CYCLIC HOMOLOGY OF MONOGENIC EXTENSIONS IN THE NONCOMMUTATIVE SETTING

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Abstract. We study the Hochschild and cyclic homology of noncommutative monogenic extensions. As an application we compute the Hochschild and cyclic homology of the rank 1 Hopf algebras introduced in [K-R].

Introduction

Let k be a commutative ring with 1. A monogenic extension of k is a k-algebra $k[x]/\langle f\rangle$, where $f \in k[x]$ is a monic polynomial. In [F-G-G] this concept was generalized to the noncommutative setting. Examples are the rank 1 Hopf algebras in characteristic zero, recently introduced in [K-R]. In the paper [F-G-G], mentioned above, it was computed the Hochschild cohomology ring of these extensions. In the present paper we study their Hochschild and cyclic homology groups. The main result obtained by us, is that, for any monogenic extension A of a k-algebra K, there exists a small mixed complex $(C^S_*(A), d_*, D_*)$, giving the Hochschild and cyclic homology of A relative to K. When K is a separable k-algebra this gives the absolute Hochschild and cyclic homology groups. The mixed complex $(C^S(A), d_*, D_*)$ is enough simple to allow us to compute the homology of each rank 1 Hopf algebras.

The paper is organized as follows: In Section 1 we give some preliminaries we need. In particular we recall the simple Υ-projective resolution of a monogenic extension A/K (where Υ is the family of all A-bimodule epimorphisms which split as K-bimodule map), built in [F-G-G]. Let M and A-bimodule (symmetric over k). In Section 2 we use the mentioned above resolution to build a complex $C^S(A, M) = (C^S_*(A, M), d_*)$ giving the Hochschild homology of A relative to K, with

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coefficients in M. Then, we obtain explicit computations under suitable hypothesis. In Section 3 we prove that $C^S(A,A)$ is the Hochschild complex of a mixed complex. This generalizes the main result of [B]. Then, we use this fact to compute the cyclic homology of A in several cases, including the rank 1 Hopf algebras.

1. Preliminaries

In this Section we recall some well known definitions and results that we will use in the rest of the paper

1.1. **A simple resolution for a noncommutative monogenic extension.** In the sequel we recall the definition of noncommutative monogenic extension and we give a brief review of some of its properties (for details and proof we refer to [F-G-G]). Let k be a commutative ring, K an associative k-algebra and α a k-algebra endomorphism of K. Consider the Ore extension $E = K[x,\alpha]$, that is the algebra generated by K and x subject to the relations

$$x\lambda = \alpha(\lambda)x \quad \text{for all } \lambda \in K.$$

Let $f = x^n + \sum_{i=1}^{n} \lambda_i x^{n-i}$ be a monic polynomial of degree $n \geq 2$, where each coefficient $\lambda_i \in K$ satisfies $\alpha(\lambda_i) = \lambda_i$ and $\lambda_i\lambda = \alpha^i(\lambda)\lambda_i$ for every $\lambda \in K$. Sometimes we will write $f = \sum_{i=0}^{n} \lambda_i x^{n-i}$, assuming that $\lambda_0 = 1$. The monogenic extension of K associated with α and f is the quotient $A = E/\langle f\rangle$. It is easy to see that $\{1, x, \dots, x^{n-1}\}$ is a left K-basis of A. Moreover, given $P \in E$, there exist unique $\overline{P}$ and $\dddot{P}$ in E such that

$$P = \overline{P} f + \dddot{P} \quad \text{and} \quad \dddot{P} = 0 \text{ or } \deg \dddot{P} < n.$$

In this paper, unadorned tensor product $\otimes$ means $\otimes_K$, all the maps are k-linear and all the K-bimodule are assumed to be symmetric over k. Given a K-bimodule M, we let $M\otimes$ denote the quotient $M/[M,K]$, where $[M,K]$ is the k-module generated by the commutators $m\lambda - \lambda m$, with $\lambda \in K$ and $m \in M$. Let $A^2_{\alpha^r} = A_{\alpha^r} \otimes A$, where A_{α^r} is A endowed with the regular left A-module structure and with the right K-module structure twisted by α^r, that is, if $a \in A_{\alpha^r}$ and $\lambda \in K$, then $a \cdot \lambda = a\alpha^r(\lambda)$. We recall that

$$\frac{T}{Tx} : E \to A^2_\alpha$$

is the unique K-derivation such that $\frac{Tx}{Tx} = 1 \otimes 1$. Notice that

$$\frac{Tx^i}{Tx} = \sum_{\ell=0}^{i-1} x^\ell \otimes x^{i-\ell-1}.$$

Let Υ be the family of all A-bimodule epimorphisms which split as K-bimodule maps.

Theorem 1.1. *The complex*

$$C'_S(A) = \ \cdots \longrightarrow A^2_{\alpha^{2n+1}} \xrightarrow{d'_5} A^2_{\alpha^{2n}} \xrightarrow{d'_4} A^2_{\alpha^{n+1}} \xrightarrow{d'_3} A^2_{\alpha^{n}} \xrightarrow{d'_2} A^2_{\alpha} \xrightarrow{d'_1} A^2\ ,$$

where

$$d'_{2m+1}\colon A^2_{\alpha^{mn+1}} \to A^2_{\alpha^{mn}} \quad \textit{and} \quad d'_{2m}\colon A^2_{\alpha^{mn}} \to A^2_{\alpha^{(m-1)n+1}},$$

are defined by

$$d'_{2m+1}(1\otimes 1) = x\otimes 1 - 1\otimes x,$$
$$d'_{2m}(1\otimes 1) = \frac{Tf}{Tx} = \sum_{i=1}^{n} \lambda_{n-i} \sum_{\ell=0}^{i-1} x^{\ell}\otimes x^{i-\ell-1},$$

is a Υ-projective resolution of A.

Let $(A\otimes \overline{A}^{\otimes^*}\otimes A, b')$ be the canonical Hochschild resolution relative to K (here $\overline{A} = A/K$).

Theorem 1.2. *There are comparison maps*

$$\phi'_*\colon C'_S(A) \to (A\otimes \overline{A}^{\otimes^*}\otimes A, b') \quad \textit{and} \quad \psi'_*\colon (A\otimes \overline{A}^{\otimes^*}\otimes A, b') \to C'_S(A),$$

which are inverse one of each other up to homotopy. These maps are given by

$$\phi'_0(1\otimes 1) = 1\otimes 1,$$
$$\phi'_1(1\otimes 1) = 1\otimes x\otimes 1,$$
$$\phi'_{2m}(1\otimes 1) = \sum_{\mathbf{i}\in \mathbb{I}_m} \boldsymbol{\lambda}_{\mathbf{n}-\mathbf{i}} \sum_{\boldsymbol{\ell}\in \mathbb{J}_{\mathbf{i}}} x^{|\mathbf{i}-\boldsymbol{\ell}|-m}\otimes \widetilde{\mathbf{x}}^{\boldsymbol{\ell}_{m,1}}\otimes 1,$$
$$\phi'_{2m+1}(1\otimes 1) = \sum_{\mathbf{i}\in \mathbb{I}_m} \boldsymbol{\lambda}_{\mathbf{n}-\mathbf{i}} \sum_{\boldsymbol{\ell}\in \mathbb{J}_{\mathbf{i}}} x^{|\mathbf{i}-\boldsymbol{\ell}|-m}\otimes \widetilde{\mathbf{x}}^{\boldsymbol{\ell}_{m,1}}\otimes x\otimes 1,$$
$$\psi'_{2m}(1\otimes \mathbf{x}^{\mathbf{i}_{1,2m}}\otimes 1) = \overline{x^{i_1+i_2}}\,\overline{x^{i_3+i_4}}\cdots \overline{x^{i_{2m-1}+i_{2m}}}\otimes 1,$$
$$\psi'_{2m+1}(1\otimes \mathbf{x}^{\mathbf{i}_{1,2m+1}}\otimes 1) = \overline{x^{i_1+i_2}}\,\overline{x^{i_3+i_4}}\cdots \overline{x^{i_{2m-1}+i_{2m}}}\frac{T(x^{i_{2m+1}})}{Tx},$$

where

- $\mathbb{I}_m = \{(i_1,\dots,i_m)\in \mathbb{Z}^m : 1\le i_j\le n \textit{ for all } j\}$,
- $\mathbb{J}_{\mathbf{i}} = \{(\ell_1,\dots,\ell_m)\in \mathbb{Z}^m : 1\le \ell_j < i_j \textit{ for all } j\}$,
- $\boldsymbol{\lambda}_{\mathbf{n}-\mathbf{i}} = \lambda_{n-i_1}\cdots\lambda_{n-i_m}$,
- $\widetilde{\mathbf{x}}^{\boldsymbol{\ell}_{m,1}} = x\otimes x^{\ell_m}\otimes\cdots\otimes x\otimes x^{\ell_1}$,

- $|\mathbf{i}-\boldsymbol{\ell}| = \sum_{j=1}^{m}(i_j-\ell_j)$.
- $\mathbf{x}^{\mathbf{i}_{1r}} = x^{i_1}\otimes\cdots\otimes x^{i_r}$,

Proposition 1.3. *$\psi'_*\phi'_* = \mathrm{id}$ and a homotopy ω'_{*+1} from $\phi'_*\psi'_*$ to id is recursively defined by $\omega'_1 = 0$ and*

$$\begin{aligned}\omega'_{r+1}(\mathbf{x}\otimes 1) &= (-1)^{r+1}(\phi'_r\psi'_r - \mathrm{id} - \omega'_r b'_r)(\mathbf{x}\otimes 1)\otimes 1\\ &= (-1)^{r+1}\phi'_r\psi'_r(\mathbf{x}\otimes 1)\otimes 1 + \omega'_r(\mathbf{x})\otimes 1,\end{aligned}$$

for $\mathbf{x}\in A\otimes\overline{A}^{\otimes^r}$.

Proof. The equality $\psi'_*\phi'_* = \mathrm{id}$ follows immediately from the definitions. For the second assertion see [G-G, Proposition 1.2.1]. □

1.2. **The suspension.** The s-th suspension of a chain complex (X,d) is the complex $(X,d)[s] = (X[s],d[s])$, defined by $X[s]_* = X_{*-s}$ and $d[s]_* = (-1)^s d_{*-s}$.

1.3. **Mixed complexes.** In this subsection we recall briefly the notion of mixed complex. For more details about this concept we refer to [K] and [Bu].

A mixed complex (X,b,B) is a graded C-module $(X_r)_{r\ge 0}$, endowed with morphisms $b\colon X_r\to X_{r-1}$ and $B\colon X_r\to X_{r+1}$, such that

$$bb=0,\quad BB=0\quad\text{and}\quad Bb+bB=0.$$

A morphism of mixed complexes $f\colon (X,b,B)\to(Y,d,D)$ is a family $f_r\colon X_r\to Y_r$ of maps, such that $df = fb$ and $Df = fB$. A mixed complex $\mathcal{X} = (X,b,B)$ determines a double complex

$$\mathrm{BP}(\mathcal{X}) = \begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots\\
 & & \downarrow b & & \downarrow b & & \downarrow b & & \downarrow b\\
\cdots & \xleftarrow{B} & X_3 & \xleftarrow{B} & X_2 & \xleftarrow{B} & X_1 & \xleftarrow{B} & X_0\\
 & & \downarrow b & & \downarrow b & & \downarrow b & & \\
\cdots & \xleftarrow{B} & X_2 & \xleftarrow{B} & X_1 & \xleftarrow{B} & X_0 & & \\
 & & \downarrow b & & \downarrow b & & & & \\
\cdots & \xleftarrow{B} & X_1 & \xleftarrow{B} & X_0 & & & & \\
 & & \downarrow b & & & & & & \\
\cdots & \xleftarrow{B} & X_0 & & & & & &
\end{array}$$

By deleting the positively numbered columns we obtain a subcomplex $\mathrm{BN}(\mathcal{X})$ of $\mathrm{BP}(\mathcal{X})$. Let $\mathrm{BN}'(\mathcal{X})$ be the kernel of the canonical surjection from $\mathrm{BN}(\mathcal{X})$ to (X,b). The quotient double complex $\mathrm{BP}(\mathcal{X})/\mathrm{BN}'(\mathcal{X})$ is denoted by $\mathrm{BC}(\mathcal{X})$. The homology goups $\mathrm{HC}_*(\mathcal{X})$, $\mathrm{HN}_*(\mathcal{X})$ and $\mathrm{HP}_*(\mathcal{X})$, of the total complexes of $\mathrm{BC}(\mathcal{X})$, $\mathrm{BN}(\mathcal{X})$ and $\mathrm{BP}(\mathcal{X})$ respectively, are called the cyclic, negative and periodic homology of $\mathcal{X}$ (the n-th module of the total complex is the product of all the modules which are in the n-th diagonal). The homology $\mathrm{HH}_*(\mathcal{X})$, of (X,b), is called the Hochschild homology of $\mathcal{X}$.

If we truncate $\mathrm{BP}(\mathcal{X})$ to the left of the p-th column we obtain a complex $\mathrm{BC}(\mathcal{X})[2p]$. Note that

$$\mathrm{BC}(\mathcal{X})[0]=\mathrm{BC}(\mathcal{X}),\quad \mathrm{Tot}(\mathrm{BC}(\mathcal{X})[2p])=\mathrm{Tot}(\mathrm{BC}(\mathcal{X}))[2p]$$

and that there is a natural epimorphism

$$S\colon\ \mathrm{BC}(\mathcal{X})[2p]\to \mathrm{BC}(\mathcal{X})[2p+2]\quad \text{for each } p.$$

It is immediate that $\mathrm{Tot}(\mathrm{BP}(\mathcal{X}))=\lim_p \mathrm{Tot}\,\mathrm{BC}(\mathcal{X})[2p]$ and that there is a diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{BN}(\mathcal{X}) & \longrightarrow & \mathrm{BP}(\mathcal{X}) & \longrightarrow & \mathrm{BC}(\mathcal{X})[2] & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & (\mathcal{X}_*,b_*) & \longrightarrow & \mathrm{BC}(\mathcal{X}) & \longrightarrow & \mathrm{BC}(\mathcal{X})[2] & \longrightarrow & 0
\end{array}$$

Taking homology in the above diagram we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{B} & \mathrm{HN}_n(\mathcal{X}) & \xrightarrow{i} & \mathrm{HP}_n(\mathcal{X}) & \xrightarrow{S} & \mathrm{HC}_{n-2}(\mathcal{X}) & \xrightarrow{B} & \mathrm{HN}_{n-1}(\mathcal{X}) & \xrightarrow{i} & \cdots\\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \downarrow & & \\
\cdots & \xrightarrow{B} & \mathrm{HH}_n(\mathcal{X}) & \xrightarrow{i} & \mathrm{HC}_n(\mathcal{X}) & \xrightarrow{S} & \mathrm{HC}_{n-2}(\mathcal{X}) & \xrightarrow{B} & \mathrm{HH}_{n-1}(\mathcal{X}) & \xrightarrow{i} & \cdots
\end{array}$$

The rows in this diagram are name the SBI Connes periodicity exact sequences of $\mathcal{X}$. Finally, it is clear that a morphism $f\colon \mathcal{X}\to\mathcal{Y}$ of mixed complexes induces a morphism from the double complex $\mathrm{BP}(\mathcal{X})$ to the double complex $\mathrm{BP}(\mathcal{Y})$. Let A be a noncommutative monogenic extension of K. The normalized mixed complex of A relative to K is

$$(A\otimes \overline{A}^{\otimes^*}\otimes, b, B),$$

where b is the canonical Hochschild boundary map and

$$B([a_0\otimes\cdots\otimes a_r])=\sum_{i=0}^{r}(-1)^{ir}[1\otimes a_i\otimes\cdots\otimes a_r\otimes a_0\otimes\cdots\otimes a_{i-1}],$$

in which $[a_0\otimes\cdots\otimes a_r]$ denotes the class of $a_0\otimes\cdots\otimes a_r$ in $A\otimes\overline{A}^{\otimes^r}\otimes$. The cyclic, negative, periodic and Hochschild homology groups $\mathrm{HC}_*^K(A)$,

$\mathrm{HN}_*^K(A)$, $\mathrm{HP}_*^K(A)$ and $\mathrm{HH}_*^K(A)$ of A, are the respective homology groups of $(A\otimes \overline{A}^{\otimes^*}\otimes, b, B)$.

1.4. **The perturbation lemma.** Next we recall the perturbation lemma. We give the version introduced in [C].

A homotopy equivalence data

$$(Y,\partial) \underset{i}{\overset{p}{\leftrightarrows}} (X,d)\ ,\quad h\colon X_*\to X_{*+1}, \tag{1}$$

consists of the following:

(1) Chain complexes (Y,∂), (X,d) and quasi-isomorphisms i and p between them,
(2) A homotopy h from ip to id.

A perturbation δ of (1) is a map $\delta\colon X_*\to X_{*-1}$ such that $(d+\delta)^2=0$. We call it small if $\mathrm{id}-\delta h$ is invertible. In this case we write

$$\Delta = (\mathrm{id}-\delta h)^{-1}\delta$$

and we consider

$$(Y,\partial^1) \underset{i^1}{\overset{p^1}{\leftrightarrows}} (X,d+\delta)\ ,\quad h^1\colon X_*\to X_{*+1}, \tag{2}$$

with

$$\partial^1=\partial+p\Delta i,\quad i^1=i+h\Delta i,\quad p^1=p+p\Delta h,\quad h^1=h+h\Delta h.$$

A deformation retract is a homotopy equivalence data such that $pi=\mathrm{id}$. A deformation retract is called special if $hi=0$, $ph=0$ and $hh=0$.

In the case considered in this paper the map δh is locally nilpotent, and so $(\mathrm{id}-\delta h)^{-1}=\sum_{j=0}^{\infty}(\delta h)^j$.

Theorem 1.4. *([C]) If δ is a small perturbation of the homotopy equivalence data (1), then the perturbed data (2) is a homotopy equivalence. Moreover, if (1) is a special deformation retract, then (2) is also.*

2. Hochschild homology of A

Let k, K, α, $f=X^n+\lambda_1X^{n-1}+\cdots+\lambda_n$ and A be as in Subsection 1.1. Given an A-bimodule M, we let $[M,K]_{\alpha^j}$ denote the k-submodule of M generated by the twisted commutators $[m,\lambda]_{\alpha^j}=m\alpha^j(\lambda)-\lambda m$. As usual, we let A^e and $\mathrm{H}_*^K(A,M)$ denote the enveloping algebra $A\otimes_k A^{\mathrm{op}}$ of A and the Hochschild homology of A relative to K, with coefficients in M, respectively. When $M=A$ we will write $\mathrm{HH}_*^K(A)$ instead of $\mathrm{H}_*^K(A,A)$.

Theorem 2.1. *Let M be an A-bimodule. With the notations introduced in Theorem 1.2, we have:*

(1) *The chain complex*

$$C^S(A,M) = \ \cdots \xrightarrow{d_4} \frac{M}{[M,K]_{\alpha^{n+1}}} \xrightarrow{d_3} \frac{M}{[M,K]_{\alpha^n}} \xrightarrow{d_2} \frac{M}{[M,K]_{\alpha}} \xrightarrow{d_1} \frac{M}{[M,K]_{\alpha^0}}\ ,$$

where the boundary maps d_* are defined by

$$d_{2m+1}([\mathfrak{m}]) = [\mathfrak{m}x - x\mathfrak{m}],$$

$$d_{2m}([\mathfrak{m}]) = \sum_{i=1}^{n}\sum_{\ell=0}^{i-1}[\lambda_{n-i}x^{i-\ell-1}\mathfrak{m}x^{\ell}],$$

in which $[\mathfrak{m}]$ denotes the class of $\mathfrak{m} \in M$ in $\frac{M}{[M,K]_{\alpha^{mn+1}}}$ and $\frac{M}{[M,K]_{\alpha^{mn}}}$ respectively, computes $\mathrm{H}^K_*(A,M)$.

(2) *The maps*

$$\phi_*\colon C^S(A,M) \to (M\otimes \overline{A}^{\otimes^*}\otimes, b_*),$$

$$\psi_*\colon (M\otimes \overline{A}^{\otimes^*}\otimes, b_*) \to C^S(A,M),$$

defined by

$$\phi_0([\mathfrak{m}]) = [\mathfrak{m}],$$

$$\phi_1([\mathfrak{m}]) = [\mathfrak{m}\otimes x],$$

$$\phi_{2m}([\mathfrak{m}]) = \sum_{\mathbf{i}\in\mathbb{I}_m}\sum_{\boldsymbol{\ell}\in\mathbb{J}_{\mathbf{i}}}[\boldsymbol{\lambda}_{\mathbf{n}-\mathbf{i}}\mathfrak{m}x^{|\mathbf{i}-\boldsymbol{\ell}|-m}\otimes \widetilde{\mathbf{x}}^{\boldsymbol{\ell}_{m,1}}],$$

$$\phi_{2m+1}([\mathfrak{m}]) = \sum_{\mathbf{i}\in\mathbb{I}_m}\sum_{\boldsymbol{\ell}\in\mathbb{J}_{\mathbf{i}}}[\boldsymbol{\lambda}_{\mathbf{n}-\mathbf{i}}\mathfrak{m}x^{|\mathbf{i}-\boldsymbol{\ell}|-m}\otimes \widetilde{\mathbf{x}}^{\boldsymbol{\ell}_{m,1}}\otimes x],$$

$$\psi_{2m}([\mathfrak{m}\otimes \mathbf{x}^{\mathbf{i}_{1,2m}}]) = [\mathfrak{m}\overline{x^{i_1+i_2}}\cdots\overline{x^{i_{2m-1}+i_{2m}}}],$$

$$\psi_{2m+1}([\mathfrak{m}\otimes \mathbf{x}^{\mathbf{i}_{1,2m+1}}]) = \sum_{\ell=0}^{i_{2m+1}-1}[x^{i_{2m+1}-\ell-1}\mathfrak{m}\overline{x^{i_1+i_2}}\cdots\overline{x^{i_{2m-1}+i_{2m}}}x^{\ell}],$$

where $[\mathfrak{m}\otimes \mathbf{x}^{\mathbf{i}_{1r}}]$ denotes the class of $\mathfrak{m}\otimes \mathbf{x}^{\mathbf{i}_{1r}}$ in $M\otimes\overline{A}^{\otimes^r}\otimes$, are chain morphisms which are inverse one of each other up to homotopy.

(3) *Let*

$$\beta\colon M\otimes_{A^e} A\otimes \overline{A}^{\otimes^{r+1}}\otimes A \to M\otimes \overline{A}^{\otimes^{r+1}}\otimes$$

be the map defined by

$$\beta_{r+1}(m\otimes x_0\otimes\cdots\otimes x_{r+2}) = [x_{r+2}mx_0\otimes x_1\otimes\cdots\otimes x_{r+1}].$$

The composition $\psi_*\phi_*$ is the identity map, and the family of maps

$$\omega_{*+1}\colon M\otimes \overline{A}^{\otimes^*}\otimes \to M\otimes \overline{A}^{\otimes^{*+1}}\otimes,$$

defined by

$$\omega_{r+1}([m\otimes \mathbf{x}]) = \beta\big(m\otimes_{A^e} \omega'_{r+1}(1\otimes \mathbf{x}\otimes 1)\big),$$

is an homotopy from $\phi_*\psi_*$ to the identity map.

Proof. For the first item, apply the functor $M\otimes_{A^e} -$ to the resolution $C'_S(A)$, and use the identification

$$\begin{aligned} M\otimes_{A^e} A^2_{\alpha^j} &\xrightarrow{\cong} \frac{M}{[M,K]_{\alpha^j}} \\ \mathfrak{m}\otimes(a\otimes b) &\longmapsto [b\mathfrak{m}a]. \end{aligned}$$

For instance

$$\begin{aligned} d_{2m}([\mathfrak{m}]) &= \sum_{i=1}^{n}\sum_{\ell=0}^{i-1}[x^{i-\ell-1}\mathfrak{m}\lambda_{n-i}x^{\ell}] \\ &= \sum_{i=1}^{n}\sum_{\ell=0}^{i-1}[x^{i-\ell-1}\mathfrak{m}x^{\ell}\lambda_{n-i}] \\ &= \sum_{i=1}^{n}\sum_{\ell=0}^{i-1}[\lambda_{n-i}x^{i-\ell-1}\mathfrak{m}x^{\ell}]. \end{aligned}$$

Let ψ_* and ϕ_* be the morphisms induced by the comparison maps ψ'_* and ϕ'_*. The second and third item follow easily from Theorem 1.2 and Proposition 1.3 in a similar way. □

When $M = A$ we will write $C^S(A)$ instead of $C^S(A, M)$. The following result will be used in the proof of Theorem 3.6.

Corollary 2.2. *There is a special deformation retract*

$$\operatorname{Tot}\operatorname{BC}(C^S_*(A), d_*, 0)) \underset{\widetilde{\Phi}}{\overset{\widetilde{\Psi}}{\rightleftarrows}} \operatorname{Tot}\operatorname{BC}(A\otimes \overline{A}^{\otimes^*}\otimes, b, 0)\ , \quad \widetilde{W},$$

where

$$\begin{aligned} \widetilde{\Phi}_n([\mathbf{a}]_n, [\mathbf{a}_{n-2}],\dots) &= (\phi_n([\mathbf{a}]_n), \phi_{n-2}([\mathbf{a}_{n-2}]),\dots) \\ \widetilde{\Psi}_n(\mathbf{x}_n, \mathbf{x}_{n-2},\dots) &= (\psi_n(\mathbf{x}_n), \psi_{n-2}(\mathbf{x}_{n-2}),\dots) \end{aligned}$$

and

$$\widetilde{W}_{n+1}(\mathbf{x}_n, \mathbf{x}_{n-2}, \dots) = (\omega_{n+1}(\mathbf{x}_n), \omega_{n-1}(\mathbf{x}_{n-2}), \dots).$$

Proof. It is immediate. $\square$

2.1. **Explicit computations.** The aim of this subsection is to compute the Hochschild homology of A relative to K, with coefficients in A, under suitable hypothesis.

Theorem 2.3. *Let $C_r^S(A)$ denote the r-th module of $C^S(A)$. If there exists $\breve{\lambda} \in \mathcal{Z}(K)$ such that*

- $\alpha^n(\breve{\lambda}) = \breve{\lambda}$,
- $\breve{\lambda} - \alpha^i(\breve{\lambda})$ *is invertible in K for $1 \le i < n$,*

then $\lambda_1 = \cdots = \lambda_{n-1} = 0$ and

$$C_r^S(A) = \begin{cases} \frac{K}{[K,K]_{\alpha^{mn}}} & \text{if } r = 2m, \\ \frac{K}{[K,K]_{\alpha^{(m+1)n}}} x^{n-1} & \text{if } r = 2m+1. \end{cases}$$

Proof. Since $\breve{\lambda}\lambda_i = \lambda_i\breve{\lambda} = \alpha^i(\breve{\lambda})\lambda_i$ and $\breve{\lambda} - \alpha^i(\breve{\lambda})$ is invertible in K for $1 \le i < n$, we have that $\lambda_1 = \cdots = \lambda_{n-1} = 0$. By item (1) of Theorem 2.1 we know that

$$C_r^S(A) = \begin{cases} \frac{A}{[A,K]_{\alpha^{mn}}} & \text{if } r = 2m, \\ \frac{A}{[A,K]_{\alpha^{mn+1}}} & \text{if } r = 2m+1. \end{cases}$$

Moreover

$$[a, \lambda]_{\alpha^r} = \sum_{i=0}^{n-1} [\lambda'_i, \lambda]_{\alpha^{r+i}} x^i$$

for each $a = \sum_{i=0}^{n-1} \lambda'_i x^i \in A$ and $\lambda \in K$. Hence, it will be sufficient to check that if i is not congruent to 0 module n, then $[K, K]_{\alpha^{mn+i}} = K$. But this follows immediately from the facts that

$$[\lambda', \breve{\lambda}]_{\alpha^{mn+i}} = \lambda'\alpha^{mn+i}(\breve{\lambda}) - \breve{\lambda}\lambda' = \lambda'(\alpha^i(\breve{\lambda}) - \breve{\lambda}),$$

since $\breve{\lambda} \in \mathcal{Z}(K)$ and $\alpha^n(\breve{\lambda}) = \breve{\lambda}$, and $\alpha^i(\breve{\lambda}) - \breve{\lambda}$ is invertible if i is not congruent to 0 module n. $\square$

Theorem 2.4. *Under the hypothesis of Theorem 2.3, the boundary maps of $C^S(A)$ are given by*

$$d_{2m+1}([\lambda]x^{n-1}) = [(\alpha(\lambda) - \lambda)\lambda_n],$$

$$d_{2m+2}([\lambda]) = \left[\sum_{\ell=0}^{n-1} \alpha^{\ell}(\lambda)\right] x^{n-1},$$

for all $m \geq 0$. Consequently, if $\lambda_n = 0$, then the odd boundary maps are zero.

Proof. By item (1) of Theorem 2.1,

$$d_{2m+1}([\lambda]x^{n-1}) = [\lambda x^n - x\lambda x^{n-1}] = [(\lambda - \alpha(\lambda))x^n] = [(\alpha(\lambda) - \lambda)\lambda_n],$$

where the last equality follows from Theorem 2.3. Again by item (1) of Theorem 2.1 and Theorem 2.3,

$$d_{2m+2}([\lambda]) = \sum_{\ell=0}^{n-1}[x^{n-\ell-1}\lambda x^{\ell}] = \left[\sum_{\ell=0}^{n-1}\alpha^{n-\ell-1}(\lambda)\right]x^{n-1},$$

as we want. □

Theorem 2.4 implies that $\lambda\lambda_n - \alpha^n(\lambda)\lambda_n \in [K,K]_{\alpha^{mn}}$ for all $\lambda \in K$ and $m \geq 0$. Indeed, this can be proved directly from the hypothesis at the beginning of this paper and then it is true with full generality. In fact,

$$\lambda\lambda_n - \alpha^n(\lambda)\lambda_n = \lambda\lambda_n - \lambda_n\lambda = \lambda\alpha^{mn}(\lambda_n) - \lambda_n\lambda.$$

Corollary 2.5. *Under the hypothesis of Theorem 2.3,*

$$\mathrm{HH}_0^K(A) = \frac{K}{[K,K] + \mathrm{Im}(\alpha - \mathrm{id})\lambda_n},$$

$$\mathrm{HH}_{2m+1}^K(A) = \frac{\{\lambda \in K : (\alpha(\lambda) - \lambda)\lambda_n \in [K,K]_{\alpha^{mn}}\}}{[K,K]_{\alpha^{(m+1)n}} + \mathrm{Im}\left(\sum_{\ell=0}^{n-1}\alpha^{\ell}\right)}x^{n-1},$$

$$\mathrm{HH}_{2m+2}^K(A) = \frac{\{\lambda \in K : \sum_{\ell=0}^{n-1}\alpha^{\ell}(\lambda) \in [K,K]_{\alpha^{(m+1)n}}\}}{[K,K]_{\alpha^{(m+1)n}} + \mathrm{Im}(\alpha - \mathrm{id})\lambda_n}.$$

Assume now that k is a field, the hypothesis of Theorem 2.3 are fulfilled, K is finite dimensional over k and α is diagonalizable. Let $\omega_1 = 1, \omega_2, \dots, \omega_s$ be the eigenvalues of α and let K^{ω_h} be the eigenspace of eigenvalue ω_h. Write

$$[K,K]_{\alpha^r}^{\omega_h} = K^{\omega_h} \cap [K,K]_{\alpha^r}.$$

Note that $1, \lambda_n \in K^1$. We assert that there is a primitive n-th root of 1 in k (which implies that the characteristic of k does not divide n), and that all the n-th roots of 1 in k are eigenvalues of α . In fact, since α is diagonalizable, we can write $\breve{\lambda} = x_1 + \dots + x_s$, where x_i is an eigenvector of eigenvalue w_i. Since

$$w_1^i x_1 + \dots + w_s^i x_s = \alpha^i(\breve{\lambda}) \neq \breve{\lambda} \quad \text{for } i < n$$

and

$$w_1^n x_1 + \dots + w_s^n x_s = \alpha^n(\breve{\lambda}) = \breve{\lambda},$$

$w_1,\dots,w_s$ are n-th roots of 1 and the least common multiple of their orders is n. Hence, there exist $i_1,\dots,i_s\in\mathbb{N}$ such that $w:=w_1^{i_1}\cdots w_s^{i_s}$ is a primitive n-th root of 1, and so $(x_1^{i_1}\cdots x_s^{i_s})^i$ is an eigenvector of eigenvalue w^i of α, because α is an algebra morphism.

Theorem 2.6. *The chain complex $C^S(A)$ decomposes as a direct sum $C^S(A)=\bigoplus_{h=1}^s C^{S,\omega_h}(A)$, where*

$$C_r^{S,\omega_h}(A)=\begin{cases}\frac{K^{\omega_h}}{[K,K]^{\omega_h}_{\alpha^{mn}}} & \text{if } r=2m,\\ \frac{K^{\omega_h}}{[K,K]^{\omega_h}_{\alpha^{(m+1)n}}}x^{n-1} & \text{if } r=2m+1.\end{cases}$$

Moreover the boundary maps $d_*^{\omega_h}$ of $C_r^{S,\omega_h}(A)$ are given by:

$$d_{2m}^{\omega_h}([\lambda])=\left(\sum_{\ell=0}^{n-1}\omega_h^\ell\right)[\lambda]x^{n-1}\quad\text{and}\quad d_{2m+1}^{\omega_h}([\lambda]x^{n-1})=(\omega_h-1)[\lambda\lambda_n].$$

Proof. It follows easily from Theorem 2.3 and 2.4, since the fact that $\lambda_n\in K^1$ implies that if $\lambda\in K^{\omega_h}$, then $\lambda\lambda_n\in K^{\omega_h}$ (and so $C^{S,\omega_h}(A)$ is a subcomplex of $C^S(A)$). □

Corollary 2.7. *Let $\mathrm{HH}_*^{K,\omega_h}(A)$ denote the homology of $C^{S,\omega_h}(A)$. By Theorem 2.1 and 2.6 we know that $\mathrm{HH}_*^K(A)=\bigoplus_{h=1}^s\mathrm{HH}_*^{K,\omega_h}(A)$. Moreover,*

$$\mathrm{HH}_0^{K,\omega_h}(A)=\begin{cases}\frac{K^1}{[K,K]^1} & \text{if } h=1,\\ \frac{K^{\omega_h}}{[K,K]^{\omega_h}+K^{\omega_h}\lambda_n} & \text{if } h\ne 1,\end{cases}$$

$$\mathrm{HH}_{2m+1}^{K,\omega_h}(A)=\begin{cases}\frac{\left\{\lambda\in K^{\omega_h}:\lambda\lambda_n\in[K,K]^{\omega_h}_{\alpha^{mn}}\right\}}{[K,K]^{\omega_h}_{\alpha^{(m+1)n}}}x^{n-1} & \text{if } h\ne 1 \text{ and } \omega_h^n=1,\\ 0 & \text{otherwise,}\end{cases}$$

$$\mathrm{HH}_{2m+2}^{K,\omega_h}(A)=\begin{cases}\frac{K^{\omega_h}}{[K,K]^{\omega_h}_{\alpha^{(m+1)n}}+K^{\omega_h}\lambda_n} & \text{if } h\ne 1 \text{ and } \omega_h^n=1,\\ 0 & \text{otherwise,}\end{cases}$$

Note that if α^n has finite order v (that is $\alpha^{nv}=\mathrm{id}$ and $\alpha^{nj}\ne\mathrm{id}$ for $0<j<v$), then

$$\mathrm{HH}_{2m+1}^{K,\omega_h}(A)=\mathrm{HH}_{2(m+v)+1}^{K,\omega_h}(A)\quad\text{and}\quad\mathrm{HH}_{2m+2}^{K,\omega_h}(A)=\mathrm{HH}_{2(m+v)+2}^{K,\omega_h}(A)$$

for all $m\ge 0$.

Example 2.8. Let k be a field, $K=k[G]$ the group k-algebra of a finite group G and $\chi\colon G\to k^\times$ a character, where $k^\times$ is the group of unities of k. Let $\alpha\colon K \to K$ be the automorphism defined by $\alpha(g) = \chi(g)g$ and let $f = x^n + \lambda_n \in K[x]$ be a monic polynomial whose coefficients satisfy the hypothesis required in the introduction. Assume that there exists $g_1 \in \mathcal{Z}(G)$ such that $\chi(g_1)$ is a primitive n-th root of 1. Here we apply the results obtained in Section 2 to compute the Hochschild homology of $A = K[x,\alpha]/\langle f\rangle$ relative to K, with coefficients in A (if the characteristic of k is relative prime to the order of G, then $k[G]$ is a separable k-algebra and so, by [G-S, Theorem 1.2], $\mathrm{HH}^K_*(A)$ coincides with the absolute Hochschild homology $\mathrm{HH}_*(A)$ of A). Note that the hypothesis of Theorem 2.3 are fulfilled, taking $\breve{\lambda} = g_1$. Since α is diagonalizable Theorem 2.6 and Corollary 2.7 apply. In this case

$$\{\omega_1,\dots,\omega_s\} = \chi(G),$$

$$K^{\omega_h} = \bigoplus_{\{g\in G:\chi(g)=\omega_h\}} kg,$$

$$[K,K]^{\omega_h}_{\alpha^j} = \sum_{\{g_1,g_2\in G:\chi(g_1g_2)=\omega_h\}} k(\chi^j(g_2)g_1g_2 - g_2g_1).$$

Next we consider another situation in which the cohomology of A can be computed. The following results are very close to the ones valid in the commutative setting.

Theorem 2.9. *If α is the identity map, then*

$$C^S_r(A) = \frac{K}{[K,K]} \oplus \frac{K}{[K,K]}x \oplus \dots \oplus \frac{K}{[K,K]}x^{n-1} = \frac{A}{[A,A]}.$$

Moreover, the odd boundary maps d_{2m+1} of $C^S(A)$ are zero, and the even boundary maps d_{2m} takes $[a]$ to $[f'a]$.

Proof. This is an immediate consequence of Theorem 2.1. □

Corollary 2.10. *If α is the identity map, then*

$$\mathrm{HH}^K_0(A) = \frac{A}{[A,A]},$$

$$\mathrm{HH}^K_{2m+1}(A) = \frac{A}{[A,A]+f'A},$$

$$\mathrm{HH}^K_{2m+2}(A) = \frac{([A,A]:f')}{[A,A]},$$

where $([A,A]:f') = \{a\in A : f'a \in [A,A]\}$.

2.2. **Hochschild homology of rank 1 Hopf algebras.** Let k be a characteristic zero field and let $n \geq 2$ be a natural number. Recall that $k^\times$ denotes the group of unities of k. Let G be a finite group and $\chi\colon G \to k^\times$ a character. Assume that there exists $g_1 \in \mathcal{Z}(G)$ such that $\chi(g_1)$ is a primitive n-th root of 1. In this section we compute the Hochschild homology of the k-algebra $A = k[G][x,\alpha]/\langle x^n - \xi(g_1^n - 1)\rangle$, where $\xi \in k$ and $\alpha \in \operatorname{Aut}(k[G])$ is defined by $\alpha(g) = \chi(g)g$. We divide the problem in three cases. The first and second ones give Hochschild homology of rank 1 Hopf algebras. For the sake of completeness we recall from [K-R] that A is the underlying algebra of a rank 1 Hopf algebra if $\xi(g_1^n - 1) = 0$ or $\chi^n = 1$. In both cases the comultiplication Δ is determined by

$$\Delta(x) = x \otimes g_1 + 1 \otimes x \quad \text{and} \quad \Delta(g) = g \otimes g \ \text{ for all } g \in G,$$

the counit ϵ by $\epsilon(x) = 0$ and $\epsilon(g) = 1$, and antipode S by $S(x) = -g_1^{-1}x$ and $S(g) = g^{-1}$, for all $g \in G$.

Let $C_n \subseteq k$ be the set of all n-th roots of 1.

$\boldsymbol{\xi = 0}$**.** In this case $A = K[x,\alpha]/\langle x^n\rangle$, where $K = k[G]$. Since K is separable over k, we know that $\mathrm{HH}_*(A) = \mathrm{HH}_*^K(A)$. So, by Corollary 2.7,

$$\begin{aligned}
\mathrm{HH}_0(A) &= \frac{K}{[K,K]},\\
\mathrm{HH}_{2m+1}(A) &= \bigoplus_{\omega \in C_n \setminus \{1\}} \frac{K^\omega}{[K,K]^\omega_{\alpha^{(m+1)n}}} x^{n-1},\\
\mathrm{HH}_{2m+2}(A) &= \bigoplus_{\omega \in C_n \setminus \{1\}} \frac{K^\omega}{[K,K]^\omega_{\alpha^{(m+1)n}}}.
\end{aligned}$$

$\boldsymbol{\xi \neq 0}$ **and** $\boldsymbol{\chi^n = 1}$**.** In this case $f = x^n - \xi(g_1^n - 1)$ satisfies the hypothesis required in the preliminaries. In fact

$$\alpha(\xi(g_1^n - 1)) = \xi(g_1^n - 1)$$

since $\alpha(g_1^n) = \chi(g_1^n)g_1^n = \chi(g_1)^n g_1^n = g_1^n$, and

$$\xi(g_1^n - 1)\lambda = \alpha^n(\lambda)\xi(g_1^n - 1) \quad \text{for all } \lambda \in k[G],$$

since $\xi(g_1^n - 1) \in \mathcal{Z}(G)$ and $\alpha^n(\lambda) = \lambda$, because $\chi^n = 1$. Note also that C_n is the set of eigenvalues of α, since G is a multiplicative basis of eigenvectors of α, the eigenvalue $\chi(g_1)$ of g_1 is a primitive n-th root of 1 and the eigenvalue $\chi(g)$ of every $g \in G$ is a n-th root of 1 (again because $\chi^n = 1$). Moreover, the algebra $K = k[G]$ is separable over k and so, $\mathrm{HH}_*(A) = \mathrm{HH}_*^K(A)$. Again by Corollary 2.7,

$$\mathrm{HH}_0(A) = \frac{K^1}{[K,K]^1} \oplus \bigoplus_{\omega\in C_n\setminus\{1\}} \frac{K^\omega}{[K,K]^\omega + K^\omega(g_1^n - 1)},$$

$$\mathrm{HH}_{2m+1}(A) = \bigoplus_{\omega\in C_n\setminus\{1\}} \frac{\{\lambda\in K^\omega : \lambda(g_1^n-1)\in [K,K]^\omega\}}{[K,K]^\omega} x^{n-1},$$

$$\mathrm{HH}_{2m+2}(A) = \bigoplus_{\omega\in C_n\setminus\{1\}} \frac{K^\omega}{[K,K]^\omega + K^\omega(g_1^n - 1)}.$$

$\boldsymbol{\xi \neq 0}$ **and** $\boldsymbol{\chi^n \neq 1}$**.** Let $g \in G$ such that $\chi^n(g) \neq 1$. Since

$$g^{-1}(x^n - \xi(g_1^n - 1))g = \chi^n(g)x^n - \xi(g_1^n - 1),$$

we conclude that the ideal $\langle x^n - \xi(g_1^n - 1)\rangle$ coincides with the ideal $\langle x^n, g_1^n - 1\rangle$. So, $A = k[G/\langle g_1^n\rangle][x,\widetilde{\alpha}]/\langle x^n\rangle$, where $\widetilde{\alpha}$ is the automorphism induced by α. We consider now $K = k[G/\langle g_1^n\rangle]$ and $f = x^n$. These data satisfy the hypothesis of Theorem 2.6 with $\breve{\lambda}$ the class of g_1 in $G/\langle g_1^n\rangle$. Moreover the algebra $K = k[G/\langle g_1^n\rangle]$ is separable over k and so, $\mathrm{HH}_*(A) = \mathrm{HH}_*^K(A)$. Thus, by Corollary 2.7,

$$\mathrm{HH}_0(A) = \frac{K}{[K,K]},$$

$$\mathrm{HH}_{2m+1}(A) = \bigoplus_{\omega\in C_n\setminus\{1\}} \frac{K^\omega}{[K,K]^\omega_{\widetilde{\alpha}^{(m+1)n}}} x^{n-1},$$

$$\mathrm{HH}_{2m+2}(A) = \bigoplus_{\omega\in C_n\setminus\{1\}} \frac{K^\omega}{[K,K]^\omega_{\widetilde{\alpha}^{(m+1)n}}}.$$

3. Cyclic homology of A

Let k, K, α, $f = X^n + \lambda_1 X^{n-1} + \cdots + \lambda_n$ and A be as in Subsection 1.1. In this section we get a mixed complex, simpler than the canonical of Tsygan, computing the cyclic homology of A relative to K.

A simple tensor $a_0 \otimes \cdots \otimes a_r \in A \otimes \overline{A}^{\otimes^r}$ will be called *monomial* if there exist $\lambda \in K\setminus\{0\}$, $0 \le i_0 < n$ and $1 \le i_1, \dots, i_r < n$ such that $a_0 = \lambda x^{i_0}$ and $a_j = x^{i_j}$ for $j > 0$. We define the *degree* of a monomial tensor

$$\lambda x^{i_0} \otimes \cdots \otimes x^{i_r} \in A \otimes \overline{A}^{\otimes^r},$$

as $\deg(\lambda x^{i_0} \otimes \cdots \otimes x^{i_r}) = i_0 + \cdots + i_r$. Since $1, x, \dots, x^{n-1}$ is a basis of A as a left K-module, each element $\mathbf{a} \in A \otimes \overline{A}^{\otimes^r}$ can be written in a unique way as a sum of monomial tensors. The *degree* $\deg(\mathbf{a})$, of $\mathbf{a}$, is the maximum of the degrees of its monomial tensors. Since $[A\otimes \overline{A}^{\otimes^r}, K]$ is an homogeneous k-submodule of $A\otimes \overline{A}^{\otimes^r}$ we have a well defined concept of degree on $A\otimes \overline{A}^{\otimes^r}\otimes$. Similarly it can be defined the *degree* $\deg(\mathbf{a})$ of an element $\mathbf{a}\in A\otimes \overline{A}^{\otimes^r}\otimes A$.

Proposition 3.1. *Let ω_{r+1} as in item (3) of Theorem 2.1. It is true that* $\deg(\omega_{r+1}(\mathbf{a}))\le \deg(\mathbf{a})$.

Proof. Let $\mathbf{x}_1 = 1\otimes x^{i_1}\otimes\cdots\otimes x^{i_r}\otimes 1\in A\otimes\overline{A}^{\otimes^r}\otimes A$. By the definition of ω_{r+1} it suffices to show that $\omega'_{r+1}(\mathbf{x}_1)$ is a sum of tensors of the form

$$\lambda' x^{j_0}\otimes x^{j_1}\otimes\cdots\otimes x^{j_{r+2}},$$

with $j_0+\cdots+j_{r+2}\le i_1+\cdots+i_r$. Using the formulas for ϕ'_r and ψ'_r establish in Theorem 1.2 it is easy to see that

$$\deg(\phi'_r\psi'_r(\mathbf{x}_1))\le \deg(\mathbf{x}_1).$$

The fact that $w'_{r+1}(x_1)$ can be expressed as a sum of simple tensors satisfying the mentioned above property follows now by induction on r, since

$$\omega'_{r+1}(\mathbf{x}_1) = (-1)^{r+1}\phi'_r\psi'_r(\mathbf{x}_1)\otimes 1 + \omega'_r(\mathbf{x}_2)x^{i_r}\otimes 1,$$

where $\mathbf{x}_2 = 1\otimes x^{i_1}\otimes\cdots\otimes x^{i_{r-1}}\otimes 1$. □

Let $D_r\colon C^S_r(A)\to C^S_{r+1}(A)$ be the composition $D_r=\psi_{r+1}B_r\phi_r$.

Theorem 3.2. $(C^S_*(A), d_*, D_*)$ *is a mixed complex, giving the Hochschild, cyclic, negative and periodic homology of A relative to K.*

Proof. By Theorem 2.1 we already know that the Hochschild homology of $(C^S_*(A), d_*, D_*)$ is the Hochschild homology of A relative to K. Let

$$\mathcal{X} = (C^S_*(A), d_*, D_*) \quad\text{and}\quad \mathcal{X}' = \mathrm{BC}(A\otimes\overline{A}^{\otimes^*}\otimes, b_*, B_*).$$

By the perturbation lema, in order to prove the assertion for the cyclic homology it suffices to check that there is a special deformation retract

$$\operatorname{Tot}\mathrm{BC}(\mathcal{X}) \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \operatorname{Tot}\mathrm{BC}(\mathcal{X}'),\quad W. \tag{3}$$

Finally, in order to prove the assertion for the periodic and negative homology it suffices to show that the maps Φ, Ψ and W commute with the canonical surjections

$$\operatorname{Tot}\mathrm{BC}(\mathcal{X})\to\operatorname{Tot}\mathrm{BC}(\mathcal{X})[2]\quad\text{and}\quad \operatorname{Tot}\mathrm{BC}(\mathcal{X}')\to\operatorname{Tot}\mathrm{BC}(\mathcal{X}')[2].$$

In fact, from this, the fact that

$$\operatorname{Tot}\mathrm{BP}(\mathcal{X}) = \lim_p \operatorname{Tot}\mathrm{BC}(\mathcal{X})[2p],\quad \operatorname{Tot}\mathrm{BP}(\mathcal{X}') = \lim_p \operatorname{Tot}\mathrm{BC}(\mathcal{X}')[2p]$$

and (3), it follows that there is a special deformation retract

$$\operatorname{Tot}\operatorname{BP}(\mathcal{X}) \underset{\widehat{\Phi}}{\overset{\widehat{\Psi}}{\rightleftarrows}} \operatorname{Tot}\operatorname{BP}(\mathcal{X}'), \quad \widehat{W},$$

which immediately implies the assertion for the periodic homology, and also for the negative homology, because of the existence of a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Tot}\operatorname{BN}(\mathcal{X}) & \longrightarrow & \operatorname{Tot}\operatorname{BP}(\mathcal{X}) & \longrightarrow & \operatorname{Tot}\operatorname{BC}(\mathcal{X})[2] & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \widehat{\Phi}} & & \downarrow{\scriptstyle \Phi} & & \\ 0 & \longrightarrow & \operatorname{Tot}\operatorname{BN}(\mathcal{X}') & \longrightarrow & \operatorname{Tot}\operatorname{BP}(\mathcal{X}') & \longrightarrow & \operatorname{Tot}\operatorname{BC}(\mathcal{X}')[2] & \longrightarrow & 0 \end{array}$$

with Φ and $\widehat{\Phi}$ quasi-isomorphisms, it follows that there is a quasi-isomorphism $\operatorname{Tot}\operatorname{BN}(\mathcal{X}) \to \operatorname{Tot}\operatorname{BN}(\mathcal{X}')$ making the diagram commutative.

Next we prove there is a special deformation retract (3) satisfying the above required conditions. Let

$$\operatorname{Tot}\operatorname{BC}(C_*^S(A), d_*, 0)) \underset{\widetilde{\Phi}}{\overset{\widetilde{\Psi}}{\rightleftarrows}} \operatorname{Tot}\operatorname{BC}(A \otimes \overline{A}^{\otimes^*} \otimes, b, 0), \quad \widetilde{W},$$

be the special deformation retract obtained in Corollary 2.2. Consider the perturbation induced by B. Applying the perturbation lemma we obtain a special deformation retract

$$(\widehat{C}_*^S(A), \widehat{d}_*) \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \operatorname{Tot}\operatorname{BC}(A \otimes \overline{A}^{\otimes^*} \otimes, b, B), \quad W,$$

where

$$\widehat{C}_n^S(A) = C_n^S(A) \oplus C_{n-2}^S(A) \oplus \cdots$$

and $\widehat{d}_n = \sum_{j\ge 0} \psi_{n-2l+2j+1}(B\omega)^j B\phi_{n-2l}$ on $C_{n-2l}^S(A)$. In order to finish the proof it suffices to show that $\psi_{r+2j+1}(B\omega)^j B\phi_r = 0$ for all $j > 0$. Assume first that $r = 2m$. By the definition of ϕ_{2m} and Proposition 3.1,

$$\deg((B\omega)^j B\phi_{2m}([\lambda' x^j]) < mn + n$$

On the other hand $\psi_{2m+2j+1}$ vanishes on elements of degree less than $(m+j)n$. The fact that $\psi_{r+2j+1}(B\omega)^j B\phi_r = 0$ for all $j > 0$ follows by combining theses facts. The case $r = 2m+1$ is similar. $\square$

Recall from Subsection 1.1, that given $P \in E$, there exist unique $\overline{P}$ and $\dddot{P}$ in E such that

$$P = \overline{P} f + \dddot{P} \quad \text{and} \quad \dddot{P} = 0 \text{ or } \deg \dddot{P} < n.$$

Theorem 3.3. *The Connes operator D_* is given by*

$$D_{2m}([\lambda x^j]) = \left[\sum_{h=0}^{j-1} \alpha^{mn+h}(\lambda) x^{j-1}\right] + \left[\sum_{i=1}^{n}\left(\sum_{u=0}^{m-1}\sum_{\ell=0}^{i-1}\alpha^{nu+\ell}(\lambda)\right)\overline{\lambda_{n-i}x^{i-1+j}}\right],$$

$$D_{2m+1}([\lambda x^j]) = \begin{cases} 0 & \text{if } j < n-1,\\ [(\mathrm{id}-\alpha)(\sum_{u=0}^{m}\alpha^{nu}(\lambda))] & \text{if } j = n-1.\end{cases}$$

Proof. Besides the notations introduced in Theorem 1.2 we use the following ones.

- $\breve{\mathbf{x}}^{\boldsymbol{\ell}_{u,1}} = x^{\ell_u}\otimes x\otimes\cdots\otimes x^{\ell_1}\otimes x$,
- $\widetilde{\mathbf{x}}^{\boldsymbol{\ell}_{m,u+1}} = x\otimes x^{\ell_m}\otimes\cdots\otimes x\otimes x^{\ell_{u+1}}$,
- $|\boldsymbol{\ell}_{u,1}| = \ell_1+\cdots+\ell_u+u$.

We shall first compute D_{2m+1}. By definition

$$B\phi_{2m+1}([\lambda x^j]) = \sum_{u=0}^{m}\sum_{\mathbf{i}\in\mathbb{I}_m}\sum_{\boldsymbol{\ell}\in\mathbb{J}_{\mathbf{i}}}\Delta^{\boldsymbol{\ell}}_{\mathbf{i},u} - \sum_{u=0}^{m}\sum_{\mathbf{i}\in\mathbb{I}_m}\sum_{\boldsymbol{\ell}\in\mathbb{J}_{\mathbf{i}}}\Gamma^{\boldsymbol{\ell}}_{\mathbf{i}},$$

where

$$\Delta^{\boldsymbol{\ell}}_{\mathbf{i},u} = [\boldsymbol{\lambda}_{\mathbf{n}-\mathbf{i}}\alpha^{|\boldsymbol{\ell}_{u,1}|}(\lambda)\otimes\breve{\mathbf{x}}^{\boldsymbol{\ell}_{u,1}}\otimes x^j x^{|\mathbf{i}-\boldsymbol{\ell}|-m}\otimes\widetilde{\mathbf{x}}^{\boldsymbol{\ell}_{m,u+1}}\otimes x]$$

and

$$\Gamma^{\boldsymbol{\ell}}_{\mathbf{i},u} = [\boldsymbol{\lambda}_{\mathbf{n}-\mathbf{i}}\alpha^{|\boldsymbol{\ell}_{u,1}|+1}(\lambda)\otimes\widetilde{\mathbf{x}}^{\boldsymbol{\ell}_{u,1}}\otimes x\otimes x^j x^{|\mathbf{i}-\boldsymbol{\ell}|-m}\otimes\widetilde{\mathbf{x}}^{\boldsymbol{\ell}_{m,u+1}}].$$

If $\psi_{2m+2}(\Delta^{\boldsymbol{\ell}}_{\mathbf{i},u})\neq 0$, then $\ell_1=\cdots=\ell_m=n-1$. So $i_1=\cdots=i_m=n$. Thus,

$$\sum_{\mathbf{i}\in\mathbb{I}_m}\sum_{\boldsymbol{\ell}\in\mathbb{J}_{\mathbf{i}}}\psi_{2m+2}(\Delta^{\boldsymbol{\ell}}_{\mathbf{i},u}) = [\alpha^{nu}(\lambda)\overline{x^{j+1}}] = \begin{cases}0 & \text{if } j<n-1,\\ [\alpha^{nu}(\lambda)] & \text{if } j=n-1.\end{cases}$$

Similarly, $\psi_{2m+2}(\Gamma^{\boldsymbol{\ell}}_{\mathbf{i},u})\neq 0$ implies that $\ell_1=\cdots=\ell_m=n-1$. Hence $i_1=\cdots=i_m=n$ and

$$\sum_{\mathbf{i}\in\mathbb{I}_m}\sum_{\boldsymbol{\ell}\in\mathbb{J}_{\mathbf{i}}}\psi_{2m+2}(\Gamma^{\boldsymbol{\ell}}_{\mathbf{i},u}) = [\alpha^{nu+1}(\lambda)\overline{x^{j+1}}] = \begin{cases}0 & \text{if } j<n-1,\\ [\alpha^{nu+1}(\lambda)] & \text{if } j=n-1.\end{cases}$$

The formula for D_{2m+1} follows immediately from these facts. We now compute D_{2m}. By definition

$$B\phi_{2m}([\lambda x^j]) = \sum_{u=0}^{m-1} \sum_{\mathbf{i}\in\mathbb{I}_m} \sum_{\boldsymbol{\ell}\in\mathbb{J}_\mathbf{i}} (\Gamma^{\boldsymbol{\ell}}_{\mathbf{i},u} + \Delta^{\boldsymbol{\ell}}_{\mathbf{i},u}) + \sum_{\mathbf{i}\in\mathbb{I}_m} \sum_{\boldsymbol{\ell}\in\mathbb{J}_\mathbf{i}} \Upsilon^{\boldsymbol{\ell}}_{\mathbf{i}},$$

where

$$\begin{aligned}
\Gamma^{\boldsymbol{\ell}}_{\mathbf{i},u} &= [\boldsymbol{\lambda}_{\mathbf{n}-\mathbf{i}}\alpha^{|\boldsymbol{\ell}_{u,1}|}(\lambda) \otimes \widetilde{\mathbf{x}}^{\boldsymbol{\ell}_{u,1}} \otimes x^j x^{|\mathbf{i}-\boldsymbol{\ell}|-m} \otimes \widetilde{\mathbf{x}}^{\boldsymbol{\ell}_{m,u+1}}],\\
\Delta^{\boldsymbol{\ell}}_{\mathbf{i},u} &= [\boldsymbol{\lambda}_{\mathbf{n}-\mathbf{i}}\alpha^{|\boldsymbol{\ell}_{u+1,1}|-1}(\lambda) \otimes x^{\ell_{u+1}} \otimes \widetilde{\mathbf{x}}^{\boldsymbol{\ell}_{u,1}} \otimes x^j x^{|\mathbf{i}-\boldsymbol{\ell}|-m} \otimes \widetilde{\mathbf{x}}^{\boldsymbol{\ell}_{m,u+2}} \otimes x],\\
\Upsilon^{\boldsymbol{\ell}}_{\mathbf{i}} &= [\boldsymbol{\lambda}_{\mathbf{n}-\mathbf{i}}\alpha^{|\boldsymbol{\ell}_{m,1}|}(\lambda) \otimes \widetilde{\mathbf{x}}^{\boldsymbol{\ell}_{m,1}} \otimes x^j x^{|\mathbf{i}-\boldsymbol{\ell}|-m}].
\end{aligned}$$

If $\psi_{2m+1}(\Gamma^{\boldsymbol{\ell}}_{\mathbf{i},u}) \neq 0$, then $\ell_1 = \cdots = \widehat{\ell_{u+1}} = \cdots = \ell_m = n-1$. In this case $i_1 = \cdots = \widehat{i_{u+1}} = \cdots = i_m = n$ and

$$\begin{aligned}
\psi_{2m+1}(\Gamma^{\boldsymbol{\ell}}_{\mathbf{i},u}) &= \sum_{h=0}^{\ell-1} \left[x^{\ell-h-1}\lambda_{n-i}\alpha^{nu}(\lambda)\, \overline{\dot{\overline{x^{j+i-\ell-1}}}\; xx^h}\right]\\
&= \sum_{h=0}^{\ell-1} \left[\lambda_{n-i}\alpha^{nu+\ell-h-1}(\lambda)x^{\ell-h-1}\, \overline{\dot{\overline{x^{j+i-\ell-1}}}\; xx^h}\right]\\
&= \sum_{h=0}^{\ell-1} \left[\lambda_{n-i}\alpha^{nu+\ell-h-1}(\lambda)x^{\ell-1}\, \overline{\dot{\overline{x^{j+i-\ell-1}}}\; x}\right]\\
&= \sum_{h=0}^{\ell-1} \left[\lambda_{n-i}\alpha^{nu+\ell-h-1}(\lambda)x^{\ell-1} \left(\overline{x^{j+i-\ell}} - \overline{x^{j+i-\ell-1}}x\right)\right].
\end{aligned}$$

In the third equality we have used that

$$\overline{\dot{\overline{x^{j+i-\ell-1}}}\; xx^h} = x^h\, \overline{\dot{\overline{x^{j+i-\ell-1}}}\; x},$$

which is valid since

$$\overline{\dot{\overline{x^{j+i-\ell-1}}}\; x} \in \mathbb{Z}[\lambda_1,\dots,\lambda_{n-1}].$$

So,

$$\begin{aligned}
\sum_{\mathbf{i}\in\mathbb{I}_m} \sum_{\boldsymbol{\ell}\in\mathbb{J}_\mathbf{i}} \psi_{2m+1}(\Gamma^{\boldsymbol{\ell}}_{\mathbf{i},u}) &= \sum_{i=1}^{n}\sum_{\ell=1}^{i-1}\sum_{h=0}^{\ell-1} \left[\lambda_{n-i}\alpha^{nu+\ell-h-1}(\lambda)x^{\ell-1}\overline{x^{j+i-\ell}}\right]\\
&\quad - \sum_{i=1}^{n}\sum_{\ell=2}^{i}\sum_{h=1}^{\ell-1} \left[\lambda_{n-i}\alpha^{nu+\ell-h-1}(\lambda)x^{\ell-1}\overline{x^{j+i-\ell}}\right]\\
&= \sum_{i=1}^{n}\sum_{\ell=1}^{i-1} \left[\lambda_{n-i}\alpha^{nu+\ell-1}(\lambda)x^{\ell-1}\overline{x^{j+i-\ell}}\right].
\end{aligned}$$

Similarly, $\psi_{2m+1}(\Delta^{\boldsymbol{\ell}}_{\mathbf{i},u}) \neq 0$ implies $\ell_2 = \cdots = \ell_m = n-1$. In this case $i_2 = \cdots = i_m = n$ and

$$\begin{aligned}\psi_{2m+1}(\Delta^{\boldsymbol{\ell}}_{\mathbf{i},u}) &= \Big[\lambda_{n-i_1}\alpha^{nu+\ell_1}(\lambda)\overline{x^{\ell_1}\ \overline{x^{j+i_1-\ell_1-1}}}\Big] \\ &= \Big[\lambda_{n-i_1}\alpha^{nu+\ell_1}(\lambda)\Big(\overline{x^{j+i_1-1}} - x^{\ell_1}\overline{x^{j+i_1-\ell_1-1}}\Big)\Big].\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{\mathbf{i}\in\mathbb{I}_m}\sum_{\boldsymbol{\ell}\in\mathbb{J}_{\mathbf{i}}}\psi_{2m+1}(\Delta^{\boldsymbol{\ell}}_{\mathbf{i},u}) &= \sum_{i=1}^{n}\left[\lambda_{n-i}\left(\sum_{\ell=1}^{i-1}\alpha^{nu+\ell}(\lambda)\right)\overline{x^{j+i-1}}\right] \\ &\quad - \sum_{i=1}^{n}\sum_{\ell=1}^{i-1}\Big[\lambda_{n-i}\alpha^{nu+\ell}(\lambda)x^{\ell}\overline{x^{j+i-\ell-1}}\Big].\end{aligned}$$

Consequently,

$$\begin{aligned}\sum_{\mathbf{i}\in\mathbb{I}_m}\sum_{\boldsymbol{\ell}\in\mathbb{J}_{\mathbf{i}}}\psi_{2m+1}(\Gamma^{\boldsymbol{\ell}}_{\mathbf{i},u} + \Delta^{\boldsymbol{\ell}}_{\mathbf{i},u}) &= \sum_{i=1}^{n}\sum_{\ell=1}^{i-1}\Big[\lambda_{n-i}\alpha^{nu+\ell-1}(\lambda)x^{\ell-1}\overline{x^{j+i-\ell}}\Big] \\ &\quad - \sum_{i=1}^{n}\sum_{\ell=2}^{i}\Big[\lambda_{n-i}\alpha^{nu+\ell-1}(\lambda)x^{\ell-1}\overline{x^{j+i-\ell}}\Big] \\ &\quad + \sum_{i=1}^{n}\left[\lambda_{n-i}\left(\sum_{\ell=1}^{i-1}\alpha^{nu+\ell}(\lambda)\right)\overline{x^{j+i-1}}\right] \\ &= \sum_{i=1}^{n}\Big[\lambda_{n-i}\alpha^{nu}(\lambda)\overline{x^{j+i-1}}\Big] \\ &\quad + \left[\sum_{i=1}^{n}\lambda_{n-i}\left(\sum_{\ell=1}^{i-1}\alpha^{nu+\ell}(\lambda)\right)\overline{x^{j+i-1}}\right] \\ &= \left[\sum_{i=1}^{n}\lambda_{n-i}\left(\sum_{\ell=0}^{i-1}\alpha^{nu+\ell}(\lambda)\right)\overline{x^{j+i-1}}\right] \\ &= \left[\sum_{i=1}^{n}\left(\sum_{\ell=0}^{i-1}\alpha^{nu+\ell}(\lambda)\right)\overline{x^{j+i-1}}\lambda_{n-i}\right] \\ &= \left[\sum_{i=1}^{n}\left(\sum_{\ell=0}^{i-1}\alpha^{nu+\ell}(\lambda)\right)\lambda_{n-i}\overline{x^{j+i-1}}\right].\end{aligned}$$

Lastly, $\psi_{2m+1}(\Upsilon_{\mathbf{i}}^{\boldsymbol{\ell}}) = 0$ except if $\ell_1 = \cdots = \ell_m = n-1$. In this last case $i_1 = \cdots = i_m = n$. So

$$\sum_{\mathbf{i}\in\mathbb{I}_m}\sum_{\boldsymbol{\ell}\in\mathbb{J}_{\mathbf{i}}} \psi_{2m+1}(\Upsilon_{\mathbf{i}}^{\boldsymbol{\ell}}) = \sum_{h=0}^{j-1}\left[x^{j-h-1}\alpha^{mn}(\lambda)x^h\right] = \left[\sum_{h=0}^{j-1}\alpha^{mn+h}(\lambda)x^{j-1}\right].$$

The expression for D_{2m} follows immediately from all these facts. □

Remark 3.4. Another formula for D_{2m} useful for some computations is the following

$$D_{2m}([\lambda x^j]) = \left[\sum_{h=0}^{mn+j-1}\alpha^h(\lambda)x^{j-1}\right] - \left[\sum_{\ell=0}^{n-1}\left(\sum_{u=0}^{m-1}\alpha^{nu+\ell}(\lambda)\right)\overline{\sum_{i=0}^{\ell}\lambda_{n-i}x^{i-1+j}}\right].$$

This follows from Theorem 3.3 and the fact that

$$\overline{\sum_{i=1}^{n}\sum_{\ell=0}^{i-1}\alpha^{nu+\ell}(\lambda)\lambda_{n-i}x^{i-1+j}} = \sum_{\ell=0}^{n-1}\alpha^{nu+\ell}(\lambda)\overline{\sum_{i=\ell+1}^{n}\lambda_{n-i}x^{i-1+j}} = \sum_{\ell=0}^{n-1}\alpha^{nu+\ell}(\lambda)\left(x^{j-1} - \overline{\sum_{i=0}^{\ell}\lambda_{n-i}x^{i-1+j}}\right).$$

3.1. **Explicit computations.** Let k, K, α, $f = X^n + \lambda_1 X^{n-1} + \cdots + \lambda_n$ and A be as above. In this subsection we compute the cyclic homology of A relative to K, under suitable hypothesis. We will freely use the notations introduced at the beginning of Section 2 and below Corollary 2.5. Recall that by Theorem 2.3, if there exists $\breve{\lambda} \in \mathcal{Z}(K)$ such that

- $\alpha^n(\breve{\lambda}) = \breve{\lambda}$,
- $\breve{\lambda} - \alpha^i(\breve{\lambda})$ is invertible in K for $1 \le i < n$,

then $\lambda_1 = \cdots = \lambda_{n-1} = 0$ and

$$C_r^S(A) = \begin{cases} \frac{K}{[K,K]_{\alpha^{mn}}} & \text{if } r = 2m,\\ \frac{K}{[K,K]_{\alpha^{(m+1)n}}}x^{n-1} & \text{if } r = 2m+1.\end{cases}$$

Moreover, by Theorem 2.4, the Hochschild boundary maps of the mixed complex $(C^S_*(A), d_*, D_*)$ are given by

$$d_{2m+1}([\lambda]x^{n-1}) = [(\alpha(\lambda) - \lambda)\lambda_n],$$

$$d_{2m+2}([\lambda]) = \left[\sum_{\ell=0}^{n-1} \alpha^\ell(\lambda)\right] x^{n-1}.$$

We now compute the Connes operator D_*.

Theorem 3.5. *Under the hypothesis of Theorem 2.3, we have:*

$$D_{2m}([\lambda]) = 0,$$

$$D_{2m+1}([\lambda]x^{n-1}) = \left[(\mathrm{id} - \alpha)\left(\sum_{u=0}^{m} \alpha^{nu}(\lambda)\right)\right].$$

Proof. If follows immediately from Theorem 3.3. □

Theorem 3.6. *Assume the hypothesis of Theorem 2.6 are fulfilled. Then the mixed complex $(C^S_*(A), d_*, D_*)$ decomposes as a direct sum*

$$(C^S_*(A), d_*, D_*) = \bigoplus_{h=1}^{s} (C^{S,\omega_h}_*(A), d^{\omega_h}_*, D^{\omega_h}_*),$$

where the Hochschild complexes $(C^{S,\omega_h}_*(A), d^{\omega_h}_*)$ are as in Theorem 2.6. Moreover the Connes operators $D^{\omega_h}_*$ satisfy $D^{\omega_h}_{2m} = 0$ and

$$D^{\omega_h}_{2m+1}([\lambda]x^{n-1}) = (1 - \omega_h)\left(\sum_{u=0}^{m} \omega_h^{nu}\right)[\lambda].$$

Proof. If follows immediately from Theorem 3.5. □

In the rest of this section we assume that k is a characteristic zero field and that hypothesis of Theorem 2.6 are fulfilled. We let $\mathrm{HC}^{K,\omega_h}_*(A)$, $\mathrm{HN}^{K,\omega_h}_*(A)$ and $\mathrm{HP}^{K,\omega_h}_*(A)$ denote the cyclic, negative and periodic homology of $(C^{S,\omega_h}_*(A), d^{\omega_h}_*, D^{\omega_h}_*)$, respectively.

Theorem 3.7. *The cyclic, negative and periodic homology of A relative to K decompose as*

$$\mathrm{HC}^K_*(A) = \bigoplus_{h=1}^{s} \mathrm{HC}^{K,\omega_h}_*(A),$$

$$\mathrm{HN}^K_*(A) = \bigoplus_{h=1}^{s} \mathrm{HN}^{K,\omega_h}_*(A)$$

and

$$\mathrm{HP}_*^K(A) = \bigoplus_{h=1}^{s} \mathrm{HP}_*^{K,\omega_h}(A).$$

Moreover, we have:

$$\mathrm{HC}_{2m}^{K,\omega_h}(A) = \begin{cases} \frac{K^1}{[K,K]^1} & \textit{if } h = 1, \\ \frac{K^{\omega_h}}{[K,K]^{\omega_h} + K^{\omega_h}\lambda_n} & \textit{if } \omega_h^n \neq 1, \\ \frac{K^{\omega_h}}{[K,K]^{\omega_h} + K^{\omega_h}\lambda_n^{m+1}} & \textit{otherwise,} \end{cases}$$

and

$$\mathrm{HC}_{2m+1}^{K,\omega_h}(A) = \begin{cases} 0 & \textit{if } h = 1 \textit{ or } \omega_h^n \neq 1, \\ \frac{\{\lambda \in K^{\omega_h} : \lambda\lambda_n^{m+1} \in [K,K]^{\omega_h}\}}{[K,K]^{\omega_h}_{\alpha(m+1)n}} x^{n-1} & \textit{otherwise,} \end{cases}$$

Proof. The first assertion is an immediate consequence of Theorems 3.2 and 3.6, and the computation of $\mathrm{HC}_*^{K,\omega_h}$ for $h = 1$ and for $\omega_h^n \neq 1$ follows from Corollary 2.7, using the spectral sequence associate with the filtration by columns of $\mathrm{BC}(C_*^{S,\omega_h}(A), d_*^{\omega_h}, D_*^{\omega_h})$, which collapse in the first step since the homology of $(C_*^{S,\omega_h}(A), d_*^{\omega_h})$ is concentrate in zero degree (it is also possible to give a direct argument that avoids any reference to spectral sequences). So, in order to finish the proof it remains to consider the case $h > 1$ and $\omega_h^n = 1$. By Theorems 2.6 and 3.6, the cyclic homology of the mixed complex $(C_*^{S,\omega_h}(A), d_*^{\omega_h}, D_*^{\omega_h})$, is the homology of

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\downarrow{\scriptstyle d^{\omega_h}} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle d^{\omega_h}} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle d^{\omega_h}} \\
X_4 & \xleftarrow{D^{\omega_h}} & X_3 & \xleftarrow{0} & X_2 & \xleftarrow{D^{\omega_h}} & X_1 & \xleftarrow{0} & X_0 \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle d^{\omega_h}} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle d^{\omega_h}} & & \\
X_3 & \xleftarrow{0} & X_2 & \xleftarrow{D^{\omega_h}} & X_1 & \xleftarrow{0} & X_0 & & \\
\downarrow{\scriptstyle d^{\omega_h}} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle d^{\omega_h}} & & & & \\
X_2 & \xleftarrow{D^{\omega_h}} & X_1 & \xleftarrow{0} & X_0 & & & & \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle d^{\omega_h}} & & & & & & \\
X_1 & \xleftarrow{0} & X_0 & & & & & & \\
\downarrow{\scriptstyle d^{\omega_h}} & & & & & & & & \\
X_0, & & & & & & & &
\end{array}$$

where

- $X_{2m} = \frac{K^{\omega_h}}{[K,K]^{\omega_h}_{\alpha^{mn}}}$ and $X_{2m+1} = \frac{K^{\omega_h}}{[K,K]^{\omega_h}_{\alpha^{(m+1)n}}} x^{n-1}$,
- $D^{\omega_h}_{2m+1}([\lambda]x^{n-1}) = (m+1)(1-\omega_h)[\lambda]$,
- $d^{\omega_h}_{2m+1}([\lambda]x^{n-1}) = (\omega_h - 1)[\lambda\lambda_n]$.

We first compute the homology in degree $2m$. Let

$$\iota\colon X_0 \to X_{2m} \oplus X_{2m-2} \oplus \cdots \oplus X_0$$

be the canonical inclusion. By using that each $D^{\omega_h}_{2i+1}$ map is an isomorphism it is easy to see that ι induces an epimorphism

$$\bar{\iota}\colon X_0 \to \mathrm{HC}^{K,\omega_h}_{2m}(A).$$

A direct computation shows now that the boundary of

$$([\zeta_{2m+1}]x^{n-1}, \dots, [\zeta_1]x^{n-1}) \in X_{2m+1} \oplus \cdots \oplus X_1$$

equals $\iota([\lambda])$ if and only if

$$[\zeta_{2i+1}] = \frac{i!}{m!}[\zeta_{2m+1}\lambda_n^{m-i}] \quad \text{for } 0 \le i \le m \tag{4}$$

and $\frac{\omega_h - 1}{m!}[\zeta_{2m+1}\lambda_n^{m+1}] = [\lambda]$. The assertion about $\mathrm{HC}^{K,\omega_h}_{2m}(A)$ follows easily from these facts. We now are going to compute the homology in degree $2m+1$. It is immediate that

$$([\zeta_{2m+1}]x^{n-1}, \dots, [\zeta_1]x^{n-1}) \in X_{2m+1} \oplus \cdots \oplus X_1$$

is a cycle of degree $2m+1$ if and only if it satisfies (4) and

$$\zeta_{2m+1}\lambda_n^{m+1} \in [K,K]^{\omega_h}.$$

So the map

$$\iota\colon X_{2m+1} \to X_{2m+1} \oplus \cdots \oplus X_1,$$

given by

$$\iota([\lambda]) = \left([\lambda]x^{n-1}, \frac{1}{m}[\lambda\lambda_n]x^{n-1}, \dots, \frac{1}{m!}[\lambda\lambda_m^n]x^{n-1}\right),$$

induce a quasi-isomorphism

$$\bar{\iota}\colon \frac{\{\lambda \in K^{\omega_h} : \lambda\lambda_n^{m+1} \in [K,K]^{\omega_h}\}}{[K,K]^{\omega_h}_{\alpha^{(m+1)n}}} x^{n-1} \to \mathrm{HC}^{K,\omega_h}_{2m+1}(A),$$

as desired. □

3.2. **Cyclic homology of rank 1 Hopf algebras.** Let k, G, χ, g_1, α and A be as in Subsection 2.2. Here we compute the cyclic homology of A. Let $C_n \subseteq k$ be the set of all n-th roots of 1. As in the above mentioned subsection we consider three cases.

$\boldsymbol{\xi = 0}$. That is $A = K[x,\alpha]/\langle x^n\rangle$, where $K = k[G]$. Since K is separable over k, from Theorem 3.7 it follows that

$$\mathrm{HC}_{2m}(A) = \frac{K}{[K,K]},$$

$$\mathrm{HC}_{2m+1}(A) = \bigoplus_{\omega\in C_n\setminus\{1\}} \frac{K^\omega}{[K,K]^\omega_{\alpha^{(m+1)n}}} x^{n-1}.$$

$\boldsymbol{\xi \neq 0}$ **and** $\boldsymbol{\chi^n = 1}$. In this case $A = K[x,\alpha]/\langle x^n - \xi(g_1^n - 1)\rangle$, where $K = k[G]$. Arguing as in Subsection 2.2, but using Theorem 3.7 instead of Corollary 2.7, we obtain

$$\mathrm{HC}_{2m}(A) = \frac{K^1}{[K,K]^1} \oplus \bigoplus_{\omega\in C_n\setminus\{1\}} \frac{K^\omega}{[K,K]^\omega + K^\omega(g_1^n-1)^{m+1}},$$

$$\mathrm{HC}_{2m+1}(A) = \bigoplus_{\omega\in C_n\setminus\{1\}} \frac{\{\lambda\in K^\omega : \lambda(g_1^n-1)^{m+1}\in [K,K]^\omega\}}{[K,K]^\omega} x^{n-1}.$$

$\boldsymbol{\xi \neq 0}$ **and** $\boldsymbol{\chi^n \neq 1}$. In this case $A = K[x,\widetilde{\alpha}]/\langle x^n\rangle$, where the algebra $K = k[G/\langle g_1^n\rangle]$ and $\widetilde{\alpha}$ is the automorphism induced by α. Since K is separable over k, from Theorem 3.7 it follows that

$$\mathrm{HC}_{2m}(A) = \frac{K}{[K,K]},$$

$$\mathrm{HC}_{2m+1}(A) = \bigoplus_{\omega\in C_n\setminus\{1\}} \frac{K^\omega}{[K,K]^\omega_{\widetilde{\alpha}^{(m+1)n}}} x^{n-1}.$$

4. The periodic and cyclic homology

Assume that k is a characteristic zero field and that the hypothesis of Theorem 2.6 are satisfied. The aim of this Section is to compute the periodic and negative homology of A when α has finite order

In the following remark we compute the maps of the SBI exact sequence of the mixed complex $(C_*^{S,\omega_h}(A), d_*^{\omega_h}, D_*^{\omega_h})$ of Theorem 3.6. We will use the notations introduced above Theorem 3.7.

Remark 4.1. From the computations of Theorem 3.7 it follows that:

(1) If $h = 1$ or $\omega_h^n \neq 1$, then the map

$$S\colon \mathrm{HC}_{2m+2}^{K,\omega_h}(A) \to \mathrm{HC}_{2m}^{K,\omega_h}(A)$$

is the identity map.

(2) If $h > 1$ and $\omega_h^n = 1$, then we have:

a. The map $S\colon \mathrm{HC}_{2m+2}^{K,\omega_h}(A) \to \mathrm{HC}_{2m}^{K,\omega_h}(A)$ is the canonical surjection.

b. The map $i\colon \mathrm{HH}_{2m}^{K,\omega_h}(A) \to \mathrm{HC}_{2m}^{K,\omega_h}(A)$ is given by

$$i([\lambda]) = \frac{1}{m!}[\lambda\lambda_n^m].$$

c. The map $B\colon \mathrm{HC}_{2m}^{K,\omega_h}(A) \to \mathrm{HH}_{2m+1}^{K,\omega_h}(A)$ is zero.

d. The map $S\colon \mathrm{HC}_{2m+3}^{K,\omega_h}(A) \to \mathrm{HC}_{2m+1}^{K,\omega_h}(A)$ is given by

$$S([\lambda]x^{n-1}) = \frac{1}{m+1}[\lambda\lambda_n]x^{n-1}.$$

e. The map $i\colon \mathrm{HH}_{2m+1}^{K,\omega_h}(A) \to \mathrm{HC}_{2m+1}^{K,\omega_h}(A)$ is the canonical inclusion.

f. The map $B\colon \mathrm{HC}_{2m+1}^{K,\omega_h}(A) \to \mathrm{HH}_{2m+2}^{K,\omega_h}(A)$ is given by

$$B([\lambda]x^{n-1}) = (m+1)(1-\omega_h))[\lambda].$$

Remark 4.2. Theorem 3.7 applies in particular to the monogenic extensions of finite group algebras $K = k[G]$ considered in Example 2.8. Note that since K is a separable k-algebra, this computes the absolute cyclic homology, as follows easily from [G-S, Theorem 1.2] using the SBI-sequence.

Theorem 4.3. *Assume the hypothesis of Theorem 2.6 are fulfilled and that there exists $m_0 \in \mathbb{N}$ such that $\alpha^{m_0} = \mathrm{id}$. Then,*

$$\mathrm{HP}_0^{K,\omega_h}(A) = \begin{cases} \frac{K^1}{[K,K]^1} & \text{if } h = 1, \\ \frac{K^{\omega_h}}{[K,K]^{\omega_h} + K^{\omega_h}\lambda_n} & \text{if } \omega_h^n \neq 1, \\ \dfrac{K^{\omega_h}}{\bigcap_{m\geq 0}\left([K,K]^{\omega_h} + K^{\omega_h}\lambda_n^{m+1}\right)} & \text{otherwise,} \end{cases}$$

$$\mathrm{HP}_1^K(A) = 0.$$

Moreover there exists a non-negative integer m_1 such that

$$\bigcap_{m\geq 0}\left([K,K]^{\omega_h} + K^{\omega_h}\lambda_n^{m+1}\right) = [K,K]^{\omega_h} + K^{\omega_h}\lambda_n^{m_1+j+1},$$

for all $j \geq 0$.

Proof. We first compute $\mathrm{HP}_0^{K,\omega_h}(A)$. By items (1) and (2a) of Remark 4.1, the sequence

$$\cdots \xrightarrow{S} \mathrm{HC}_4^{K,\omega_h}(A) \xrightarrow{S} \mathrm{HC}_2^{K,\omega_h}(A) \xrightarrow{S} \mathrm{HC}_0^{K,\omega_h}(A)$$

satisfies the Mittag-Leffler condition. So,

$$\mathrm{HP}_0^{K,\omega_h}(A) = \varprojlim_S \mathrm{HC}_{2m}^{K,\omega_h}(A).$$

If $h = 1$ or $\omega_h^n \neq 1$, then by item (1) of Remark 4.1,

$$\mathrm{HP}_0^{K,\omega_h}(A) = \mathrm{HC}_0^{K,\omega_h}(A) = \frac{K^{\omega_h}}{[K,K]^{\omega_h} + K^{\omega_h}\lambda_n}.$$

If $h \neq 1$ and $\omega_h^n = 1$, then by item (2a) of Remark 4.1,

$$\mathrm{HP}_0^{K,\omega_h}(A) = \frac{K^{\omega_h}}{\bigcap_{m\geq 0}\bigl([K,K]^{\omega_h} + K^{\omega_h}\lambda_n^{m+1}\bigr)}.$$

Moreover, since K^{ω_h} is a finite dimensional k-vector space, there exists a non-negative integer m_1 such that

$$\bigcap_{m\geq 0}\bigl([K,K]^{\omega_h} + K^{\omega_h}\lambda_n^{m+1}\bigr) = [K,K]^{\omega_h} + K^{\omega_h}\lambda_n^{m_1+j+1},$$

for all $j \geq 0$. We now compute $\mathrm{HP}_1^{K,\omega_h}(A)$. Since $\mathrm{HC}_{2n+1}^{K,\omega_h}(A)$ is a finite dimensional k-vector space, the sequence

$$\cdots \xrightarrow{S} \mathrm{HC}_5^{K,\omega_h}(A) \xrightarrow{S} \mathrm{HC}_3^{K,\omega_h}(A) \xrightarrow{S} \mathrm{HC}_1^{K,\omega_h}(A)$$

satisfies the Mittag-Leffler condition. Thus,

$$\mathrm{HP}_1^{K,\omega_h}(A) = \varprojlim_S \mathrm{HC}_{2m+1}^{K,\omega_h}(A).$$

If $h = 1$ or $\omega_h^n \neq 1$, then by Theorem 3.7, we have $\mathrm{HP}_1^{K,\omega_h}(A) = 0$. Assume now that $h \neq 1$ and $\omega_h^n = 1$. By Theorem 3.7,

$$\mathrm{HC}_{2m_0m-1}^{K,\omega_h}(A) = \frac{\{\lambda \in K^{\omega_h} : \lambda\lambda_n^{m_0m} \in [K,K]^{\omega_h}\}}{[K,K]^{\omega_h}} x^{n-1}. \tag{5}$$

Again since K^{ω_h} is a finite dimensional k-vector space, there exists m_2 such that

$$\mathrm{HC}_{2m_0(m_2+j)-1}^{K,\omega_h}(A) = \mathrm{HC}_{2m_0m_2-1}^{K,\omega_h}(A) \quad \text{for all } j \geq 0. \tag{6}$$

Let $m \geq m_2$ arbitrary. By item (2d) of Remark 4.1, the map

$$S^{m_0m_2}\colon \mathrm{HC}_{2m_0(m_2+m)-1}^{K,\omega_h}(A) \to \mathrm{HC}_{2m_0m-1}^{K,\omega_h}(A),$$

is given by

$$S^{m_0m_2}([\lambda]x^{n-1}) = \frac{1}{m(m+1)\cdots(m+m_2-1)}[\lambda\lambda_n^{m_0m_2}]x^{n-1}. \tag{7}$$

Since, by (5) and (6) with $j = m - m_2$,

$$\mathrm{HC}_{2m_0m-1}^{K,\omega_h}(A) = \frac{\{\lambda \in K^{\omega_h} : \lambda\lambda_n^{m_0m_2} \in [K,K]^{\omega_h}\}}{[K,K]^{\omega_h}}x^{n-1},$$

using (7) we obtain that $S^{m_0m_2}([\lambda]x^{n-1}) = 0$, and so

$$\mathrm{HP}_1^{K,\omega_h}(A) = \varprojlim_S \mathrm{HC}_{2m_0m-1}^{K,\omega_h}(A) = 0,$$

as desired. □

Theorem 4.4. *Assume the hypothesis of Theorem 4.3 are fulfilled. Then,*

$$\mathrm{HN}_{2m}^{K,\omega_h}(A) = \begin{cases} \mathrm{HC}_{2m-1}^{K,\omega_h}(A) & \text{if } h = 1 \text{ or } \omega_h^n \neq 1, \\ \mathrm{HC}_{2m-1}^{K,\omega_h}(A) \oplus L_m & \text{otherwise,} \end{cases}$$

$$\mathrm{HN}_{2m+1}^{K}(A) = 0,$$

where

$$L_m = \frac{[K,K]^{\omega_h} + K^{\omega_h}\lambda_n^m}{\bigcap_{l\ge 0}\big([K,K]^{\omega_h} + K^{\omega_h}\lambda_n^{l+1}\big)}.$$

Proof. Consider the canonical exact sequence

$$\mathrm{HP}_0^K(A) \xrightarrow{S} \mathrm{HC}_{2m}^K(A) \xrightarrow{B} \mathrm{HN}_{2m+1}^K(A) \xrightarrow{i} \mathrm{HP}_1^K(A)\ .$$

Since $\mathrm{HP}_1^K(A) = 0$ and S is an epimorphism, $\mathrm{HN}_{2m+1}^K(A) = 0$. Now, for each ω_h consider the exact sequence

$$\mathrm{HP}_1^{K,\omega_h}(A) \xrightarrow{S} \mathrm{HC}_{2m-1}^{K,\omega_h}(A) \xrightarrow{B} \mathrm{HN}_{2m}^{K,\omega_h}(A) \xrightarrow{i} \mathrm{HP}_0^{K,\omega_h}(A) \xrightarrow{S} \mathrm{HC}_{2m-2}^{K,\omega_h}(A)\ .$$

Since $\mathrm{HP}_1^{K,\omega_h}(A) = 0$, we have

$$\mathrm{HN}_{2m}^{K,\omega_h}(A) \simeq \mathrm{HC}_{2m-1}^{K,\omega_h}(A) \oplus \mathrm{Ker}(S\colon \mathrm{HP}_0^{K,\omega_h}(A) \to \mathrm{HC}_{2m-2}^{K,\omega_h}(A)).$$

The theorem follows now from Theorems 3.7 and 4.3. □

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