*J. of Mult.-Valued Logic & Soft Computing.,* Vol. X, pp. 01–10
Reprints available directly from the publisher
Photocopying permitted by license only



# The Lattice of Subvarieties of Monadic $n$-valued Łukasiewicz-Moisil Algebras

M. Abad, J. P. Díaz Varela, L. A. Rueda, and A. M. Suardíaz

*Departamento de Matemática, Universidad Nacional del Sur, 8000 Bahía Blanca, Argentina*
*Departamento de Matemática, Universidad Nacional del Comahue, 8300 Neuquén Argentina,*
*E-mail: imabad, usdiavar, larueda@criba.edu.ar*

In this paper we describe the lattice of subvarieties of the variety of monadic Łukasiewicz-Moisil algebras. We then characterize all these subvarieties by means of identities.

## 1 INTRODUCTION AND PRELIMINARIES

The notion of an $n$-valued Łukasiewicz-Moisil algebra was introduced by Gr. C. Moisil [17], and was developed and investigated further by several authors such as A. Monteiro [13], L. Monteiro [16], R. Cignoli [6], A. Iorgulescu [9] [10] and J. Varlet [19]. An extensive monograph on Łukasiewicz-Moisil algebras was written by V. Boicescu et all. in [4]. We assume that the reader is familiar with the theory of $n$-valued Łukasiewicz-Moisil algebras. For the basic properties, the reader is referred to [3], [4] and [6].

An *$n$-valued Łukasiewicz-Moisil algebra* is an algebra $(L, \wedge, \vee, \sim, s_1, s_2, \ldots, s_{n-1}, 0, 1)$, $n$ an integer, $n \geq 2$, of type $(2, 2, 1, 1, 1, \ldots, 1, 0, 0)$, such that $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\sim, s_1, s_2, \ldots, s_{n-1}$ satisfy

1. $\sim\sim x = x$,
2. $\sim (x \wedge y) = \sim x \vee \sim y$,
3. $s_i(x \vee y) = s_i x \vee s_i y$,
4. $s_i x \vee \sim s_i x = 1$,
5. $s_i s_j x = s_j x$,
6. $s_i \sim x = \sim s_{n-i} x$,
7. $s_1 x \leq s_2 x \leq \ldots \leq s_{n-1} x$,

8. If $s_i x = s_i y$ for $i = 1, 2, \ldots, n-1$, then $x = y$.

The equational class of all $n$-valued Łukasiewicz-Moisil algebras will be denoted by $\mathcal{L}_n$.

A *monadic n-valued Łukasiewicz-Moisil algebra* ([1], [4]) is an algebra $(A, \exists)$ such that $A$ is an $n$-valued Łukasiewicz-Moisil algebra and $\exists$ is a unary operation defined on $A$ fulfilling the following properties:

1. $\exists 0 = 0$,
2. $x \leq \exists x$,
3. $\exists(x \wedge \exists y) = \exists x \wedge \exists y$,
4. $\exists s_i x = s_i \exists x$.

Monadic $n$-valued Łukasiewicz-Moisil algebras are a straight generalization of the notion of monadic Boolean algebras [8] and three-valued monadic Łukasiewicz-Moisil algebras investigated by L. Monteiro in [16]. They form an equational class $\mathcal{M}L_n$, the main properties of which were established in [1] (see also [4]). In this paper we consider the lattice of subvarieties of this variety. We describe the structure of the poset of its join irreducible elements and we find equational bases for each subvariety of $\mathcal{M}L_n$.

We introduce some notation. The $n$-element chain $0 < 1/(n-1) < \ldots < (n-2)/(n-1) < 1$, with the natural lattice structure and the operations $\sim$ and $s_i$, $1 \leq i \leq n-1$, defined as

$$\sim\ (j/(n-1)) = 1 - j/(n-1) = (n-1-j)/(n-1)$$

$$s_i\ (j/(n-1)) = \begin{cases} 0 \text{ if } & i + j < n \\ 1 \text{ if } & i + j \geq n \end{cases}$$

will be denoted by $C_n$.

It is well known that $C_n$ and its subalgebras are the subdirectly irreducible algebras of $\mathcal{L}_n$ and that they are simple ([6]).

Let $S_2 = \emptyset$, and, for $n \geq 3$, let $S_n = \{1, 2, \ldots, (n-2)/2\}$ when $n$ is even and $S_n = \{1, 2, \ldots, (n-1)/2\}$ when $n$ is odd.

If $J \subseteq S_n$, then $A_J = \{0\} \cup \{j/(n-1) : j \in J\} \cup \{1 - j/(n-1) : j \in J\} \cup \{1\}$ is a subalgebra of $C_n$, and it is easy to check that the correspondence $J \to A_J$ establishes an isomorphism from the Boolean algebra $2^{S_n}$ onto the lattice of subalgebras of $C_n$ (see [6] and [2]). In particular, $A_\emptyset = \{0, 1\}$ and $A_{S_n} = C_n$.

In any algebra $A \in \mathcal{L}_n$, it is defined a binary operation $\mapsto$, called weak implication, by $x \mapsto y = s_{n-1} \sim x \vee y$. This operation was introduced by A. Monteiro in [14] for $n = 3$ and generalized by R. Cignoli in [6] for arbitrary $n$.

A subset $D \subseteq A \in \mathcal{L}_n$ is said to be a *deductive system* of $A$ if $1 \in D$ and if $x \in D$ and $x \mapsto y \in D$, then $y \in D$. If $A \in \mathcal{M}L_n$, a subset $D \subseteq A$ is a *monadic deductive system* of $A$ if $D$ is a deductive system of the Łukasiewicz-Moisil algebra $A$ and $D$ satisfies in addition that $\forall x \in D$, whenever $x \in D$, where $\forall x = \sim \exists \sim x$.

For $A \in \mathcal{M}L_n$, the set $\exists(A) = \{x \in A : \exists x = x\}$ is a Łukasiewicz-Moisil subalgebra of $A$. If $\mathbf{M}$ is the set of all monadic deductive systems of $A$ and $\mathbf{D}$ is the set of all deductive systems of $\exists(A)$, then it is not difficult to prove that $\varphi : \mathbf{M} \to \mathbf{D}$ defined by $\varphi(D) = D \cap \exists(A)$, $D \in \mathbf{M}$, is a lattice isomorphism ([1]).

On the other hand, congruences on a Łukasiewicz-Moisil algebra (monadic Łukasiewicz-Moisil algebra) are determined by deductive systems [6] (monadic deductive systems [1] respectively), so we have the following lemma, where $Con\, A$ and $Con\, \exists(A)$ respectively denote the lattice of congruences of the algebras $A$ and $\exists(A)$.

**Lemma 1.1** *The lattices $Con\, A$, $\mathbf{M}$, $Con\, \exists(A)$ and $\mathbf{D}$ are isomorphic.*

It is easily seen that the set $B(A)$ of complemented elements of $A$ is a monadic Boolean algebra, and consequently, $B(A) \cap \exists(A)$ is a Boolean algebra. Also, it was proved in [1] that the lattices $\mathbf{M}$ and $\mathbf{D}$ are also isomorphic to the lattice of monadic filters of $B(A)$ and to the lattice of filters of $B(A) \cap \exists(A)$.

As an immediate consequence, it follows that $A \in \mathcal{M}L_n$ is simple if and only if $\exists(A)$ is simple in $\mathcal{L}_n$ if and only if $B(A)$ is a simple monadic Boolean algebra if and only if $B(A) \cap \exists(A)$ is a simple Boolean algebra. And if $A \in \mathcal{M}L_n$ is subdirectly irreducible, then $\exists(A)$ is simple in $\mathcal{L}_n$, and consequently, $A$ is simple.

On the Łukasiewicz-Moisil algebra $C_n^X$ of all functions $f$ from a given set $X$ into $C_n$ we define the following unary operation: $(\exists f)(x) = \bigvee\{f(x) : x \in X\}$. It is known (see [1]) that $(C_n^X, \exists)$ is a monadic Łukasiewicz-Moisil algebra, which we denote $C_{n,X}$. It is clear that $f \in B(C_{n,X})$ if and only if $f(X) \subseteq \{0, 1\}$, and $f \in \exists(C_{n,X})$ if and only if $f$ is a constant function. Since the constant functions in $C_{n,X}$ are the funtions $e_j$, $0 \leq j \leq n-1$, defined by $e_j(x) = j/(n-1)$ for every $x$, it follows that $\exists(C_{n,X}) = \{e_j, 0 \leq j \leq n-1\}$. So $B(C_{n,X}) \cap \exists(C_{n,X}) = \{e_0, e_{n-1}\} = \{\mathbf{0}, \mathbf{1}\}$. So the algebra $C_{n,X}$ is simple.

A similar argument proves that every subalgebra of $C_{n,X}$ is simple. The most important thing is that these are, up to isomorphism, the only simple algebras of the variety $\mathcal{M}L_n$. Indeed, it is not difficult to prove that if $A$ and $B$ are simple algebras in $\mathcal{M}L_n$ and $f : A \to B$ is an injective Łukasiewicz-Moisil homomorphism, then $h(\exists a) = \exists h(a)$, for all $a \in A$. As a consequence we have the following

**Theorem 1.2** [1] *If $A \in \mathcal{M}L_n$ is simple, then $A$ is isomorphic to a subalgebra of $C_{n,X}$.*

**Proof.** From Moisil Representation Theorem (see [6], [18] p. 134), there exists a (Łukasiewicz-Moisil) embedding $h : A \to C_n^X$. If we consider the algebra $C_{n,X}$, since $A$ and $C_{n,X}$ are simple, $h$ is a (monadic) embedding. ■

According to this theorem, it is important to obtain a characterization of the subalgebras of the algebra $C_{n,X}$. Since the variety $\mathcal{M}L_n$ is locally finite [1], it is sufficient for our purposes to have a characterization of the subalgebras of $C_{n,X}$ for $X$ finite, say $X = \{1, 2, \ldots, m\}$. We denote in this case $C_{n,m}$ instead of $C_{n,X}$.

The following characterization of the subalgebras of $C_{n,m}$ can be found in [1]. Let $A$ be a (Łukasiewicz-Moisil) subalgebra of $C_n$ and let $B$ be a (Boolean) subalgebra of $B(C_{n,m})$. Let $\{P_1, \ldots, P_r\}$ be the partition of $X$ associated to $B$. Then $S(A, B) = \{f \in C_{n,m} : f(i) \in A, 1 \leq i \leq m$, and $f(i) = f(j)$ if $i, j \in P_t\}$ is a subalgebra of $C_{n,m}$. Let us see that these are all the subalgebras of $C_{n,m}$. For a given subalgebra $S$ of $C_{n,m}$, consider $A = \{f(1) : f \in S\}$, $B = S \cap B(C_{n,m})$ and $\{P_1, \ldots, P_r\}$ the partition associated to $B$. We are going to prove that $S = S(A, B)$. Let $f \in S$ and let $f(i) = k/(n-1) \in C_n$. The function $g = \exists(f \wedge \sim s_{n-k-1} f) \in S$ and $g(1) = k/(n-1)$, which implies that $f(i) \in A$. In addition, if $f(i) = k/(n-1) < l/(n-1) = f(j)$, $i, j \in P_t$, then $s_{n-l} f(i) < s_{n-l} f(j)$, and this contradicts that $s_{n-l} f \in B$. Hence $f \in S(A, B)$. For the other inclusion, if $f \in S(A, B)$ and $i \in P_t$, as $f(i) \in A$, there exists $g \in S$ such that $g(1) = f(i)$, and for some $k$, $g_t = \exists(g \wedge \sim s_k g)$ is such that $g_t(j) = f(i)$ for every $j \in X$. Consequently, if $h_1, \ldots, h_r$ are the atoms of $B$, $f = (g_1 \wedge h_1) \vee \ldots \vee (g_r \wedge h_r)$, and then $f \in S$.

Now, $A = A_J$, for some $J \subseteq S_n$ and $B \cong \mathbf{2}^r$, for some $1 \leq r \leq m$. Then if $A_{J,r}$ is the monadic Łukasiewicz-Moisil algebra of all functions from $Y = \{1, 2, \ldots, r\}$ into $A_J$ with the operation $(\exists f)(x) = \bigvee\{f(x) : x \in Y\}$, then $S(A, B) \cong A_{J,r}$ by means of the mapping $g \to f$, where $f(i) = g(t)$ whenever $i \in P_t$. Hence, identifying isomorphic algebras, we can state the following theorem ([1]).

**Theorem 1.3** *The subalgebras of the algebra $C_{n,m}$ are the algebras $A_{J,r}$, with $J \subseteq S_n$ and $1 \leq r \leq m$.*

Consequently, the subdirectly irreducible finite algebras of the variety $\mathcal{M}L_n$ are the algebras $A_{J,r}$, with $J \subseteq S_n$ and $1 \leq r \leq m$.

**Remark 1.4** *If $A_J \neq A_L$ then $A_{J,i}$ is not isomorphic to $A_{L,k}$, for every positive integers $i, k$.*

**Theorem 1.5** [1] *$A_{J,i}$ is a subalgebra of $A_{L,k}$ $\Leftrightarrow$ $J \subseteq L$ and $i \leq k$.*

## 2 SUBVARIETIES

The purpose of this section is to investigate properties of the lattice $\Lambda(\mathcal{M}L_n)$ of subvarieties of the variety of monadic $n$-valued Łukasiewicz-Moisil algebras.

Observe that $\mathcal{M}L_n$ is clearly congruence-distributive, so by well-known results of Jónsson [12] the lattice $\Lambda(\mathcal{M}L_n)$ is also distributive. Then in order to determine $\Lambda(\mathcal{M}L_n)$ it suffices to characterize the ordered set $\mathcal{J}(\Lambda(\mathcal{M}L_n))$ of its join irreducible elements.

If $K$ is a class of algebras in a variety $V$, $V(K)$ denotes the subvariety of $V$ generated by $K$. $\mathbf{Si}(K)$ and $\mathbf{Si}_{fin}(K)$ respectively denote the classes of isomorphic types of subdirectly irreducible algebras and finite subdirectly irreducible algebras in $K$. The class of algebras that are homomorphic images of algebras in $K$ will be denoted by $\mathbf{H}(K)$, and the class of algebras that are subalgebras of algebras in $K$ will be denoted by $\mathbf{S}(K)$. In what follows we identify isomorphic algebras.

Recall that for $A, B \in \mathbf{Si}(\mathcal{M}L_n)$, we may define a partial order: $A \leq B$ if and only if $A \in \mathbf{H}(\mathbf{S}(B))$, so that $V(A) \leq V(B)$ if and only if $A \leq B$. Since $\mathcal{M}L_n$ is locally finite [1], by the results of Davey [7] the lattice $\Lambda(\mathcal{M}L_n)$ is completely distributive and isomorphic to $\mathcal{O}(\mathbf{Si}_{fin}(\mathcal{M}L_n))$, the lattice of down-sets (order-ideals) of the ordered set $\mathbf{Si}_{fin}(\mathcal{M}L_n)$. Moreover, a finitely generated subvariety $U \in \Lambda(\mathcal{M}L_n)$ is join irreducible if and only if $U = V(A)$, for some subdirectly irreducible algebra $A$.

If $A$ is a subdirectly irreducible algebra in $\mathcal{M}L_n$, then $\mathbf{H}(\mathbf{S}(A)) \setminus \{\text{trivial algebras}\} = \mathbf{S}(A)$. Then $A_{J,i} \leq A_{L,k}$ if and only if $J \subseteq L$ and $i \leq k$, so the ordered set $\mathbf{Si}_{fin}(\mathcal{M}L_n)$ is isomorphic to $\mathbf{2}^{[(n-1)/2]} \times C$, where $C$ is a chain of type $\omega$, $[x]$ denotes the integral part of the number $x$ and $\mathbf{2}^{[(n-1)/2]}$ is the Boolean algebra with $[(n-1)/2]$ atoms. Similarly, the ordered set of join irreducible elements of the lattice of subvarieties of $\mathcal{V}_{J,i} = V(A_{J,i})$ is isomorphic to $\mathbf{2}^{|J|} \times C_i$, where $C_i$ is an $i$-element chain.

Consider now, for each $J \subseteq S_n$, the subvarieties:

$$M_J = V(\{A_{J,i},\ i \geq 1\}) = \bigvee_{i \geq 1} \mathcal{V}_{J,i}.$$

Observe that $M_\emptyset$ is the variety of monadic Boolean algebras and $M_{S_n} = \mathcal{M}L_n$.

**Lemma 2.1** $M_J \in \mathcal{J}(\Lambda) = \mathcal{J}(\Lambda(\mathcal{M}L_n))$.

**Proof.** Suppose that $M_J = V_1 \vee V_2$. The sets $N_l = \{i \in \mathbf{N} : A_{J,i} \in \mathbf{Si}_{fin}(V_l)\}$, $l = 1, 2$, cannot be bounded, since $\mathbf{Si}_{fin}(M_J) = \mathbf{Si}_{fin}(V_1) \cup \mathbf{Si}_{fin}(V_2)$ is not finite. So, for some $l = 1, 2$, say $l = 1$, $A_{J,i} \in \mathbf{Si}_{fin}(V_1)$, for all $i \in \mathbf{N}$, and then, $A_{L,i} \in \mathbf{Si}_{fin}(V_1)$ for all $L \subseteq J$ and $i \in \mathbf{N}$. Hence $\mathbf{Si}_{fin}(V_1) = \mathbf{Si}_{fin}(M_J)$ and $M_J = V_1$. ■

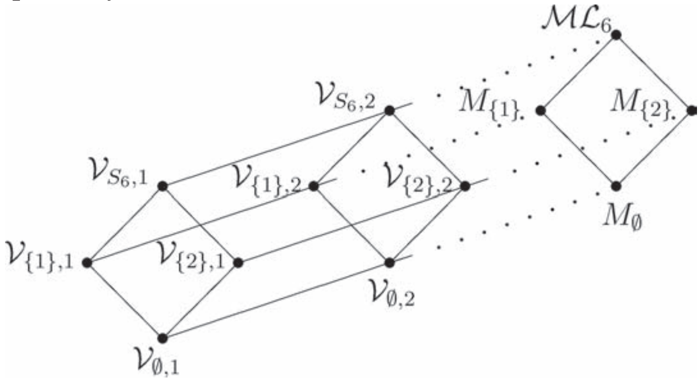
**Theorem 2.2** $\mathcal{J}(\Lambda) = \{\mathcal{V}_{J,i}\}_{J \subseteq S_n, i \geq 1} \cup \{M_J\}_{J \subseteq S_n}$.

**Proof.** Let $V \in \mathcal{J}(\Lambda)$ and let $S = \cup_{J \subseteq S_n} F_J$, where $F_J = \{i \in \mathbf{N} : A_{J,i} \in \mathbf{Si}_{fin}(V)\}$. If $\mathbf{Si}_{fin}(V)$ is finite, then so are the sets $F_J$. Thus $V = \bigvee_{J \subseteq S_n} \bigvee_{i \in F_J} \mathcal{V}_{J,i}$. Since $V$ is join irreducible, then $V = \mathcal{V}_{J_0,i_0}$ for some $J_0 \subseteq S_n$ and $i_0 \in F_{J_0}$. If $\mathbf{Si}_{fin}(V)$ is not finite there exists $J \subseteq S_n$ such that $F_J$ is infinite. Let $J_1, \cdots, J_k$ be such that $F_{J_t}$, $1 \leq t \leq k \leq 2^e$, where $e = [(n-1)/2]$, are all the infinte sets, $J_{k+1}, \cdots, J_d$, $d \leq 2^e$ such that $F_{J_i}$ are non empty finite sets, $k < i \leq d$ and let $m_i = max(F_{J_i})$ for $k < i \leq d, m_i \geq 1$. Then, since $V$ is locally finite,

$$V = \bigvee_{1 \leq t \leq k} M_{J_t} \vee \bigvee_{k < i \leq d} \mathcal{V}_{J_i,m_i}.$$

Since $V \in \mathcal{J}(\Lambda)$, then either $V = M_{J_t}$ for some $1 \leq t \leq k$ or $V = \mathcal{V}_{J_i,m_i}$, for some $k < i \leq d$, but $V$ cannot be $\mathcal{V}_{J_i,m_i}$, as $V$ is not finitely generated, Hence $V = M_{J_t}$ for some $1 \leq t \leq k$. ■

**Example 2.3** *If $n = 6$, then $S_6 = \{1, 2\}$, and $\mathcal{J}(\Lambda(\mathcal{M}L_6))$ is*




**Theorem 2.4** *If $V \in \Lambda(\mathcal{M}L_n)$, then $V$ is a finite join of elements in $\mathcal{J}(\Lambda(\mathcal{M}L_n))$.*

**Proof.** It is a consequence of the proof of Theorem 2.2. ■

**Theorem 2.5** $\mathcal{J}(\Lambda(\mathcal{M}L_n)) \simeq \mathbf{2}^e \times (C+1)$*, where $C$ is a chain of type $\omega$ and $e = [(n-1)/2]$.*

**Proof.** The isomorphism from $\mathbf{2}^e \times (C+1)$ into $\mathcal{J}(\Lambda(\mathcal{M}L_n))$ is the mapping $\phi$ such that $\phi(J,i) = \mathcal{V}_{J,i}$ for $i \neq \omega + 1$, $\phi(J,i) = M_J$ if $i = \omega + 1$. ■

## 3 EQUATIONAL BASES

The aim of this section is to find equational bases for each subvariety of $\mathcal{M}L_n$.

Recall that if $A \in \mathcal{L}_n$, the operation

$$x \to y = y \vee \bigwedge_{i=1}^{n-1} (\sim s_i x \vee s_i y)$$

is an intuitionistic implication, that is, $(A, \to)$ is a Heyting algebra (see [4], p. 204).

Consider the following term:

$$\gamma_p(x_0, \ldots, x_{p+1}) = \bigvee_{i=0}^{p} \forall \left( x_{i+1} \to \bigvee_{j=0}^{i} x_j \right).$$

**Theorem 3.1** *The following identities characterize the subvarieties $\mathcal{V}_{\emptyset,p}$, $p \geq 1$, within $\mathcal{M}L_n$:*

(1) $\exists x = x$ *and* $x \wedge \sim x = 0$, *for* $p = 1$.
(2) $\gamma_p(x_0, \ldots, x_{p+1}) = 1$ *and* $x \wedge \sim x = 0$, *for* $p > 1$.

**Proof.** The case $p = 1$ is immediate. Suppose that $p > 1$ and let $a_0, \ldots, a_{p+1} \in A_{\emptyset,p}$. Consider the elements $b_0 = a_0, b_1 = a_0 \vee a_1, \ldots, b_p = \bigvee_{j=0}^{p} a_j$. It is clear that $b_0 \leq b_1 \leq \ldots \leq b_p$. If $b_i < b_{i+1}$ for $i = 0, \ldots, p-1$, then $b_p = 1$, as $A_{\emptyset,p}$ is a $p$-atom Boolean algebra. So $\forall(a_{p+1} \to \bigvee_{j=0}^{p} a_j) = \forall(a_{p+1} \to b_p) = \forall(a_{p+1} \to 1) = \forall 1 = 1$, and consequently, $\gamma_p(a_0, \ldots, a_{p+1}) = 1$. If $b_i = b_{i+1}$, for some $i$, then $a_{i+1} \leq b_i = \bigvee_{j=0}^{i} a_i$. So $\forall(a_{i+1} \to \bigvee_{j=0}^{i} a_j) = \forall 1 = 1$. Thus $\gamma_p(a_0, \ldots, a_{p+1}) = 1$. Therefore $\gamma_p(x_0, \ldots, x_{p+1}) = 1$ holds in $A_{\emptyset,p}$.

Let $A$ be a finite subdirectly irreducible algebra in $\mathcal{M}L_n$ and suppose that the identities (2) hold in $A$. Since $x \wedge \sim x = 0$ holds in $A$, it follows that $A$ is a Boolean algebra. So $A = A_{\emptyset,q}$. Suppose that $q > p$ and let $a_1, \ldots, a_q$ be the atoms of $A$. Consider the elements $b_0 = 0$, $b_1 = a_1$, $b_2 = a_1 \vee a_2$, $\ldots$, $b_p = \bigvee_{i=1}^{p} a_i$ and $b_{p+1} = 1$. We have that $b_p \neq 1$, since $p < q$. Since $b_{i+1} \to b_i \neq 1$, it follows that $\forall(b_{i+1} \to b_i) = 0$. Thus $\gamma_p(b_0, \ldots, b_{p+1}) = \bigvee_{i=0}^{p} \forall(b_{i+1} \to \bigvee_{j=0}^{i} b_j) = \bigvee_{i=0}^{p} \forall(b_{i+1} \to b_i) = 0$, a contradiction. So $q \leq p$, and consequently, by Theorem 1.5, $A \in \mathcal{V}_{\emptyset,p}$. ■

In what follows, $\gamma_p(s_1(x_0), \ldots, s_1(x_{p+1}))$ will be abbreviated by $\gamma_p(s_1(\vec{x}))$.

Consider now the following unary operators $H_0, H_1, \ldots, H_{n-1}$ introduced by M. Adams and R. Cignoli in [2]: $H_0(x) = s_{n-1}(x)$, $H_{n-1}(x) = \sim s_1(x)$, and $H_i(x) = \sim s_{n-i}(x) \vee s_{n-i-1}(x)$ for $0 < i < n-1$. Note that in the algebra $C_n$, $H_i(j/(n-1)) = 0$ if $i = j$ and $H_i(j/(n-1)) = 1$ when $i \neq j$, $0 \leq i \leq n-1$.

**Theorem 3.2** *The following identities characterize the subvarieties $\mathcal{V}_{J,p}$, $\emptyset \subset J \subset S_n$, within $\mathcal{M}L_n$:*

(3) $\bigwedge_{i \notin J, i \in S_n} H_i(x) = 1$ *and* $\exists s_1 x = s_1 x$, *for* $\emptyset \subset J \subset S_n$, $p = 1$.
(4) $\gamma_p(s_1(\vec{x})) = 1$ *and* $\bigwedge_{i \notin J, i \in S_n} H_i(\exists x) = 1$, *for* $\emptyset \subset J \subset S_n$ *and* $p > 1$.

**Proof.** Suppose that $p = 1$. Then $\mathcal{V}_{J,1} = V(A_{J,1})$, and it is easy to see that $A_{J,1}$ satisfies equations (3). If $U$ is a subvariety such that $U \not\subseteq \mathcal{V}_{J,1}$, then there exists $A_{I,l} \in U$ such that either $l > 1$ or $I \not\subseteq J$. In the first case, there exists $x \in A_{I,l}$, $x \notin \{0, 1\}$ such that $\exists s_1 x = \exists x = 1$ and $s_1 x = x \neq 1$. In the other case, since also $A_{I,1} \in U$, there exists $i \in I$, $i \notin J$ such that $x_0 = i/(n-1) \in A_{I,1}$ and $H_i(x_0) = 0$.

Suppose now that $p > 1$, $\mathcal{V}_{J,p} = V(A_{J,p})$, and let us prove that the equations (4) hold in $A_{J,p}$. Observe that $s_1(x) \in B(A_{J,p}) = A_{\emptyset,p}$. So $\gamma_p(s_1(\vec{x})) = 1$ holds in $A_{J,p}$. On the other hand, since $\exists(A_{J,p}) = A_{J,1}$, $A_{J,p}$ satisfies $\bigwedge_{i \notin J, i \in S_n} H_i(x) = 1$.

Let $A$ be a finite subdirectly irreducible algebra in $\mathcal{M}L_n$ and suppose that $A$ satisfies the identities (4). Then $A = A_{I,q}$ and consequently $B(A) = A_{\emptyset,q}$ and $B(A)$ satisfies $\gamma_p(s_1(\vec{x})) = 1$, which in $B(A)$ is equivalent to $\gamma_p(\vec{x}) = 1$. Thus by Theorem 3.1, $q \leq p$. Observe that $\exists(A) = \exists(A_{I,q}) = A_{I,1}$, so by (3), $I \subseteq J$. Hence, by Theorem 1.5, $A_{I,q}$ is a subalgebra of $A_{J,p}$. ■

**Theorem 3.3** *The following equations characterize the subvarieties $\mathcal{V}_{S_n,p}$ within $\mathcal{M}L_n$:*

(5) $\exists s_1 x = s_1 x$, *for* $p = 1$.
(6) $\gamma_p(s_1(\vec{x})) = 1$, *for* $p > 1$.

**Theorem 3.4** *The subvarieties $M_J$ of $\mathcal{M}L_n$, $J \subset S_n$, are characterized by:*

(7) $x \wedge \sim x = 0$, *for* $J = \emptyset$.
(8) $\bigwedge_{i \notin J, i \in S_n} H_i(\exists x) = 1$, *for* $\emptyset \subset J \subset S_n$.

**Proof.** It is a consequence of Theorem 3.2. ■

If $t_1(\vec{x})$, $t_2(\vec{x})$ are terms in the language of $\mathcal{M}L_n$, then $t_1(\vec{x}) = t_2(\vec{x})$ is equivalent to $(t_1(\vec{x}) \to t_2(\vec{x})) \wedge (t_2(\vec{x}) \to t_1(\vec{x})) = 1$. In addition, $t_1(\vec{x}) = 1$ and $t_2(\vec{x}) = 1$, is equivalent to $t_1(\vec{x}) \wedge t_2(\vec{x}) = 1$. In this way, every join irreducible variety $V_i$ can be characterized by a single equation $\gamma_{V_i}(\vec{x}) = 1$.

We want to characterize the subvarieties of $\mathcal{M}L_n$ that are not join irreducible in the lattice $\Lambda(\mathcal{M}L_n)$.

**Theorem 3.5** *If $V = \bigvee_{i=1}^{s} V_i$, where $V_i$ are join irreducible varieties in $\Lambda(\mathcal{M}L_n)$, then $\gamma_V(\vec{x}) = \bigvee_{i=1}^{s} \forall(\gamma_{V_i}(\vec{x})) = 1$ is a characteristic equation for $V$ relative to $\mathcal{M}L_n$.*

**Proof.** If $A$ is a subdirectly irreducible algebra in $V$, then $A \in V_i$ for some $i$ and so $\forall(\gamma_{V_i}(\vec{a})) = 1$ for every $\vec{a} \in A^m$. Then $\gamma_V(\vec{a}) = 1$, for every $\vec{a} \in A^m$, that is, $A$ satisfies the equation $\gamma_V(\vec{x}) = 1$. Now, if $A \cong A_{J,p}$ is a subdirectly irreducible algebra and $A \notin V$, then $A \notin V_i$, for all $i$. Thus there exist $\vec{a} = (a_1, \ldots, a_r) \in A^r$, $r$ as needed, such that $\gamma_{V_i}(\vec{a}) \neq 1$ for all $i$. Then $\forall(\gamma_{V_i}(\vec{a})) \leq c$ for all $i$, where $c$ is greatest element in $\exists(A) \setminus \{1\}$. Consequently, $\bigvee_{i=1}^{s} \forall(\gamma_{V_i}(\vec{a})) \leq c \neq 1$, and the equation $\gamma_V(\vec{x}) = \bigvee_{i=1}^{s} \forall(\gamma_{V_i}(\vec{x})) = 1$ fails in the algebra $A$. ■

In particular, from this theorem and Theorem 2.4 it follows that every subvariety of $\mathcal{M}L_n$ is finitely axiomatizable, in fact, is axiomatizable by a single equation relative to $\mathcal{M}L_n$.

## REFERENCES

[1] Abad, M. (1988). Estructuras Cíclicas y Monádicas de un Algebra de Łukasiewicz n-Valente, Notas de Lógica Matemática No. 36, Universidad Nacional del Sur, Bahía Blanca.

[2] Adams, M. E., and Cignoli, R. (1990). *A note on the Axiomatization of Equational Classes of n-Valued Łukasiewicz Algebras*, Notre Dame Journal of Formal Logic, *31,* No 2, Spring, 304–307.

[3] Balbes, R., and Dwinger, P. (1974). Distributive Lattices, University of Missouri Press, Columbia, MO.

[4] Boicescu, V., Filipoiu, A., Georgescu, G., and Rudeanu, S., (1991). Łukasiewicz-Moisil Algebras, Annals of Discrete Mathematics, 49.

[5] Burris, S., and Sankappanavar, H.P. (1981). A Course in Universal Algebra, Graduate Texts in Mathematics, *78* Springer, Berlin.

[6] Cignoli, R., and Algebras, Moisil. (1970). Notas de Lógica Matemática *27,* Universidad Nacional del Sur, Bahía Blanca.

[7] Davey, B.A. (1979). *On the lattice of subvarieties*. Houston J. Math. *5,* 183–192.

[8] Halmos, P.R. (1955). *Algebraic Logic I (Monadic Boolean algebras)*, Compositio Mathematica, 12, 217–249.

[9] Iorgulescu, A. (1998). *Connections between* $\mathrm{MV}_n$*-algebras and n-valued Łukasiewicz-Moisil algebras, Part I*, Discrete Math. *181,* 155–177.

[10] Iorgulescu, A. (1999). *Connections between* $\mathrm{MV}_n$*-algebras and n-valued Łukasiewicz-Moisil algebras, Part II*, Discrete Math. *202,* 113–134.

[11] Iorgulescu, A. (2000). *Connections between* $\mathrm{MV}_n$*-algebras and n-valued Łukasiewicz-Moisil algebras, Part IV*, Journal of Universal Computer Science, *6*(1), 139–154.

[12] Jónsson, B. (1967). *Algebras whose congruence lattices are distributive*, *Math. Scand. 21,* 110–121.

[13] Monteiro, A. (1963). *Sur la définition des algèbres de Łukasiewicz trivalentes*, Bull. Math. Soc. Sc. Phys. R.P. Roum. *7*(55) 1–2, 3–12.

[14] Monteiro, A. (1967). *Construction des algèbres de Łukasiewicz trivalentes dans les algèbres de Boole monadiques*, Math. Japon. *12,* 1–23.

[15] Monteiro, L. (1963). *Axiomes indépendents pour les algèbres de Łukasiewicz trivalentes*, Bull. Math. Soc. Sc. Phys. R.P. Roum. *7*(55), 199–202.

[16] Monteiro, L., and Algebras de Łukasiewicz Trivalentes Monádicas, (1974). Notas de Lógica Matemática *32*, Universidad Nacional del Sur, Bahía Blanca.

[17] Moisil, Gr.C. (1940). *Recherches sur les logiques non-chrysippiennes*, Ann. Sci. Univ. Jassy, *26,* 431–436.

[18] Moisil, Gr.C. (1963). *Le algebre di Łukasiewicz*, Acta Logica, Bucarest, *6,* 97–135.

[19] Varlet, J. C. (1968). *Algèbres de Łukasiewicz trivalentes*, Bull. Sc. Liège, 9–10, 281–290.