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Maxwell–Higgs self-dual solitons on an infinite cylinder

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We have studied the Maxwell–Higgs model on the surface of an infinite cylinder. In particular, we show that this model supports self-dual topological soliton solutions on the infinite tube. Finally, the Bogomol'nyi-type equations are studied from theoretical and numerical point of view.

Keywords: Soliton solution; magnetic monopole.

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It is well known that the Abelian Higgs models in $(2+1)$ dimensions with the Maxwell term present topological stable vortex solution[1] (for review see Refs. 2–5). These models have the particularity to became auto-dual when the self-interactions are suitably chosen. When this occur, the model presents particular mathematical and physics properties, such as the supersymmetric extension of the model,[6] and the reduction of the motion equation to first-order derivative equation.[7,8] Characteristically, these vortex solutions carry magnetic flux but are electrically neutral. They correspond to vortex-like object carrying a magnetic field in its core, where the scalar field vanishes. Also, the magnetic flux is quantized.

Recently, there has been interest in the study of gauge theories on infinite tubes.[9] Particularly, it was studied SU(2) Yang–Mills model within closed and open tubes, showing the existence of magnetic monopoles and dyons living within this tube-shaped domain.

Here, we are interested on the study of the Abelian Maxwell–Higgs model within an infinite cylinder. Our goal is to show that the Abelian Maxwell–Higgs model support topological solitons when it is defined within an infinite cylinder. Specifically, we propose an ansatz such that the model is reduced to a theory within an infinite cylinder surface. We will show that for the model defined in the cylinder, it is possible to obtain self-dual or Bogomol'nyi equations by minimizing the energy functional of the model. The model is then bounded below by topological number. Finally, we analyze the numerical solutions of the field equations showing the behavior of matter and gauge fields as well as their energy density. We also analyze the behavior of the magnetic vortex.

We start by considering the Maxwell–Higgs electrodynamics in a $(2+1)$-dimensional curved manifold described by the following metric

$$ds^2 = dt^2 - \rho^2\, d\phi^2 - dz^2\,, \tag{1}$$

(for solitons on curved manifold see Refs. 10–13) where the azimuthal angle runs over $0 \le \phi \le 2\pi$ and the z-coordinate varies between $0 \le z_+ < +\infty$. Here, ρ is a constant and in the following we set $\rho = 1$, for simplicity. Such a manifold is a semi-infinite cylinder immersed in a $(3+1)$-dimensional spacetime. The action describing the Maxwell electrodynamics coupled to complex scalar field $\psi(x)$ is

$$S = \int d^3x \left[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + |D_\mu \psi|^2 - U(|\psi|) \right]. \tag{2}$$

The covariant derivative is defined as $D_\mu = \partial_\mu + ieA_\mu$ and the metric tensor is $g^{\mu\nu} = (1, -1, -1)$.

By varying with respect to A_μ and $\psi^\dagger$, we obtain the field equations

$$\partial^\mu F_{\mu\nu} + J_\nu = 0\,, \tag{3}$$

$$D_\mu D_\mu \psi + \frac{\partial U}{\partial \psi^*} = 0\,, \tag{4}$$

where $J_\nu = -ie[\psi^*(D_\nu\psi) - \psi(D_\nu\psi)^*]$.

The energy associated with the action (2) is

$$E = \int d^2x \left[\frac{1}{2} F_{0i}^2 + \frac{1}{4} F_{ji}^2 + |D_0\psi|^2 + |D_i\psi|^2 + U(|\psi|) \right]. \tag{5}$$

From (3), the stationary Gauss law reads

$$\partial_j \partial_j A_0 + 2e^2 A_0 |\psi|^2 = 0\,, \tag{6}$$

it is identically satisfied by the configuration $A_0 = 0$, thus, in the static regimen the model describes pure magnetic configurations.

Here, we are interested in the particular case of the static field configurations with $A_0 = 0$. In this context, the energy in (5) reduces to:

$$E = \int d^2x \left[\frac{1}{2}\mathbf{B}^2 + |D_i\psi|^2 + U(|\psi|) \right], \tag{7}$$

where $\mathbf{B} = \nabla \times \mathbf{A}$.

Let us consider the following ansatz:

$$A_{z_+} = 0\,, \qquad A_\phi = \frac{h(z_+)}{e}\,, \qquad \psi = vf(z_+)e^{-in\phi}\,, \tag{8}$$

where n is a topological number called the winding number. With such conditions we see that the magnetic field restricted to the cylinder surface reduce to:

$$\mathbf{B} = \nabla \times \mathbf{A} = -\partial_{z_+} A_\phi \hat{\rho} = -\frac{\partial_{z_+} h(z_+)}{e}\hat{\rho}\,. \tag{9}$$

In addition the ϕ-component of the covariant derivative takes the form,

$$D_\phi \psi = \partial_\phi \psi + ieA_\phi \psi\,, \tag{10}$$

where we have used $\rho = 1$. Using the ansatz (8) we can replace ψ and A_ϕ, so that,

$$\begin{aligned} D_\phi \psi &= -invf(z_+)e^{-in\phi} + ivf(z_+)h(z_+)e^{-in\phi} \\ &= ivf(z_+)[-n + h(z_+)]e^{-in\phi}\,. \end{aligned} \tag{11}$$

Thus, we may develop the term $|D_\phi \psi|^2$ to be

$$|D_\phi \psi|^2 = v^2 f(z_+)^2 [h(z_+) - n]^2\,. \tag{12}$$

On the other hand, the component $\hat{\mathbf{z}}_+$ of the covariant derivative reduce to

$$D_{z_+}\psi = \partial_{z_+}\psi = ve^{-in\phi}\partial_{z_+} f(z_+)\,, \tag{13}$$

so that

$$|D_{z_+}\phi|^2 = v^2[\partial_{z_+} f(z_+)]^2\,. \tag{14}$$

Thus, it is possible, in order to establish the suitable boundary conditions, rewrite the energy (7) in terms of the ansatz (8)

$$\begin{aligned} E = \int dz_+\, d\phi \Bigg[\frac{1}{2}\left(\frac{\partial_{z_+} h(z_+)}{e} \right)^2 + U(f) \\ + v^2[\partial_{z_+} f(z_+)]^2 + v^2 f(z_+)^2 [h(z_+) - n]^2 \Bigg]. \end{aligned} \tag{15}$$

The field equations (3) and (4) may also be written in terms of the ansatz. The set of Eq. (3) reduce to one equation:

$$\partial^2_{z_+} h(z_+) = 2v^2 e^2 [f(z_+)]^2 [h(z_+) - n]\,, \tag{16}$$

whereas Eq. (4) reads as:

$$\partial_{z_+}^2 f(z_+) - f(z_+)[h(z_+) - n]^2 - \frac{1}{2v}\frac{\partial U}{\partial f} = 0\,. \tag{17}$$

Appropriate boundary conditions for a soliton solution of finite energy should take the form

$$f(+\infty) = 1\,, \qquad f(0) = \gamma_0^{(n)}\,, \tag{18}$$

$$h(+\infty) = n\,, \qquad h(0) = 0\,, \tag{19}$$

where $\gamma_0^{(n)}$ is related to the winding number and therefore is uniquely determined for each topological sector. In particular, we will show by numerical analysis that, $0 < \gamma_0^{(n)} < 1$. Note that this requirement does not affect the regularity of the field $\psi(\phi, z)$ at $z_+ = 0$, since $z_+ = 0$ is a circle S_1 and then $\psi(\phi, 0)$ is a map $S_1 \to S_1$, i.e. for each angle ϕ we have a complex number $v\gamma_0^{(n)} e^{-in\phi}$.

The boundary conditions (19) allow one to compute the magnetic flux, the important observable for the solitons described by the Maxwell–Higgs electrodynamics. So, by using the formulas (9) and (19), the magnetic flux reads

$$\begin{aligned}\Phi &= \int dz_+\, d\phi\, B_\rho \\ &= 2\pi \int_0^\infty dz_+ \left(-\frac{\partial_{z_+} h(z_+)}{e}\right) = -\frac{2\pi}{e} n\,,\end{aligned} \tag{20}$$

as expected, it becomes a quantized quantity, i.e. proportional to the winding number n describing the respective topological sector.

In order to find the static field configurations that are stationary points of the energy, it is convenient to rewrite the expression of the energy as

$$\begin{aligned}E = 2\pi \int_0^\infty dz_+ \Bigg[&\frac{1}{2}\left(\frac{\partial_{z_+} h(z_+)}{e} \pm \sqrt{2U}\right)^2 + v^2(\partial_{z_+} f(z_+) \pm f(z_+)[h(z_+) - n])^2 \\ &\mp \frac{\partial_{z_+} h(z_+)}{e}\sqrt{2U} \mp 2v^2[h(z_+) - n] f(z_+)\partial_{z_+} f(z_+)\Bigg].\end{aligned} \tag{21}$$

Let us now define the potential term $U(f)$ to be the usual symmetry breaking one of the Maxwell–Higgs model in its self-dual form,

$$U(f) = \frac{e^2 v^4}{2}(f^2 - 1)^2\,. \tag{22}$$

Now it is not difficult to see that

$$\partial_{z_+}\left(\frac{\sqrt{2U}}{e}\right) = 2v^2 f(z_+)\partial_{z_+} f(z_+)\,, \tag{23}$$

which allows the third line of Eq. (21) to be rewritten in the following way:

$$\mp \frac{\partial_{z_+} h(z_+)}{e}\sqrt{2U} \mp 2v^2[h(z_+) - n]f(z_+)\partial_{z_+} f(z_+)$$
$$= \mp \frac{1}{e}\partial_{z_+}\left([h(z_+) - n]\sqrt{2U}\right). \tag{24}$$

Therefore, the energy (21) may be rewritten as a sum of squared terms plus a total derivative,

$$E = \pi \int_0^\infty dz_+ \left[\frac{1}{2}\left(\frac{\partial_{z_+} h(z_+)}{e} \pm \sqrt{2U}\right)^2 + v^2(\partial_{z_+} f(z_+) \pm f(z_+)[h(z_+) - n])^2 \right.$$
$$\left. \mp \frac{1}{e}\partial_{z_+}\left((h(z_+) - n)\sqrt{2U}\right)\right]. \tag{25}$$

The total derivative may be explicitly evaluated by using the expression (22) and the boundary conditions (18) and (19). In such case we have,

$$\mp 2\pi v^2 \int_0^\infty dz_+\ \partial_{z_+}[(h(z_+) - n)(f(z_+)^2 - 1)] = \pm 2\pi v^2 n[1 - (\gamma_0^{(n)})^2]. \tag{26}$$

Therefore, we see that the energy is bounded below by a multiple of the magnitude of the winding number (for positive $n[1-\gamma_0^2]$ we choose the upper sign, and for negative $n[1-\gamma_0^2]$ we choose the lower sign). Here, it is interesting to note that the topological bound (26) consist on two terms. The first of these terms is the topological bound of the energy associated to the $(2+1)$-planar Maxwell–Higgs model, i.e.

$$\pm 2\pi v^2 n\,, \tag{27}$$

which is proportional to the magnetic flux (20). The second term is

$$\mp 2\pi v^2 n(\gamma_0^{(n)})^2\,, \tag{28}$$

and it is a novel term that emerges in our theory. As we mention $\gamma_0^{(n)}$ depends on the winding number n, so that the topological bound (26) is unequivocally determined for each topological sector. In particular we will show, by numerical analysis, that $\gamma_0^{(n)} \to 0$ for large values of n, such that

$$\pm 2\pi v^2 n[1 - (\gamma_0^{(n)})^2] \to \pm 2\pi v^2 n\,. \tag{29}$$

Thus, our topological bound tends, for large values of n, to the topological bound (27) of the Maxwell–Higgs model.

The bound (26) is saturated by fields satisfying the first-order Bogomol'nyi self-duality equations,

$$\frac{\partial_{z_+} h(z_+)}{e} \pm ev^2(f^2 - 1) = 0\,, \tag{30}$$

$$\partial_{z_+} f(z_+) \pm f(z_+)(h(z_+) - n) = 0\,. \tag{31}$$

It is easily verified that BPS equations (31) and (30) solve the second-order equations of motion given in (16) and (17) if the potential is given by (22).

On the other hand, by using self-dual equations, the BPS energy density $\varepsilon(z_+)$, $[E = 2\pi\rho_0 \int_0^\infty dz_+\, \varepsilon(z_+)]$, is expressed as

$$\varepsilon(z_+) = e^2v^4(f^2-1)^2 + 2v^2f^2(h-n)^2\,, \tag{32}$$

which is a positive-definite quantity.

Let us now concentrate in the solution of Eqs. (31) and (30) at $z_+ \to +\infty$. From the preceding arguments, it is not difficult to find their asymptotic behavior

$$\begin{aligned} f(z_+) &\simeq 1 - \lambda_1 e^{-ev\sqrt{2}z_+}\,, \\ h(z_+) &\simeq n - \lambda_1 ev\sqrt{2}e^{-ev\sqrt{2}z_+}\,, \end{aligned} \tag{33}$$

with λ_1 a real number. From these solutions, we may evaluate the magnetic field at $z_+ \to +\infty$. By substituting $h(z_+)$ in expression (9) for the magnetic field we have

$$B|_{z_+\to\infty} = -2ev^2\lambda_1 e^{-ev\sqrt{2}z_+}\,, \tag{34}$$

which shows the asymptotic behavior of the magnetic field.

We also analyze the behavior of the solution at $z_+ \to 0$. In such case we have for $n > 0$:

$$\begin{aligned} f(z_+) &= \gamma_0^{(n)} - n\gamma_0^{(n)}z_+ + \cdots\,, \\ h(z_+) &= e^2v^2[1-(\gamma_0^{(n)})^2]z_+ - e^2v^2(\gamma_0^{(n)})^2 n z_+^2 + \cdots\,, \end{aligned} \tag{35}$$

where γ_0 is a real number depending on the value of n and it is determined numerically. Finally, we can evaluate the magnetic field near $z_+ = 0$ by substituting the last equation of (35) in expression (9),

$$B|_{z_+\to 0} = -ev^2[1-(\gamma_0^{(n)})^2]\,. \tag{36}$$

By comparing the expression (36) with (34) we see that the absolute value of the magnetic field has a maximum at $z_+ = 0$ and decrease exponentially as $z_+ \to +\infty$.

We have verified that the self-dual equations solved with boundary conditions (18) and (19) provide well-behaved solutions both at origin as at infinity.

Below we show the numerical solutions of the self-dual equations (30) and (31) with boundary conditions given by (18) and (19) for some values of $n > 0$. To perform the numerical analysis, we have considered $e = v = \rho_0 = 1$ and $n = 1,2,3,4,5$.

Figure 1 depicts the profiles of $f(z_+)$ which show that $\gamma_0^{(n)} = f(0) < 1$ for all n. The plots also show that $\gamma_0^{(n)}$ decreases when the values of n increase: $\gamma_0^{(1)} = 0.398239039446192$, $\gamma_0^{(2)} = 0.0823639322747160$, $\gamma_0^{(3)} = 0.00673809869063151$, $\gamma_0^{(4)} = 0.000203456264553003$, $\gamma_0^{(5)} = 0.00000225915820929528, \ldots$. The numerical analysis shows that $\gamma_0^{(n)} \to 0$ for large values of n. For all values of n and for large values of z_+ the profiles attain their asymptotic value 1.

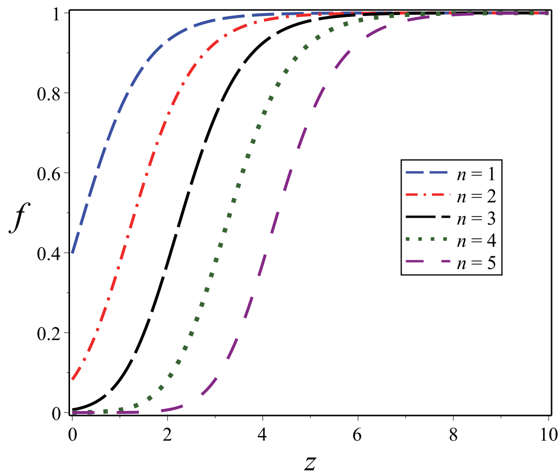

Fig. 1. The scalar field profile $f(z_+)$.

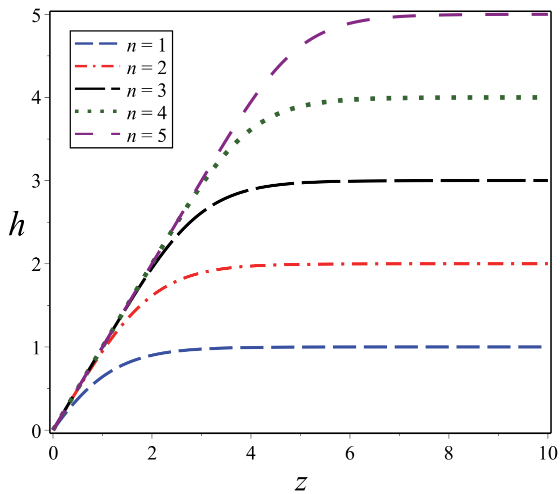

Fig. 2. The gauge field profile $f(z_+)$.

Figure 2 shows the profiles of the gauge field $h(z_+)$. As expected, they are null in $r = 0$. The behavior, near the origin, for sufficiently large values of n, is linear in z_+, in concordance with (35). For large values of z_+, the profiles reach their asymptotic value n.

Figure 3 shows the profiles of the magnetic field $B(z_+)$. The profiles, for low values of n (here $n = 1, 2$), are lumps concentrated in the origin, whose amplitude is $|B(0)| = 1 - (\gamma_0)^2 < 1$. However, for sufficiently large values of n (here $n > 3$), the profiles develop a plateau whose width increases when n increases. Such behavior resembles that of the magnetic field of the Maxwell–Higgs vortices.

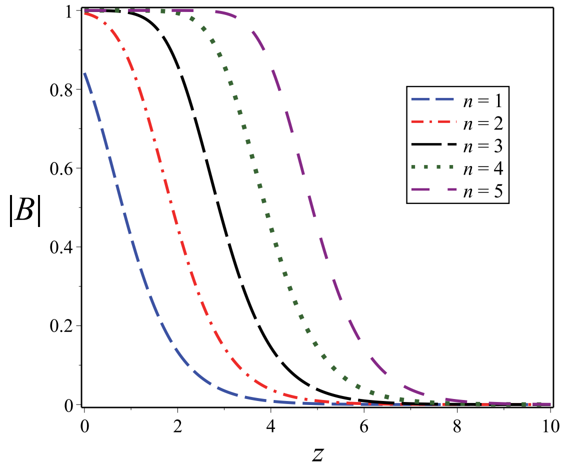

Fig. 3. The magnetic field $B(z_+)$.

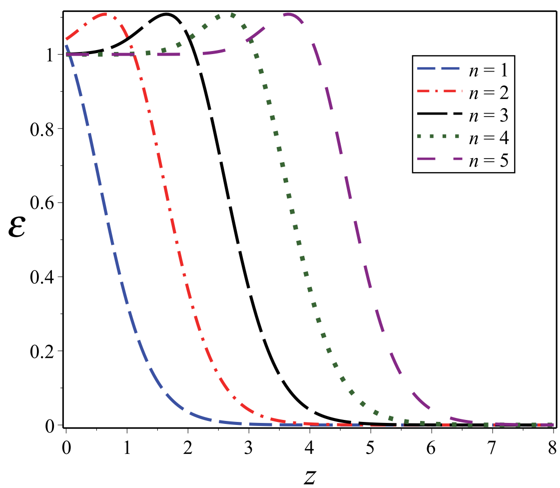

Fig. 4. The BPS energy density $\varepsilon(z_+)$.

Figure 4 describes the BPS energy density profiles $\varepsilon(z_+)$. For $n = 1$ is a lump centered at origin. For sufficiently large values of n (here $n > 2$), the profiles form a plateau, starting at origin, with amplitude 1 whose width increases when n increases. At the end of the plateau, the profile increases its values reaching a maximum value. After this, the profiles decreases rapidly to zero. Such behavior is also shown in the BPS energy density of the Maxwell–Higgs vortices.

In summary, we study the Abelian Maxwell–Higgs model on an infinite cylinder surface. We explore the Bogomol'nyi framework of the model in such surface, showing that it is possible to construct a topological soliton solutions. These solutions present interesting features, such as the behavior of the field ψ at $z_+ = 0$, which is different from zero, in contrast to Nielsen–Olesen vortex solution,[1] where $\phi(0) = 0$. We also analyze carefully the numerical solutions for the matter and gauge fields and for the magnetic field and the energy density. Specifically, we have analyzed solutions for $n > 0$ however the solutions for $n < 0$ (anti-soliton solutions) can be easily visualized by doing: $h \to -h$, $f \to f$ and consequently $B \to -B$; it gives opposite magnetic flux, as expected for an anti-soliton solution. It is worthwhile to point out that there are also similar soliton solutions for $z < 0$.

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