (where the action on M is the quotient action coming from $R \otimes_k L$).* □

In two cases, however, more precise information is available.

First, consider the situation where L is perfect. That is, if the submodule spanned by all $[x, y]$ is equal to L. In this case any L-equivariant k-linear map from L into an $R-L$-module M is actually R-linear: indeed, if $\varphi\colon L \to M$ is L-equivariant, then

$$\varphi(r[x, y]) = \varphi([(rx), y]) = (rx)\varphi(y) = r(x\varphi(y)) = r\varphi([x, y])$$

and since commutators span L, we see that R-linearity holds.[9] Note that this means that $C_k(M) = \mathrm{Hom}_L(L, M) = \mathrm{Hom}_{R-L}(L, M) = C_R(M)$. In particular, the left- and right-module structures on $C_R(M)$ coincide, any derivation $\delta\colon R \to C_k(M)$ is a derivation of the left-module $C_k(M)$. As an immediate consequence, we see that the map $\Omega^b_{R/k} \to C_k(M) = C_R(M)$ actually factors through $\Omega_{R/k}$, the module of Kähler differentials of R over k.

Proposition 4.8 *Suppose L is perfect. Then $T\left(L, \Omega^b_{R/k}\right) \simeq \Omega_{R/k} \otimes_R L$.*

Here, the right hand side is the usual tensor product of left-R-modules, and the isomorphism is an isomorphism of R-bimodules.

Proof $\Omega_{R/k}$ is canonically isomorphic to the "restriction of $\Omega^b_{R/k}$ to the diagonal," i.e. $\Omega_{R/k} \simeq \Omega^b_{R/k} \otimes_{R\otimes_k R} R$, where R is an $R \otimes_k R$-extension by means of the multiplication map $R \otimes_k R \to R$.

Next, recall that the kernel of $\Omega^b_{R/k} \boxtimes_R L \to T\left(L, \Omega^b_{R/k}\right)$ is spanned by all

$$r\omega s \otimes [x, y] - \omega \otimes [(rxs)y] = r\omega s \otimes [x, y] - \omega \otimes (rs)[x, y].$$

As L is perfect, this is precisely the span of all

$$r\omega s \otimes x - \omega \otimes rsx = r\omega \otimes sx - \omega r \otimes sx.$$

So the quotient is $\Omega^b_{R/k} \otimes_{R\otimes_k R} L \simeq \Omega^b_{R/k} \otimes_{R\otimes_k R} R \otimes_R L = \Omega_{R/k} \otimes_R L$. □

In this situation we see that $\Gamma_{R,L/k}$ is a quotient of $\Omega_{R/k} \otimes_R L$, a module that seems not too hard to understand. In general, we have no good understanding about the nature of this kernel.

The second situation, where we can say more, is the case where L is isomorphic to $\mathfrak{g} \otimes_k R$ for some Lie algebra $\mathfrak{g}$ over k.

Proposition 4.9 *Let $L = \mathfrak{g} \otimes_k R$. Then the natural transformation of Proposition 4.4 is an isomorphism.*

[9]This result is a special case of Lemma 1.10 in [13]

Proof We will show that if $L = \mathfrak{g} \otimes_k R$, then $\Gamma_{R,L/k} \simeq T\left(L, \Omega^b_{R/k}\right)$.

For this, consider the map $\delta\colon L \to \Omega^b_{R/k} \boxtimes_R L$ defined on pure tensors as

$$\delta(x \otimes r) = dr \otimes (x \otimes 1)$$

This is actually a k-derivation since

$$\begin{aligned}\delta([x, y] \otimes rs) &= d(rs) \otimes [x, y] \otimes 1 = rds \otimes [x, y] \otimes 1 + (dr)s \otimes [x, y] \otimes 1\\ &= (x \otimes r) \otimes y \otimes 1 - (y \otimes s)dr \otimes x \otimes 1.\end{aligned}$$

Note that this uses the fact that $\Omega^b_{R/k} \boxtimes_R L$ is a tensor product over R. Combining this with the projection to $T\left(L, \Omega^b_{R/k}\right)$ we end up with a derivation $L \to T\left(L, \Omega^b_{R/k}\right)$, also denoted δ, and hence an $R - L$ linear map $\sigma\colon \Omega_{R,L/k} \to T\left(L, \Omega^b_{R/k}\right)$. By definition σ restricted to $\Gamma_{R,L/k}$ is a section of the surjective map $T\left(L, \Omega^b_{R/k}\right) \to \Gamma_{R,L/k}$: indeed, the generator $d_{R,L,k}(x \otimes rs) - rd_{R,L,k}(x \otimes s)$ is mapped to the equivalence class of $dr \otimes x \otimes s$, which is mapped back to $d_{R,L,k}(x \otimes rs) - rd_{R,L,k}(x \otimes s)$. The section being $R - L$-linear it is surjective as the elements of the form $dr \otimes (x \otimes s)$ generate $\Omega^b_{R/k} \boxtimes_R L$ as an $R - L$-module. □

Combining all of the above we obtain a complete description of $\Gamma_{R,L/k}$ for a very important class of Lie algebras.

Corollary 4.10 *Suppose $L = \mathfrak{g} \otimes_k R$, where $\mathfrak{g}$ is a perfect Lie algebra over k. Then $\Gamma_{R,L/k}$ is canonically isomorphic to $\Omega_{R/k} \otimes_R L = \Omega_{R/k} \otimes_k \mathfrak{g}$.*

We conclude this section with an interesting consequence of Propositon 4.8.

Proposition 4.11 *Suppose R is a finitely generated k-algebra, and that L is a perfect Lie algebra which is finitely generated as an R-module.*

Then $\Gamma_{R,L/k}$ is a finitely generated R-module and $\Omega_{R,L/k}$ is a finitely generated $R - L$-module.

Proof By Proposition 4.8 have an exact sequence

$$\Omega_{R/k} \otimes_R L \to \Omega_{R,L/k} \to \Omega_{R,L/R} \to 0.$$

The claim now follows immediately from the fact that the left hand side and $\Omega_{R,L/R}$ are finitely generated R-modules, respectively, finitely generated $R - L$-modules. (This uses the fact that $\Omega_{R,L/R} \simeq \mathfrak{M}_{L,R}$ is a finitely generated $R - L$-module.) □

5 Base Change

Henceforth k will denote a field of characteristic 0. Our main interest lies in the situation where L is an R-form of some finite dimensional Lie algebra $\mathfrak{g}$ over k. Since the automorphism group of $\mathfrak{g}$ is a smooth algebraic group over k, every form of $\mathfrak{g} \otimes_k R$ is split by a faithfully flat étale extension of R. The goal of this section is therefore to see how $\Omega_{R,L/k}$ changes when passing from R to S and L to $L_S = L \otimes_R S$ whenever S/R is étale. It turns out that $\Omega_{R,L/k}$ behaves as it should. The crucial role in understanding base change is played by the defect module $\Gamma_{R,L/k}$.

Let $R \to S$ be an arbitrary ring extension. Let L be an R-Lie algebra and M an S-Lie algebra. View M as an R-Lie algebra by means of $R \to S$. Let $\varphi\colon L \to M$ be any homomorphism of R-Lie algebras. The derivation $d_{S,M,k}\colon M \to \Omega_{S,M/k}$ combines with φ to give a k-derivation $L \to \Omega_{S,M/k}$ and hence an $R-L$-linear map $\Omega_{R,L/k} \to \Omega_{S,M/k}$. As the right hand side is an $S-M$-module, we get an $S-L_S$-linear map.[10]

$$d\varphi\colon \Omega_{R,L/k} \otimes_R S \to \Omega_{S,M/k}.$$

Lemma 5.1 *$d\varphi$ maps the image of $\Gamma_{R,L/k} \otimes_R S$ in $\Omega_{R,L/k} \otimes_R S$ to $\Gamma_{S,M/k}$*

Note that since S/R is not assumed to be $\Gamma_{R,L/k} \otimes_R S$ may not be viewed as a submodule of $\Omega_{R,L/k} \otimes_R S$.

Proof By definition, $(d_{R,L,k}(rx) - rd_{R,L,k}x) \otimes s$ is mapped to $s(d_{S,M,k}(r\varphi(x)) - d_{S,M,k}(\varphi(x)))$ which is an element of $\Gamma_{S,M/k}$. Since these elements generate $\Gamma_{R,L/k} \otimes_R S$ as an $S-L_S$-module, this concludes the proof. □

It follows that $d\varphi$ descends to the quotients, i.e. $d\varphi$ induces the map

$$d\varphi\colon \Omega_{R,L/R} \otimes_R S \to \Omega_{S,M/S}.$$

(this map is $d\varphi_{S,L,R/R}$ using the full notation mentioned above). Right now this is mainly interesting in case $M = L_S$, and $\varphi\colon L \to L_S$ the canonical map.

Lemma 5.2 *The canonical map $d\varphi\colon \Omega_{R,L/R} \otimes_R S \to \Omega_{S,L_S/S}$ is an isomorphism.*

Proof From the explicit construction of $\mathrm{U}_S(L_S)$ (or by the universal property of the universal enveloping algebra) it follows that $\mathrm{U}_S(L_S) \simeq \mathrm{U}_R(L) \otimes_R S$. Now $\mathrm{U}_R(L)$, as an R-module, is isomorphic to $R \oplus \mathfrak{M}_{L,R}$, whereas $\mathrm{U}_S(L_S)$, as an S-module is $S \oplus \mathfrak{M}_{L_S,S}$. Since $\mathfrak{M}_{L,R}$ is a direct summand of $\mathrm{U}_R(L)$, we see that $\mathfrak{M}_{L,R} \otimes_R S$ is a submodule of $\mathfrak{M}_{L_S,S}$, that is also a supplement of S. Hence $\mathfrak{M}_{L,R} \otimes_R S = \mathfrak{M}_{L_S,S}$. The claim now follows from the fact that $\Omega_{R,L/R} = \mathfrak{M}_{L,R}$, whereas $\Omega_{S,L_S/S} = \mathfrak{M}_{L_S,S}$. □

Substantially more subtle is the question of passing from R to S for k-differentials. Here, no general results can be stated unless we put restrictions on L and the extension $R \to S$. To avoid overly complex notation, we write r also for the image of $r \in R$ in S, even if $R \to S$ is not injective.

Lemma 5.3 (Étale base change) *Let L be a perfect Lie algebra, and $R \to S$ an étale extension. Then $d\varphi$ is an isomorphism.*

The condition that L is perfect is necessary: if L is not perfect, there is generally no simple étale base change formula, even in the case where L is abelian. This is mainly due to the fact that $\Gamma_{R,L/k}$ in these cases is not isomorphic to $\Omega_{R/k} \otimes_R L$. For abelian L (say, projective of finite rank over R), $\Gamma_{R,L/k} \simeq T\left(L, \Omega^b_{R/k}\right)$. Now this module does not localize "nicely": $\Gamma_{R,L/k} \otimes_R R_f \neq \Gamma_{R_f,L_f/k}$ for general $f \in R$. But of course $R \to R_f$ is étale.

[10] To avoid any confusion we should use the notation $d\varphi_{S,L,R/k}$. This is cumbersome and we trust that the reader will be able to identify which case we are on.

Proof of Lemma 5.3 Recall that if L is perfect we have an exact sequence

$$\Omega_{R/k} \otimes_R L \to \Omega_{R,L/k} \to \Omega_{R,L/R} \to 0.$$

Also, since S/R is étale, $(\Omega_{R/k} \otimes_R L) \otimes_R S \simeq \Omega_{S/k} \otimes_S L_S$ (where the isomorphism is the canonical one). L_S is of course perfect as well, so $d\varphi$ (and the canonical map $\Omega_{R/k} \otimes_R L \to \Omega_{S/k} \otimes_S L_S$) gives rise to the following commutative diagram

$$\begin{tikzcd}
(\Omega_{R/k} \otimes_R L) \otimes_R S \arrow[r, "\sigma"] \arrow[d, "\varepsilon"] & \Omega_{R,L/k} \otimes_R S \arrow[r] \arrow[d, "d\varphi"] & \Omega_{R,L/R} \otimes_R S \arrow[r] \arrow[d, "\overline{d\varphi}"] & 0 \\
\Omega_{S/k} \otimes_S L_S \arrow[r] & \Omega_{S,L_S/k} \arrow[r] & \Omega_{S,L_S/S} \arrow[r] & 0
\end{tikzcd}$$

where the bottom row comes from Proposition 4.8. By Lemma 5.2, the map ε is an isomorphism. Since the right vertical map is an isomorphism as well, it follows that the middle map $\Omega_{R,L/k} \otimes_R S \to \Omega_{S,L_S/k}$ is surjective.

As for injectivity, we construct a section much like in the proof of Proposition 4.9. Define $\delta\colon L \otimes_k S \to \Omega_{R,L/k} \otimes_R S$ by

$$\delta(x \otimes s) = (d_{R,L,k}x) \otimes s + \sigma\varepsilon^{-1}(ds \otimes (x \otimes 1)).$$

For $x \in L$ and $s \in S$. For all $r \in R$,

$$\begin{aligned}
\delta(x \otimes rs) &= (d_{R,L,k}x) \otimes rs + \sigma\varepsilon^{-1}(d(rs) \otimes (x \otimes 1)) \\
&= rd_{R,L,k}x \otimes s + \sigma\varepsilon^{-1}((rds + sdr) \otimes (x \otimes 1)) \\
&= rd_{R,L,k}x \otimes s + \sigma(\varepsilon^{-1}(ds \otimes (rx \otimes 1)) + s\sigma\varepsilon^{-1}(dr \otimes (x \otimes 1)) \\
&= rd_{R,L,k}x \otimes s + \sigma\varepsilon^{-1}(ds \otimes (rx \otimes 1)) + (d_{R,L,k}(rx) - rd_{R,L,k}x) \otimes s \\
&= \delta(rx \otimes s).
\end{aligned}$$

So δ, which is defined on $L \otimes_k S$, factors through $L_S = L \otimes_R S$ as a k-linear map which we still denote by δ. This map is in fact a k-derivation. Indeed

$$\begin{aligned}
\delta([x, y] \otimes st) &= d_{R,L,k}([x, y]) \otimes st + \sigma\varepsilon^{-1}(dt \otimes [x, y] \otimes s) - \sigma\varepsilon^{-1}(ds \otimes [y, x] \otimes t) \\
&= (x \otimes s)\delta(y \otimes t) - (y \otimes t)\delta(x \otimes r).
\end{aligned}$$

It follows that we have an induced map $\Omega_{S,L_S/k} \to \Omega_{R,L/k} \otimes_R S$. Since it maps $d_{S,L_S,k}(x \otimes 1)$ to $d_{R,L,k}x \otimes 1$, it is a section. □

We can now recover one of the main results of [13].

Corollary 5.4 *Let L be a perfect lie algebra over R and S/R an étale extension. For every $S-L_S$-module M, the canonical map $\mathrm{Der}_{S,k}(L_S, M) \to \mathrm{Der}_{R,k}(L, M)$ is an isomorphism.*

Proof The isomorphism $\Omega_{R,L/k} \otimes S \to \Omega_{S,L_S/k}$ induces an isomorphism

$$\mathrm{Hom}_{R-L}(\Omega_{R,L/k}, M) = \mathrm{Hom}_{S-L_S}(\Omega_{R,L/k} \otimes S, M) \to \mathrm{Hom}_{S-L_S}(\Omega_{S,L_S/k}, M).$$

The left hand side is $\mathrm{Der}_{R,k}(L, M)$ whereas the right hand side is $\mathrm{Der}_{S,k}(L_S, M)$. It remains to check that this isomorphism is the canonical map coming from restriction from L_S to L. But this is clear. □

Remark 5.5 Suppose S is a scheme over k, and $\mathcal{L}$ is a quasi-coherent sheaf of Lie algebras on S, that is, a sheaf of Lie algebras over $\mathcal{O}_S$, such that for every open affine subset U, $\mathcal{L}|_U$ is isomorphic to the sheaf associated to the $\mathcal{O}_S(U)$-Lie algebra $\mathcal{L}(U)$. If $\mathcal{L}$ is perfect (that is, $\mathcal{L}(U)$ is a perfect Lie algebra for every open affine subset of S), the étale base change lemma implies that there is a well-defined quasi-coherent sheaf $\Omega_{\mathcal{O}_X,\mathcal{L}/k}$, such that for each open affine subset U of S, $\Omega_{\mathcal{O}_X,\mathcal{L}/k}(U) \simeq \Omega_{\mathcal{O}_S(U),\mathcal{L}(U)/k}$. Indeed, for ever open affine sub-scheme U, let Ω_U be the sheaf associated to the $\mathcal{O}_S(U)$-module $\Omega_{\mathcal{O}_S(U),\mathcal{L}(U)/k}$. Because differentials localize nicely, the sheaves Ω_U glue together to a global sheaf, denoted $\Omega_{\mathcal{O}_S,\mathcal{L}/k}$. $\Omega_{\mathcal{O}_X,\mathcal{L}/k}$ is the sheaf associated to the pre-sheaf $U \rightsquigarrow \Omega_{\mathcal{O}_S(U),\mathcal{L}(U)/k}$. This sheaf (together with the canonical map $d_{\mathcal{O}_X,\mathcal{L},k}\colon \mathcal{L} \to \Omega_{\mathcal{O}_X,\mathcal{L}/k}$) is then universal with respect to k-derivations $\mathcal{L} \to \mathcal{M}$, where $\mathcal{M}$ is a $\mathcal{O}_S - \mathcal{L}$-module (which means $\mathcal{M}(U)$ is a $\mathcal{O}_S(U) - \mathcal{L}(U)$-module for every open subset $U \subset S$). A k-derivation $\mathcal{L} \to \mathcal{M}$ is a morphism of abelian sheaves, that is over every open subset $U \subset S$ a k-derivation for the Lie algebra $\mathcal{L}(U)$. We leave the details to the reader.

6 Forms of Perfect Lie Algebras

Recall that k is a field of characteristic 0, and that R is a k-algebra. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over k which we assume is perfect. We now proceed with our main application of the étale base change lemma, namely to the case of R-forms of $\mathfrak{g}$.[11] So far, we have determined the structure of $\Gamma_{R,L/k}$ if L is isomorphic to $\mathfrak{g} \otimes_k R$: in this case $\Gamma_{R,L/k}$ is isomorphic to $\Omega_{R/k} \otimes_R L$. Using base change this translates to forms as well.

Proposition 6.1 *Let L be a Lie algebra over R which is a form of $\mathfrak{g}$. Then L is perfect and the canonical map*

$$\Omega_{R/k} \otimes_R L \to \Gamma_{R,L/k}$$

is an isomorphism.

Proof That L is perfect is well-known (see the proof of Lemma 4.6 of [4]). The map is surjective because of Proposition 4.8. It maps $d_{R/k}r \otimes x$ to $d_{R,L,k}rx - rd_{R,L,k}x$.

To see that it is injective, let S/R be a faithfully flat étale extension for which $L \otimes_R S \simeq \mathfrak{g} \otimes_k S$.[12] Tensoring the exact sequence $\Omega_{R/k} \otimes_R L \to \Gamma_{R,L/k} \to 0$ by S over R, we get

$$\Omega_{R/k} \otimes_R L \otimes_R S \to \Gamma_{R,L/k} \otimes_R S \to 0. \tag{6}$$

On the one hand, $\Gamma_{R,L/k} \otimes_R S$ is canonically isomorphic to $\Gamma_{S,L_S/k}$ by Lemma 5.3. On the other hand, $\Omega_{R/k} \otimes_R L \otimes_R S \simeq \Omega_{S/k} \otimes_S L_S$, and on generators this isomorphism is given by the map $d_{R/k}r \otimes x \otimes s \mapsto d_{S/k}r \otimes (x \otimes s)$. Using these identifications, $d_{S/k}r \otimes (x \otimes s)$ is mapped to

$$sd_{S,L_S,k}rx \otimes 1 - srd_{S,L_S,k}x \otimes 1.$$

So on S-module generators of $\Omega_{S/k} \otimes_S L_S$ our map coincides with the map $d_{S/k}s \otimes x \mapsto d_{S,L_S,k}sx - sd_{S,L_S,k}x$, and hence it is that map. But this map is an isomorphism by Corollary 4.10. By descent of isomorphism under faithfully flat base change, (6) is left-exact as well, and we are done. □

[11] We remind the reader that "forms of $\mathfrak{g}$" stands for forms of the R-Lie algebra $\mathfrak{g} \otimes_k R$.

[12] Recall that $\mathbf{Aut}(\mathfrak{g})$ is smooth. We can thus replace the fppf by the étale topology.

Corollary 6.2 *If L is an R-form of a perfect Lie algebra $\mathfrak{g}$ over k, the fundamental exact sequence* (2) *becomes*

$$0 \to \Omega_{R/k} \otimes_R L \to \Omega_{R,L/k} \to \Omega_{R,L/R} \to 0.$$

As we will see now, this sequence is actually split in most cases of interest. First, let us treat the case of a "split form" (which is also a consequence of Lemma 2.2 in [13]).

Lemma 6.3 *Suppose $L \simeq \mathfrak{g} \otimes_k R$. Then the sequence* (2) *is split as a sequence of $R-L$-modules.*

Proof There is no loss of generality is assuming that $L = \mathfrak{g} \otimes_k R$. We then have $\mathfrak{M}_{L,R} = \Omega_{k,\mathfrak{g}/k} \otimes_k R$. This is a consequence of the fact that $\mathrm{U}_R(L) = \mathrm{U}_k(\mathfrak{g}) \otimes_k R$ (cf. Lemma 5.2).

The canonical map $\Omega_{k,\mathfrak{g}/k} \otimes_k R \to \Omega_{R,L/k}$ provides then the desired splitting. It maps $dx \otimes 1$ to $d(x \otimes 1)$, which is mapped to $dx \otimes 1$ under the map $\Omega_{R,L/k} \to \Omega_{R,L/R} = \Omega_{k,\mathfrak{g}/k} \otimes_k R$. $\square$

Remark This splitting is not canonical. It depends on the choice of isomorphism $L \simeq \mathfrak{g} \otimes_k R$.

The following theorem requires that R be of finite type over k. However, in light of Remark 2.4, this may not be a severe restriction.

Theorem 6.4 *Let R be a k-algebra of finite type, and let L be an R-form of a perfect, finite dimensional Lie algebra $\mathfrak{g}$ over k. Then* (2) *splits as a sequence of $R-L$-modules.*

The proof requires some preparation. Recall the submodule $M \subset \Omega_{R,L/k}$ generated by $d_{R,L,k}(L)$, and that under these assumption, $d_{R,L,k}$ is injective. As a k-vector space $M = \Gamma_{R,L/k} \oplus d_{R,L,k}(L)$. Also, by construction $rdx - dry \in \Gamma_{R,L/k}$, so $M/\Gamma_{R,L/k} \simeq d_{R,L,R}(L) \simeq L$ is a finitely generated R-module (assuming the hypotheses of the theorem).

Lemma 6.5 *With the hypotheses of Theorem 6.4 on L and R, if N is an $R-L$-module, finitely generated as an R-module, and S/R is a flat ring extension, the canonical map*

$$\mathrm{Hom}_{R-L}(\Omega_{R,L/k}, N) \otimes_R S \to \mathrm{Hom}_{S-L_S}(\Omega_{R,L/k} \otimes_R S, N \otimes_R S)$$

is an isomorphism.

Proof While the canonical map is indeed the one coming from commutative algebra, this is somewhat subtle because the usual result for homomorphism groups under flat ring extensions only applies if the source module is finitely presented, which is not true here (as an R-module, anyway).

By the assumptions on R and L, and the remarks preceding this proof, we have that $\Gamma_{R,L/k} \simeq \Omega_{R/k} \otimes_R L$, and these are finitely generated R-modules. Thus M is finitely generated as an R-module, and hence finitely presented because R is Noetherian. But then the canonical map $\mathrm{Hom}_R(M, N) \otimes_R S \to \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)$ is an isomorphism.

Consider the restriction map $\mathrm{Hom}_{R-L}(\Omega_{R,L/k}, N) \to \mathrm{Hom}_R(M, N)$ with image U. Since M generates $\Omega_{R,L/k}$ as an $R-L$-module, this map is injective. Then $\varphi \in \mathrm{Hom}_R(M, N)$ is in U, if and only if $\varphi \circ d_{R,L,k}$ is a k-derivation. For simplicity, throughout this proof, we will denote $d_{R,L,k}$ simply by d. Then U is given by linear equations of the form $f_{x,y}(\varphi) = \varphi(d[x, y]) - x\varphi(dy) + y\varphi(dx) = 0$ where $x, y \in L$.

Now $\mathrm{Hom}_R(M, N) \otimes_R S$ is isomorphic to $\mathrm{Hom}_S(M \otimes_R S, N \otimes_R S) = \mathrm{Hom}_R(M, N \otimes_R S)$. Under this isomorphism $U \otimes_R S$ is mapped to the image U' of the restriction map $\mathrm{Hom}_{S-L_S}(\Omega_{R,L/k} \otimes_R S, N \otimes_R S) \to \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)$ (this uses the flatness of S/R so that $M \otimes_R S$ is the S-module generated by the image of $d(L)$ in $\Omega_{R,L/k} \otimes_R S$): if $\sigma : \Omega_{R,L/k} \to N$ is any $R-L$-map, it induces canonically a map $\Omega_{R,L/k} \otimes_R S \to N \otimes_R S$, whose restriction to $M \otimes_R S$ is exactly the image of the restriction of σ to M.

Let $H = \mathrm{Hom}_R(M, N)$, which is a finitely generated (and presented) R-module. Then $\mathrm{Hom}_R(H, N) \otimes_R S$ is isomorphic to

$$\mathrm{Hom}_S(H \otimes_R S, N \otimes_R S) \simeq \mathrm{Hom}_S(\mathrm{Hom}_S(M \otimes_R S, N \otimes_R S), N \otimes_R S).$$

All these isomorphisms are canonical. Now an element $\varphi\colon M \otimes_R S \to N \otimes_R S$ is an element of U', if and only if $\varphi \circ (d \otimes 1)$ is a derivation (where $(d \otimes 1)(x) = dx \otimes 1$). Indeed, in this case we get an $R-L$-linear map $\Omega_{R,L/k} \to N \otimes_R S$ and hence an $S-L_S$-linear map $\Omega_{R,L/k} \otimes_R S \to N \otimes_R S$, which restricts to the original φ on $M \otimes_R S$. Then $\varphi = \sum_i \varphi_i \otimes s_i \in H \otimes_R S$ is an element of U' if and only if when viewed as an R-linear map $M \to N \otimes_R S$, for all $x, y \in L$,

$$0 = \sum_i (\varphi_i(d[x, y]) \otimes s_i - (x\varphi_i(dy)) \otimes s_i + (y\varphi_i(dy)) \otimes s_i = \sum_i f_{x,y}(\varphi_i) \otimes s_i.$$

Thus

$$U' = \{\varphi \in H \otimes_R S \mid (f_{x,y} \otimes 1)(\varphi) = 0 \text{ for all } x, y \in L\}$$

where we identify $\mathrm{Hom}_S(H \otimes_R S, N \otimes_R S)$ with $\mathrm{Hom}_R(H, N) \otimes_R S$. Let V be the span of all $f_{x,y}$ in $\mathrm{Hom}_R(H, N)$, and let $f_1, f_2, \ldots, f_p$ be some generators as an R-module. Then $(f_1, f_2, \ldots, f_p)$ defines a map $\psi\colon H \to N^p$ with kernel U. By the above, U' is the kernel of $\psi \otimes \mathrm{id}_S\colon H \otimes_R S \to N \otimes_R S$ (since the f_i also generate $V \otimes_R S$ as an S-module). Since S/R is flat, this is precisely $U \otimes_R S \subset H \otimes_R S$. □

We will also need the following related result, which is an adapted version of the well known base change results for flat ring extensions.

Lemma 6.6 *Let M, N be $R-L$-modules with M finitely presented as an R-module, and L finitely generated as an R-module. Let S/R be a flat ring extension, then the canonical map*

$$\mathrm{Hom}_{R-L}(M, N) \otimes_R S \to \mathrm{Hom}_{S-L_S}(M \otimes_R S, N \otimes_R S)$$

is an isomorphism of S-modules.

Proof Notice that $\mathrm{Hom}_{R-L}(M, N) = \mathrm{Hom}_R(M, N)^L$, where L acts on $\mathrm{Hom}_R(M, N)$ by $x(\varphi) = x\varphi - \varphi x$. By hypothesis, the natural map $\mathrm{Hom}_R(M, N) \otimes_R S \to \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)$ is an isomorphism. If we let act L trivially on S, L acts on $\mathrm{Hom}_R(M, N) \otimes S$, and the action extends canonically to L_S. The isomorphism is clearly L_S-equivariant, so it identifies L_S-fixed points. It remains to see that these fixed points are precisely the elements of $\mathrm{Hom}_R(M, N)^L \otimes_R S$, which by flatness is a submodule of $\mathrm{Hom}_R(M, N) \otimes_R S$. The assertion of the next lemma is precisely this statement in a more general context. □

Lemma 6.7 *Let M be any $R-L$-module, where L is finitely generated as an R-module. If $R \to S$ is any flat ring extension, then the canonical map*

$$M^L \otimes_R S \to (M \otimes_R S)^{L_S}$$

is an isomorphism.

Proof For $m \in M$, we define $\tilde{m}: L \to M$ as $\tilde{m}(x) = xm$ (in fact, this is what is called an inner derivation). As M is an $R-L$-module, $\tilde{m}$ is R-linear. Then M^L is by definition the kernel of the map $M \to \mathrm{Hom}_R(L, M)$ defined by $m \mapsto \tilde{m}$. Thus, we obtain an exact sequence of R-modules

$$0 \to M^L \to M \to \mathrm{Hom}_R(L, M).$$

After applying $\otimes_R S$, this sequence remains exact, so we have an exact sequence of S-modules

$$0 \to M^L \otimes_R S \to M \otimes_R S \to \mathrm{Hom}_R(L, M) \otimes_R S.$$

Since L is finitely generated as an R-module, standard commutative algebra tells us that the canonical map $\mathrm{Hom}_R(L, M) \otimes_R S \to \mathrm{Hom}_S(L_S, M_S)$ is injective. Moreover, under this map $\tilde{m} \otimes s$ is mapped to $\widetilde{m \otimes s}$, where $\widetilde{m \otimes s}$ is defined in the obvious way as $\widetilde{m \otimes s}(x \otimes t) = (x \otimes t)m \otimes s$. But this means the kernel of the map $M_S \to \mathrm{Hom}_S(L_S, M_S)$ defined by $m \otimes s \mapsto \widetilde{m \otimes s}$ is precisely the image of $M^L \otimes_R S$ in M_S. By definition again, this kernel is $M_S^{L_S}$. □

With this preparation in place, we are ready to prove the main result of this section.

Proof of Theorem 6.4 The hypotheses imply that L and $\Omega_{R/k}$, and hence $\Gamma_{R,L/k}$ are finitely generated R-modules. In fact, they are finitely presented as R is Noetherian.

Let S/R be a faithfully flat extension splitting L. Let us apply the functor $\mathrm{Hom}_{R-L}(\cdot, \Gamma_{R,L/k})$ to the exact sequence

$$0 \to \Gamma_{R,L/k} \to \Omega_{R,L/k} \to \Omega_{R,L/R} \to 0$$

to obtain a left exact sequence

$$0 \to \mathrm{Hom}_{R-L}(\Omega_{R,L/R}, \Gamma_{R,L/k}) \to \mathrm{Hom}_{R-L}(\Omega_{R,L/k}, \Gamma_{R,L/k}) \to \mathrm{End}_{R-L}(\Gamma_{R,L/k})$$

We need to show that this sequence is also right-exact, i.e. that

$$\mathrm{Hom}_{R-L}(\Omega_{R,L/k}, \Gamma_{R,L/k}) \to \mathrm{End}_{R-L}(\Gamma_{R,L/k})$$

is surjective. Any preimage of the identity in $\mathrm{End}_{R-L}(\Gamma_{R,L/k})$ provides the desired splitting. By Lemmas 6.5 and 6.6 tensoring with S over R results in the canonical map

$$\mathrm{Hom}_{S-L_S}(\Omega_{R,L/k} \otimes_R S, \Gamma_{R,L/k} \otimes_R S) \to \mathrm{End}_{S-L_S}(\Gamma_{R,L/k} \otimes_R S)$$

which is surjective by base change (Lemma 5.3) and Lemma 6.3. Since S/R is faithfully flat, the original map was onto as well. □

We obtain one of the main results of [13], with the added benefit that we have no condition on the splitting étale extension $R \to S$.

Corollary 6.8 (of Theorem 6.4, see also Proposition 3.2 in [13]) *Let L be an R-form of a finite dimensional perfect Lie algebra $\mathfrak{g}$ over k. Assume that R is of finite type over k.*

Then for any $R-L$-module M we have an isomorphism of $R-L$-modules

$$\mathrm{Der}_k(L, M) \simeq \mathrm{Der}_R(L, M) \oplus \mathrm{Der}_k\big(R, C_R(M)\big).$$

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