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Symmetry results in the half-space for a semi-linear fractional Laplace equation

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Abstract In this paper, we analyze the semi-linear fractional Laplace equation

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^N_+, \quad u = 0 \quad \text{in } \mathbb{R}^N \setminus \mathbb{R}^N_+,$$

where $\mathbb{R}^N_+ = \{x = (x', x_N) \in \mathbb{R}^N : x_N > 0\}$ stands for the half-space and f is a locally Lipschitz nonlinearity. We completely characterize one-dimensional bounded solutions of this problem, and we prove among other things that if u is a bounded solution with $\rho := \sup_{\mathbb{R}^N} u$ verifying $f(\rho) = 0$, then u is necessarily one dimensional.

Keywords Fractional Laplacian · Symmetry solutions · One-dimensional anaysis · Energy formulas

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1 Introduction

In this paper, we study existence and qualitative properties of positive bounded solution of the semi-linear nonlocal equation:

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \mathbb{R}^N_+, \end{cases} \tag{P_N}$$

where $\mathbb{R}^N_+ = \{x = (x', x_N) \in \mathbb{R}^N : x_N > 0\}$ is the half-space and f is a locally Lipschitz function. Here $(-\Delta)^s$ denotes the *fractional laplacian*, which is defined on smooth functions as

$$(-\Delta)^s u(x) = c(N,s) \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \mathrm{d}y, \tag{1.1}$$

where $c(N,s)$ is a normalization constant given by

$$c(N,s) = 4^s s(1-s)\pi^{-\frac{N}{2}} \frac{\Gamma\left(s + \frac{N}{2}\right)}{\Gamma(2-s)} \tag{1.2}$$

(cf. Lemma 5.1 in [37]). The integral in (1.1) has to be understood in the principal value sense.

Before stating our results, let us briefly discuss the known achievements for the local case $s = 1$, which motivates our study. The more relevant references in the subject are a series of papers by Berestycki, Caffarelli and Nirenberg, [4–7], where qualitative properties of solutions of

$$\begin{cases} -\Delta u = f(u) \text{ in } \mathbb{R}^N_+, \\ u = 0 \qquad \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{1.3}$$

were obtained. The two main properties analyzed there are the monotonicity of solutions of (1.3) and their one-dimensional symmetry (sometimes called rigidity). In some of these papers, some more general unbounded domains were also considered.

With regard to monotonicity properties in the case $s = 1$, the first known result in the half-space seems to be due to Dancer in [18], although monotonicity in some coercive epigraphs was shown before in [19]. The more general case where f is a Lipschitz function and $f(0) \geq 0$ is solved in [5,6]. It is shown there that all positive solutions u of (1.3), not necessarily bounded, are monotone in the x_N direction. The case $f(0) < 0$ is more delicate, and nowadays still not completely solved. See [21] for several achievements in $N = 2$, and [17] for some partial results in higher dimensions. The main reason is the existence of a one-dimensional, periodic solution of (1.3) which is not strictly positive.

As for the symmetry of solutions of (1.3), it is only conjectured that all bounded solutions are necessarily one dimensional, see [6]. This conjecture was shown to be true when $N = 2$ or when $N = 3$ and $f(0) \geq 0$ in [6]. In higher dimensions, the only general result in this direction at the best of our knowledge is the one in [4], where it is proved that if $\rho := \sup u$ verifies $f(\rho) \leq 0$, then u is symmetric and one additionally has $f(\rho) = 0$. A slightly more restrictive version of this result had been previously proved by Angenent in [2] and Clément and Sweers in [15].

Back to our nonlocal problem (P_N), the question of monotonicity for positive bounded solution has been addressed before in some works. We mention preliminary results obtained in [20,27] for special nonlinearities, and a fairly general recent result by the authors in [3], where it is shown that nonnegative bounded solutions of (P_N) are increasing in the x_N direction even in the more delicate case $f(0) < 0$. The only additional requirement is that f needs to be C^1. Let us also mention the paper [36], where some monotonicity properties are obtained for some more general unbounded domains and some special nonlinearities.

Nevertheless, the question of symmetry for positive bounded solution of (P_N) is far from being completely analyzed. We are only aware of Corollary 1.2 in [36], where some special nonlinearities are dealt with.

Next we describe our main results. First, let us comment that with the exception of Sect. 2, we will be mainly dealing with classical solutions of (P_N). However, it can be seen with the use of the regularity theory developed in [13,14,35] and bootstrapping arguments that bounded, viscosity solutions of (1.1) in the sense introduced in [13] are automatically classical. See "Appendix" for a definition of viscosity solution.

We begin by considering the one-dimensional version of problem (P_N), that is

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \mathbb{R}_+, \\ u = 0 & \text{in } \mathbb{R}\backslash\mathbb{R}_+. \end{cases} \tag{P_1}$$

At the best of our knowledge, this problem is not very well understood at present. The fact that $N = 1$ does not substantially simplify the expression of the operator $(-\Delta)^s$ seems to be responsible for this lack of knowledge. In spite of this, when the problem is posed in $\mathbb{R}$ and special solutions are taken into account, there has been some progress in [9,10].

When $s = 1$, however, the corresponding problem

$$\begin{cases} -u'' = f(u) & \text{in } \mathbb{R}_+, \\ u(0) = 0 \end{cases} \tag{1.4}$$

has been extensively studied, and it is easy to see that there exists a positive bounded solution of (1.4) if and only if $\rho = \|u\|_{L^\infty(\mathbb{R}_+)}$ verifies $f(\rho) = 0$ and

$$F(t) < F(\rho) \text{ for all } t \in [0, \rho), \tag{F}$$

where F is the primitive of f vanishing at zero, $F(t) = \int_0^t f(\tau)\mathrm{d}\tau$. Moreover, the solutions are increasing in x, and there exists a unique solution with a prescribed value of ρ. Thus problem (1.4) admits as many solutions as zeros of f verifying condition (F). This is an immediate consequence of the existence of an energy for solutions of (1.4). As can be directly checked, if u is a solution of $-u'' = f(u)$ in $(0, +\infty)$, the function

$$E(x) := \frac{u'(x)^2}{2} + F(u(x)), \qquad x > 0,$$

is constant. It is also important to remark that the uniqueness of solutions for initial value problems associated to the equation plays also an important role in this characterization.

On the contrary, for the nonlocal problem (P_1), no energy is known to exist for the moment despiste the Hamiltonian identity obtained in [9,11] for layer solutions using the extension tool [12], and of course initial value problems have no sense in its context. Thus existence and uniqueness of solutions and their monotonicity have to be shown in an alternative way.

Indeed, we will prove that problem (P_1) possesses the same features as the local version, by constructing solutions in a different way. In addition, we will also obtain a nonlocal energy which can be used to show the uniqueness of solutions with a prescribed maximum. We strongly believe that this energy could be useful in other one-dimensional problems.

Before stating our main result, we remark that solutions of (P_1) are expected to have a singular derivative at $x = 0$, and we need to consider the 'fractional derivative'

$$\ell_0 := \lim_{x\to 0^+} \frac{u(x)}{x^s}. \tag{1.5}$$

The existence of this limit for solutions of (P_1) is consequence of the regularity results in [29].

We will establish now the main results of the work:

Theorem 1.1 *Assume f is locally Lipschitz and $\rho > 0$ is such that $f(\rho) = 0$ and condition (F) is verified. Then there exists a unique positive solution u of (P_1) with the property*

$$\|u\|_{L^\infty(\mathbb{R})} = \rho.$$

Moreover, u is strictly increasing and ℓ_0 in (1.5) is given by

$$\ell_0 = \frac{(2F(\rho))^{\frac{1}{2}}}{\Gamma(1+s)}. \tag{1.6}$$

Finally, all positive bounded solutions of (P_1) are of the above form.

For every positive ρ verifying (F), we denote the unique positive solution given by Theorem 1.1 by u_ρ.

Once solutions of the one-dimensional problem are completely understood, we expect them to give rise to special solutions of (P_N). While in the local case $s = 1$ this is immediate, it is not completely straightforward when $s \in (0, 1)$, due to the presence of a constant in the definition of $(-\Delta)^s$ which depends on the dimension N. We are unaware if this fact is already present somewhere in the literature, but we include a proof for completeness. That is, we have the following.

Proposition 1.1 *Under the conditions of Theorem 1.1, let u_ρ be the positive bounded solution of the one-dimensional problem (P_1). Then the function*

$$u(x) = u_\rho(x_N), \qquad x \in \mathbb{R}^N \tag{1.7}$$

is a positive bounded solution of (P_N). Conversely, if u is a bounded solution of (P_N) which depends only on x_N, then (1.7) holds for some $\rho > 0$ such that $f(\rho) = 0$ and (F) is verified.

In the light of Proposition 1.1, it is natural to ask as in the local case whether all positive bounded solution of (P_N) come from solutions of (P_1). Thus we pose the following.

Conjecture: assume f is locally Lipschitz and let u be a positive bounded solution of (P_N). Then u is one dimensional.

We are unable to prove this conjecture in its full generality, but we will address some particular instances which are generalizations of some known facts in the local case. We begin by considering the case where the maximum of u is a zero of f, as in [4]. To be more precise, it was assumed there that $f(\|u\|_{L^\infty(\mathbb{R}^N)}) \le 0$, but it is easily seen that this condition is equivalent to $f(\|u\|_{L^\infty(\mathbb{R}^N)}) = 0$. Then we have the next.

Theorem 1.2 *Assume f is locally Lipschitz and let u be a positive bounded solution of (P_N). Suppose in addition that $\rho = \|u\|_{L^\infty(\mathbb{R}^N)}$ verifies $f(\rho) = 0$. Then f verifies (F) and u is one dimensional. More precisely,*

$$u(x) = u_\rho(x_N), \quad x \in \mathbb{R}^N.$$

The proof of Theorem 1.2 ultimately relies in obtaining good lower bounds for the solutions u which allow us to construct a one-dimensional solution below it. It is precisely in this step when the condition $f(\rho) = 0$ is important. When this condition is not assumed, we can still say something by placing some restriction on the behavior of f at zero. The usual condition

$$\liminf_{t\to 0^+} \frac{f(t)}{t} > 0 \tag{1.8}$$

has been considered at several places in the literature of local problems with the same objective (cf. for instance [7]).

A generalization of the results in [7] has been recently obtained in [36]. When it comes to the half-space, it was shown there that if f is a function that has a unique positive zero ρ, that initially it does not have to be the supremum of the solution, that verifies (1.8) and is negative for values larger than ρ and nonincreasing near ρ, then every positive bounded solution of (P_N) is one dimensional. We improve Corollary 1.2 there, in the sense that we do not require the monotonicity condition on f and we show moreover that the solution is unique.

Theorem 1.3 *Assume f is locally Lipschitz and verifies $f > 0$ in $(0, \rho)$, $f < 0$ in $(\rho, +\infty)$ and (1.8). Then the unique positive bounded solution of (P_N) is*

$$u(x) = u_\rho(x_N), \quad x \in \mathbb{R}^N,$$

where u_ρ is the unique solution of (P_1) with $\|u\|_{L^\infty(\mathbb{R}^N)} = \rho$ given by Theorem 1.1.

As a corollary of Theorem 1.3, we obtain a Liouville theorem for a particular class of nonlinearities.

Corollary 1 *Assume f is locally Lipschitz and verifies $f > 0$ in $(0, +\infty)$ and (1.8). Then problem (P_N) does not admit any positive bounded solution.*

To conclude the introduction, we will briefly comment on our methods of proof. With regard to the one-dimensional problem (P_1), the existence of solutions follows by means of sub- and supersolutions. It is worthy of mention that precise subsolutions have to be constructed in order to ensure that the solutions so obtained have the desired L^∞ norm. These subsolutions are shown to exist with an adaptation of the results in [15]. As for uniqueness, it is obtained thanks to Hopf's Lemma and the characterization (1.6). This characterization follows because of our nonlocal energy, furnished by Theorem 3.1 below. The energy is obtained by direct integration of the expression $u'(x)(-\Delta)^s u(x)$, with a careful analysis of all the appearing terms. It is to be noted that the same expression can be obtained with the results in [30], which however need the restriction $f(u) \in L^1$. This could not hold in general.

As for the rest of our theorems, most of them follow with the use of the well-known sliding method, see [8]. However, some additional care is needed because the subsolutions we slide do not have a compact support, which is the usual situation. The method of sub- and supersolutions, providing with a maximal solution in each case is the other essential tool in our approach.

The rest of the paper is organized as follows: Sect. 2 is dedicated to the existence of solutions for problems (P_N) and (P_1). In Sect. 3, we obtain our nonlocal energy and use it to prove the uniqueness of solutions of (P_1). Section 4 is devoted to the proof of our main results, and an "Appendix" is included dealing with the method of sub- and supersolutions.

2 Existence of solutions

In this section, we are concerned with the existence of positive solutions of the problem

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \mathbb{R}_+, \\ u = 0 & \text{in } \mathbb{R}\backslash\mathbb{R}_+. \end{cases} \tag{P_1}$$

More precisely, if the function f is locally Lipschitz and $\rho > 0$ is such that $f(\rho) = 0$ and (F) is satisfied, then we will show that there exists a positive, viscosity solution of (P_1) which is increasing in x and verifies in addition $\lim_{x\to+\infty} u(x) = \rho$. Recall that viscosity solutions are automatically classical.

To simplify the notation, throughout this section we will omit the normalization constant $c(N, s)$ in the definition of the fractional laplacian.

2.1 Existence of solutions in a ball

Although we will primarily deal with the one-dimensional problem (P_1), in the procedure, we need to consider several related problems which are posed in finite domains. For its use in Sect. 4, we will analyze the N-dimensional problem

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } B_R, \\ u = 0 & \text{in } B_R^c = \mathbb{R}^N\backslash B_R, \end{cases} \tag{2.1}$$

where $B_R \subset \mathbb{R}^N$, $N \geq 1$, stands for the ball of radius R centered at the origin. However, all the results in this section are directly generalized to problems where the underlying domain is a dilation of a fixed one.

In general, there is no hope that problem (2.1) admits nonnegative solutions. This is the reason why we are imposed in a first stage the additional assumption $f(0) \geq 0$.

Lemma 2.1 *Assume f is locally Lipschitz in $\mathbb{R}$ and $\rho > 0$ is such that $f(\rho) = 0$ and (F) is satisfied, together with $f(0) \geq 0$. Then for every $\varepsilon > 0$ there exists a positive number $R_0 = R_0(\varepsilon)$ such that for $R \geq R_0$, problem (2.1) admits a positive viscosity solution $u_R \in C^s(\mathbb{R}^N)$, verifying in addition*

$$\rho - \varepsilon \leq \|u_R\|_{L^\infty(B_R)} < \rho. \tag{2.2}$$

Proof The proof is an adaptation of that of Lemma 2.1 in [15], where the local case $s = 1$ was analyzed. We split it in two steps.
Step 1. First we show that for every $R > 0$ there exists a viscosity solution $u_R \in C^s(\mathbb{R}^N)$ of (2.1) such that $0 \leq u_R \leq \rho$ in B_R. For this aim, we define an auxiliary function $\widetilde{f}$ by setting $\widetilde{f}(t) = f(t)$ in $[0, \rho]$,

$$\widetilde{f}(t) = 0 \quad \text{for } t > \rho$$

and extend it to negative values by means of

$$\widetilde{f}(t) = 2f(0) - \widetilde{f}(-t) \quad \text{if } t < 0.$$

Observe that the function $\widetilde{f}$ is bounded in $\mathbb{R}$, and $\widetilde{f}(t) - f(0)$ is odd by its very definition. Denote

$$\widetilde{F}(t) := \int_0^t \widetilde{f}(s)\mathrm{d}s. \tag{2.3}$$

Next, in the Hilbert space

$$\widetilde{H}(B_R) := \{u \in H^s(\mathbb{R}^N) \colon u = 0 \text{ a.e. in } B_R^c\}$$

we define the following functional:

$$J(v) := \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}}\mathrm{d}x\mathrm{d}y - \int_{B_R} \widetilde{F}(v)\mathrm{d}x,$$

(we refer the reader to [32] or [33] for a definition of $H^s(\mathbb{R}^N)$ and the use of variational methods for boundary value problems involving the fractional laplacian).

Observe that J is sequentially weakly lower semicontinuous and the boundedness of $\widetilde{f}$ implies that it is also coercive in $\widetilde{H}^s(B_R)$. Thus it possesses a global minimizer $u_R \in \widetilde{H}^s(B_R)$. We claim that indeed u_R can be chosen to verify

$$0 \le u_R \le \rho. \tag{2.4}$$

To prove the first inequality in (2.4), we will show that for every $v \in \widetilde{H}^s(B_R)$ we have

$$J(|v|) \le J(v) \tag{2.5}$$

which clearly implies that u_R can be taken to be nonnegative. To show (2.5) it is enough to notice that, since $\widetilde{f}(t) - f(0)$ is an odd function, then its primitive $\widetilde{F}(t) - f(0)t$ is even, so that for $t > 0$:

$$\widetilde{F}(-t) = \widetilde{F}(t) - 2f(0)t \le \widetilde{F}(t),$$

owing to our extra condition $f(0) \ge 0$. This immediately yields $\widetilde{F}(t) \le \widetilde{F}(|t|)$ for $t \in \mathbb{R}$. Since it is also well known that

$$||v(x)| - |v(y)|| \le |v(x) - v(y)| \quad \text{for every } x, y \in \mathbb{R}^N,$$

then (2.5) follows.

To show the second inequality in (2.4), we define $w(x) = \min\{u_R(x), \rho\}$. Observing that $F(t) = F(\rho)$ whenever $t > \rho$ and that

$$|w(x) - w(y)| \le |u_R(x) - u_R(y)| \quad \text{for every } x, y \in \mathbb{R}^N,$$

it directly follows that $J(w) \le J(u_R)$. Thus replacing u_R by w, we may always assume that the second inequality in (2.4) holds.

By a standard argument, a minimizer of J in $\widetilde{H}^s(B_R)$ is a weak solution of (2.1). In addition, since $f(u_R) \in L^\infty(B_R)$, we deduce using Proposition 1.1 in [29] that $u_R \in C^s(\mathbb{R}^N)$. Moreover, since then the right-hand side of (2.1) is a continuous function, then u_R is a viscosity solution of (2.1) (cf. Remark 2.11 in [29] or Remark 6 in [34]).
Step 2. We prove that for any $\varepsilon > 0$ there is a positive number $R_0 = R_0(\varepsilon)$ such that u_R is positive in B_R and (2.2) holds if $R > R_0$.

We begin by observing that the scaled function $w_R(x) = u_R(Rx)$ is a minimizer of

$$J_R(v) := \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}}\mathrm{d}x\mathrm{d}y - R^{2s}\int_{B_1} \widetilde{F}(v)\mathrm{d}x$$

in $\widetilde{H}^s(B_1)$. As a first step in proving (2.2), we will show that given $\varepsilon > 0$ there is a positive number R_0 such that

$$\rho - \varepsilon \le \|w_R\|_{L^\infty(B_1)} \le \rho \quad \text{for every } R > R_0. \tag{2.6}$$

Suppose that (2.6) does not hold. Then there exist $\varepsilon > 0$ and a sequence $R_n \to +\infty$ such that $\|w_n\|_{L^\infty(B_1)} < \rho - \varepsilon$, where $w_n = w_{R_n}$. Define

$$\alpha = \min\{F(\rho) - F(r)\colon 0 \le r \le \rho - \varepsilon\},$$
$$\beta = \max\{F(\rho) - F(r)\colon 0 \le r \le \rho\}.$$

Since, by (F), $\alpha > 0$, we can choose $\delta > 0$ small enough to have

$$|B_1^\delta|\beta < |B_1|\alpha, \tag{2.7}$$

where $B_1^\delta = \{x \in B_1 \colon \mathrm{dist}(x, \partial B_1) < \delta\}$ and $|\cdot|$ stands for the Lebesgue measure.

We next choose a function $w \in C_0^\infty(B_1)$ satisfying $0 \le w(x) \le \rho$ in B_1^δ and $w \equiv \rho$ in $B_1 \backslash B_1^\delta$. Then for a positive constant C:

$$\begin{aligned} J_{R_n}(w) - J_{R_n}(w_n) &\le \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y - R_n^{2s}\int_{B_1}(F(w) - F(w_n)) \\ &= C - R_n^{2s}\left(\int_{B_1}(F(\rho) - F(w_n)) - \int_{B_1^\delta}(F(\rho) - F(w))\right) \\ &\le C - R_n^{2s}(\alpha|B_1| - \beta|B_1^\delta|) < 0 \end{aligned}$$

for large n, thanks to (2.7). This is a contradiction with the fact that w_n is a minimizer of J_{R_n}, which shows that (2.6) must be true.

Coming back to the functions u_R, we see that (2.2) holds except for the strict inequality. However, since u_R is a viscosity solution of (2.1), f is locally Lipschitz and $f(\rho) = 0$, it is standard by the strong maximum principle that $u_R < \rho$ in B_R. Observe that the strong maximum principle also implies that $u_R > 0$ in B_R, concluding the proof of the lemma. □

It is now the turn to remove the extra assumption $f(0) \ge 0$. As observed before, without this hypothesis we can not guarantee the existence of a positive solution. However, it will be enough for our purposes in the near future to obtain slightly negative solutions. To this aim, we will redefine the function f for negative values when $f(0) < 0$. Observe that for small enough positive δ we have

$$\frac{f(0)}{2}\delta + F(\rho) > 0. \tag{2.8}$$

We define the function f_δ in $[-\delta, \rho]$ by setting

$$f_\delta(t) := \begin{cases} \dfrac{f(0)}{\delta}(t + \delta) & \text{if } t \in [-\delta, 0), \\ f(t) & \text{if } t \in [0, \rho]. \end{cases} \tag{2.9}$$

In the case $f(0) \ge 0$, we simply take $\delta = 0$ and $f_0 = f$. We now consider a slight variant of problem (2.1) with f replaced by f_δ and a negative datum outside B_R, namely:

$$\begin{cases} (-\Delta)^s u = f_\delta(u) & \text{in } B_R, \\ u = -\delta & \text{in } \mathbb{R}^N \backslash B_R. \end{cases} \tag{2.10}$$

Note that the function $g_\delta(t) = f_\delta(t - \delta)$ is locally Lipschitz and satisfies (F) with ρ replaced by $\rho + \delta$. Hence we can apply Lemma 2.1 to obtain that, for every $\varepsilon > 0$, there exists $R_0 > 0$

such that for $R \geq R_0$, problem (2.1) with f replaced by g_δ admits a positive viscosity solution $w_{\delta,R} \in C^s(\mathbb{R}^N)$ verifying

$$\rho + \delta - \varepsilon \leq \|w_{\delta,R}\|_{L^\infty(B_R)} < \rho + \delta.$$

Setting $u_{\delta,R} = w_{\delta,R} - \delta$, we get the following result:

Lemma 2.2 *Assume f is locally Lipschitz in $\mathbb{R}$ and $\rho > 0$ is such that (F) is verified. If $\delta > 0$ is small enough so that* (2.8) *holds, then for every $\varepsilon > 0$ there exists a positive number $R_0 = R_0(\varepsilon)$ such that for $R \geq R_0$, problem (2.10) admits a viscosity solution $u_{\delta,R} \in C^s(\mathbb{R}^N)$, verifying $u_{\delta,R} > -\delta$ in B_R and*

$$\rho - \varepsilon \leq \|u_{\delta,R}\|_{L^\infty(B_R)} < \rho.$$

We next observe that, thanks to Theorem A.1 in "Appendix", whenever a viscosity solution u of (2.10) exists with the property $u \leq \rho$ in $\mathbb{R}^N$, then a maximal viscosity solution $\widetilde{u}$ of the same problem and with the same property also exists. Here and in what follows, by 'maximal' we mean maximal with respect to the supersolution $\bar{u} = \rho$, that is, if v is any viscosity solution of (2.10) with $v \leq \rho$ in $\mathbb{R}^N$ then we have $v \leq \widetilde{u}$ in $\mathbb{R}^N$.

On the other hand, by Theorem 1.1 in [22], every positive solution of (2.1) with f replaced by g_δ is radially symmetric and radially decreasing. Thus we immediately have:

Lemma 2.3 *Under the same assumptions as in Lemma* 2.2*, for every $\varepsilon > 0$ there exists a positive number $R_0 = R_0(\varepsilon)$ such that for $R \geq R_0$, problem* (2.10) *admits a maximal viscosity solution $\widetilde{u}_{\delta,R} \in C^s(\mathbb{R}^N)$, verifying $\widetilde{u}_{\delta,R} > -\delta$ in B_R and*

$$\rho - \varepsilon \leq \|\widetilde{u}_{\delta,R}\|_{L^\infty(B_R)} < \rho.$$

Moreover, $\widetilde{u}_{\delta,R}$ is radially symmetric and radially decreasing.

Remark 2.1 Let $R_1 < R_2$ and denote by $\widetilde{u}_{\delta,1}, \widetilde{u}_{\delta,2}$ the maximal viscosity solutions of (2.10) with $R = R_1$ and $R = R_2$,, respectively. Then $w(x) = \max\{\widetilde{u}_{\delta,1}(x), \widetilde{u}_{\delta,2}(x)\}$ is a viscosity subsolution of (2.10) with $R = R_2$. By Theorem A.1 in the Appendix, there exists a solution in the ordered interval $[w, \rho]$, and therefore the maximal solution lies in that interval, that is $w \leq \widetilde{u}_{\delta,2}$ in B_{R_2}, in fact in $\mathbb{R}^N$. Hence

$$\widetilde{u}_{\delta,1} \leq \widetilde{u}_{\delta,2} \text{ in } \mathbb{R}^N.$$

With a similar argument, and taking into account that f_δ is decreasing with respect to δ we can deduce that, if $\delta_1 < \delta_2$ then

$$\widetilde{u}_{1,R} \geq \widetilde{u}_{2,R} \text{ in } \mathbb{R}^N$$

where now $\widetilde{u}_{1,R}$ and $\widetilde{u}_{2,R}$ are the maximal viscosity solutions of (2.10) with $\delta = \delta_1$ and $\delta = \delta_2$, respectively.

2.2 Existence of solutions in $\mathbb{R}_+$

Next we consider again the one-dimensional problem (P_1). The purpose of this subsection is to obtain the following existence result:

Theorem 2.1 *Assume f is locally Lipschitz in $\mathbb{R}$ and $\rho > 0$ is such that $f(\rho) = 0$ and (F) is verified. Then problem (P_1) admits a maximal viscosity solution $u \in C^s(\mathbb{R})$, which is positive and verifies*

$$\|u\|_{L^\infty(\mathbb{R})} = \rho.$$

In addition, u is strictly increasing for $x > 0$ and

$$\lim_{x\to+\infty} u(x) = \rho.$$

The way to achieve existence of solutions of (P_1) is to establish it first for a δ−variation of this problem, that is,

$$\begin{cases} (-\Delta)^s u = f_\delta(u) & \text{in } \mathbb{R}_+, \\ u = 0 & \text{in } \mathbb{R}\backslash\mathbb{R}_+, \end{cases} \tag{2.11}$$

where f_δ is given by (2.8), and then pass to the limit as $\delta \to 0^+$. Let us recall that we take $\delta = 0$ and $f_\delta = f$ when $f(0) \geq 0$.

Lemma 2.4 *With the same assumptions as in Theorem 2.1, there exists a viscosity solution u_δ of (2.11) such that $-\delta < u_\delta < \rho$ in $\mathbb{R}_+$ and $\|u_\delta\|_{L^\infty(\mathbb{R})} = \rho$.*

Proof Fix $\varepsilon > 0$. By Lemma 2.3, there exists $R > 0$ such that problem (2.10) with $N = 1$ and $B_R = (0, 2R)$ admits a maximal viscosity solution u_R which verifies $\rho - \varepsilon \leq \|u_R\|_{L^\infty(\mathbb{R})} < \rho$.

However, it is easily seen that the function u_R is a subsolution of problem (2.11). Thus by Theorem A.2 in "Appendix" (see also Remark A.1), there exists a maximal solution u_δ of (2.11) relative to ρ, which verifies $\rho - \varepsilon \leq \|u_\delta\|_{L^\infty(\mathbb{R})} < \rho$. Since u_δ does not depend on ε, it immediately follows that

$$\|u_\delta\|_{L^\infty(\mathbb{R})} = \rho.$$

Finally, since $f_\delta(-\delta) = f_\delta(\rho) = 0$ and f is locally Lipschitz, we deduce from the strong maximum principle that $-\delta < u_\delta(x) < \rho$ in $\mathbb{R}_+$. □

Proof of Theorem 2.1 Remember that, when $f(0) \geq 0$ we are simply choosing $\delta = 0$, so that there exists a solution of (P_1) by Lemma 2.4. Therefore, regarding existence, only the case $f(0) < 0$ needs to be dealt with.

By the second part of Remark 2.1, we have that if $\delta_1 < \delta_2$ then $u_{\delta_1} \geq u_{\delta_2}$ in $\mathbb{R}$. Therefore

$$v(x) := \lim_{\delta\to 0^+} u_\delta(x) = \sup\{u_\delta(x) \colon \delta > 0\}.$$

Observe that $0 \leq v \leq \rho$ in B_R and $\|v\|_{L^\infty(\mathbb{R})} = \rho$. We next prove that v is a solution of (P_1).

Choose $\delta_n \to 0^+$ and let $u_n = u_{\delta_n}$. First, observe that for any $n \in \mathbb{N}$

$$\|u_n\|_{L^\infty(\mathbb{R})} \leq \rho \quad \text{and} \quad \|f_\delta(u_n)\|_{L^\infty(\mathbb{R}_+)} \leq \|f\|_{L^\infty(0,\rho)}.$$

With the use of standard interior regularity (see for instance Theorem 12.1 in [13]) we can obtain appropriate interior bounds for the Hölder norms of the solutions. More precisely, for every $b > a > 0$ we have

$$\|u_n\|_{C^s[a,b]} \leq C\left(\|f_\delta(u_n)\|_{L^\infty(\mathbb{R}_+)} + \|u_n\|_{L^\infty(\mathbb{R})}\right) \leq C\left(\|f\|_{L^\infty(0,\rho)} + \rho\right)$$

for some positive constant $C = C(a, b)$. Hence, we can conclude that $\{u_n\}_{n\in\mathbb{N}}$ is an equicontinuous and uniformly bounded sequence. It follows that $u_n \to v$ locally uniformly in $\mathbb{R}_+$, so that $v \in C(\mathbb{R}\backslash\{0\})$ and $v = 0$ in $(-\infty, 0)$. Observe that with this procedure it is not immediate that $v(0) = 0$ and v is continuous at zero. However, we can argue as in Theorem A.2 in the Appendix to obtain that actually $v \in C(\mathbb{R})$ and $v(0) = 0$.

We can now use Lemma 4.7 in [13], which shows that v is indeed a viscosity solution of (P_1) with $0 \le v < \rho$. By Theorem A.2 in the Appendix, there exists a maximal viscosity solution $u \in C(\mathbb{R})$ of (P_1), which of course verifies $0 \le u < \rho$ and $\|u\|_{L^\infty(\mathbb{R})} = \rho$.

Thus to conclude the proof, only the strict monotonicity of u in $\mathbb{R}_+$ remains to be shown, since it will imply $u > 0$ in $(0, +\infty)$. We mention in passing that the monotonicity of u is a consequence of Lemma 3.1 below, but we are providing an independent proof of this fact.

Choose $\lambda > 0$ and consider the function $v_\delta(x) = u_\delta(x - \lambda)$. It is easily seen that v_δ is a subsolution of (2.11). By Theorem A.2 in the Appendix, there exists a solution w_δ of (2.11) verifying $v_\delta \le w_\delta$ in $\mathbb{R}$. Arguing exactly as in the first part of the proof, we can show that $w_\delta \to w$ locally uniformly in $(0, +\infty)$, where $w \in C(\mathbb{R})$ is a positive viscosity solution of (P_1). It follows that

$$u(x - \lambda) \le w(x) \le u(x) \qquad \text{in } \mathbb{R},$$

since u is the maximal solution. This shows that u is monotone. Moreover, arguing as in Step 3 in the proof of Theorem 1 in [3], we can show that $u' > 0$ in $(0, +\infty)$, so that u is strictly monotone. The proof is concluded. □

3 Uniqueness

Our main objective in this section is the uniqueness of positive solutions of the one-dimensional problem

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \mathbb{R}_+, \\ u = 0 & \text{in } \mathbb{R}\setminus\mathbb{R}_+. \end{cases} \qquad (P_1)$$

In the procedure of proving this uniqueness, we will obtain a nonlocal energy for the problem which we believe is interesting in its own right, and could be further exploited to analyze other related one-dimensional problems.

3.1 A nonlocal energy for one-dimensional solutions

The following is the main result of this subsection:

Theorem 3.1 *Assume f is locally Lipschitz and let u be a positive bounded solution of (P_1). Then u is strictly monotone in $(0, +\infty)$. Moreover, for every $a > 0$ we have*

$$F(u(a)) - \frac{c(1,s)}{2}\left(\int_{-\infty}^{+\infty} \frac{(u(a)-u(y))^2}{|a-y|^{1+2s}}dy - (1+2s)\int_a^{+\infty}\int_{-\infty}^{a} \frac{(u(x)-u(y))^2}{|x-y|^{2+2s}}dydx\right)$$
$$= F(\rho),$$

where $\rho = \lim_{x\to+\infty} u(x)$. In addition, if ℓ_0 is given in (1.5), then

$$\ell_0 = \frac{(2F(\rho))^{\frac{1}{2}}}{\Gamma(1+s)}.$$

Remark 3.1 It can be seen with a little effort that the energy given by Theorem 3.1 converges, as $s \to 1^-$, to the usual one for the local problem $E(x) := u'(x)^2/2 + F(u(x))$.

The proof of Theorem 3.1 will be split in several lemmas for convenience. We begin by showing the monotonicity of solutions of (P_1). We remark that the main result in [3] could be easily modified to include the case $N = 1$. If we adapted the proof presented in this work, we notice that the additional hypothesis $f \in C^1$, required there to obtain the monotonicity property of the solutions, will be not needed in the simpler situation of dimension one. However, we give here an alternative proof that avoids the introduction of the notation established in [3] regarding Green's function in half-spaces.

Lemma 3.1 *Assume f is locally Lipschitz and let u be a positive bounded solution of (P_1). Then $u' > 0$ in $(0, +\infty)$.*

Sketch of proof The proof follows with the use of the moving planes method. We borrow the notation from [3], which is for the most part standard. For $\lambda > 0$, let

$$\begin{aligned}
&\Sigma_\lambda := (0, \lambda)\\
&x^\lambda := 2\lambda - x \text{ (the reflection of } x \text{ with respect to the point } \lambda)\\
&w_\lambda(x) = u(x^\lambda) - u(x), \quad x \in \mathbb{R}\\
&D_\lambda = \{x \in \Sigma_\lambda : \ w_\lambda(x) < 0\}\\
&v_\lambda = w_\lambda \chi_{D_\lambda}.
\end{aligned}$$

Observe that by Lemma 5 in [3] we obtain $(-\Delta)^s v_\lambda \geq L v_\lambda$ in the viscosity sense in D_λ, while $v_\lambda = 0$ outside D_λ. Here L stands for the Lipschitz constant of f in the interval $[0, \|u\|_{L^\infty(\mathbb{R})}]$.

As a consequence of the maximum principle in narrow domains (which follows for instance from Theorem 2.4 in [27]) we deduce that $D_\lambda = \emptyset$ if λ is small enough. Thus $w_\lambda \geq 0$ in Σ_λ if λ is small. Define

$$\lambda^* = \sup\{\lambda > 0 : w_\lambda \geq 0 \text{ in } \Sigma_\lambda\}.$$

If we assume that $\lambda^* < +\infty$, then there exist sequences $\lambda_n \downarrow \lambda^*$ and $x_n \in [0, \lambda_n]$ such that $w_{\lambda_n}(x_n) < 0$. The maximum principle in narrow domains also implies that the points x_n can be chosen indeed in some interval $[\delta, \lambda^* - \delta]$. Thus we may assume $x_n \to x_0 \in [\delta, \lambda^* - \delta]$.

Passing to the limit we see that $w_{\lambda^*} \geq 0$ in $[0, \lambda^*]$, with $w_{\lambda^*}(x_0) = 0$. The strong maximum principle then gives $w_{\lambda^*} \equiv 0$ in $[0, \lambda^*]$, that is, u is symmetric with respect to the point $x = \lambda^*$. However, this contradicts Theorem 8 in [3], whose proof can be seen to be valid when $N = 1$ as well.

The contradiction shows that $\lambda^* = +\infty$, that is, $w_\lambda \geq 0$ in $[0, \lambda]$ for every $\lambda > 0$. Thus u is nondecreasing. Finally, arguing as in Step 3 in the proof of Theorem 1 in [3], we see that $u' > 0$ in $(0, +\infty)$, as wanted. □

Next, we will give the first step in obtaining our energy. The following result is somehow related to the ones obtained in [30] regarding Pohozaev's identity for the fractional laplacian.

Lemma 3.2 *Let $u \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C^1(0, +\infty)$ be such that $u' \in L^1(b_0, +\infty)$ for some $b_0 > 0$ and $\|u\|_{C^{2s+\beta}[b,+\infty)}$ is finite for every $b > 0$ and some $\beta \in (0, 1)$. Then*

$$\begin{aligned}
\int_a^{+\infty} u'(x)(-\Delta)^s u(x)dx = -\frac{c(1,s)}{2}\Bigg(&\int_{-\infty}^{+\infty} \frac{(u(a)-u(y))^2}{|a-y|^{1+2s}}dy \\
&- (1+2s)\int_a^{+\infty}\int_{-\infty}^{a} \frac{(u(x)-u(y))^2}{|x-y|^{2+2s}}dydx\Bigg)
\end{aligned} \tag{3.1}$$

for every $a > 0$. The first integral above is absolutely convergent. In particular, if u is a positive bounded solution of (P_1) with a locally Lipschitz f then

$$F(u(a)) - \frac{c(1,s)}{2}\left(\int_{-\infty}^{+\infty}\frac{(u(a)-u(y))^2}{|a-y|^{1+2s}}dy - (1+2s)\int_a^{+\infty}\int_{-\infty}^{a}\frac{(u(x)-u(y))^2}{|x-y|^{2+2s}}dydx\right)$$
$$= F(\rho), \tag{3.2}$$

for every $a > 0$, where $\rho = \lim_{x\to+\infty} u(x)$ and F is a primitive of f.

Proof Fix $a > 0$ and choose δ and M with the restrictions $0 < \delta < a$ and $M > a + \delta$. We first consider the integral

$$I_{\delta,M} = \int_a^M u'(x) \int\limits_{\substack{-M\\ |y-x|\geq\delta}}^{M} \frac{u(x)-u(y)}{|x-y|^{1+2s}}\mathrm{d}y\mathrm{d}x = \iint_{A_{\delta,M}} u'(x)\frac{u(x)-u(y)}{|x-y|^{1+2s}}\mathrm{d}y\mathrm{d}x, \tag{3.3}$$

where $A_{\delta,M} = ([a, M] \times [-M, M]) \cap \{(x, y) \in \mathbb{R}^2 : |y - x| \geq \delta\}$ (see Fig. 1). It is not hard to see that

$$I_{\delta,M} = \frac{1}{2}\iint_{A_{\delta,M}} \frac{\left((u(x)-u(y))^2\right)_x}{|x-y|^{1+2s}}\mathrm{d}y\mathrm{d}x$$
$$= \frac{1}{2}\iint_{A_{\delta,M}}\left(\frac{(u(x)-u(y))^2}{|x-y|^{1+2s}}\right)_x \mathrm{d}y\mathrm{d}x + \frac{1+2s}{2}\iint_{A_{\delta,M}}\frac{(x-y)(u(x)-u(y))^2}{|x-y|^{3+2s}}\mathrm{d}y\mathrm{d}x.$$

We now split $A_{\delta,M} = A^1_{\delta,M} \cup A^2_{\delta,M} \cup A^3_{\delta,M}$, where

$$A^1_{\delta,M} = \{(x, y) \in A_{\delta,M} : y \geq x + \delta\}$$
$$A^2_{\delta,M} = \{(x, y) \in A_{\delta,M} : a \leq y \leq x - \delta\}$$
$$A^3_{\delta,M} = \{(x, y) \in A_{\delta,M} : y \leq a\}.$$

Since the region $A^1_{\delta,M}$ is the reflection of $A^2_{\delta,M}$ with respect to the line $y = x$ and the integrand in the last integral above is antisymmetric, we immediately deduce that

$$I_{\delta,M} = \frac{1}{2}\iint_{A_{\delta,M}}\left(\frac{(u(x)-u(y))^2}{|x-y|^{1+2s}}\right)_x \mathrm{d}y\mathrm{d}x + \frac{1+2s}{2}\iint_{A^3_{\delta,M}}\frac{(u(x)-u(y))^2}{(x-y)^{2+2s}}\mathrm{d}y\mathrm{d}x$$
$$= \frac{1}{2}\oint_{\partial A_{\delta,M}}\frac{(u(x)-u(y))^2}{|x-y|^{1+2s}}\mathrm{d}y + \frac{1+2s}{2}\iint_{A^3_{\delta,M}}\frac{(u(x)-u(y))^2}{(x-y)^{2+2s}}\mathrm{d}y\mathrm{d}x.$$

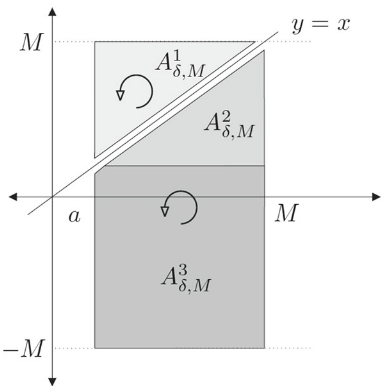

Fig. 1 The region $A_{\delta,M}$ and its subregions

We have made use of Green's formula, hence the line integral is to be taken in the positive sense. Parameterizing the line integral we deduce

$$\begin{aligned} I_{\delta,M} &= -\frac{1}{2}\int\limits_{\substack{-M\\|y-a|\geq\delta}}^{M}\frac{(u(a)-u(y))^2}{|a-y|^{1+2s}}\mathrm{d}y+\frac{1}{2}\int_{-M}^{M-\delta}\frac{(u(M)-u(y))^2}{|M-y|^{1+2s}}\mathrm{d}y\\ &\quad+\frac{1}{2}\int_a^{M-\delta}\frac{(u(x)-u(x+\delta))^2}{\delta^{1+2s}}\mathrm{d}x-\frac{1}{2}\int_a^{M}\frac{(u(x)-u(x-\delta))^2}{\delta^{1+2s}}\mathrm{d}x\\ &\quad+\frac{1+2s}{2}\iint_{A^3_{\delta,M}}\frac{(u(x)-u(y))^2}{(x-y)^{2+2s}}\mathrm{d}y\mathrm{d}x\\ &= -\frac{1}{2}\int\limits_{\substack{-M\\|y-a|\geq\delta}}^{M}\frac{(u(a)-u(y))^2}{|a-y|^{1+2s}}\mathrm{d}y+\frac{1}{2}\int_{-M}^{M-\delta}\frac{(u(M)-u(y))^2}{|M-y|^{1+2s}}\mathrm{d}y\\ &\quad-\frac{1}{2}\int_{a-\delta}^{a}\frac{(u(x+\delta)-u(x))^2}{\delta^{1+2s}}\mathrm{d}x+\frac{1+2s}{2}\iint_{A^3_{\delta,M}}\frac{(u(x)-u(y))^2}{(x-y)^{2+2s}}\mathrm{d}y\mathrm{d}x.\\ &=: I_1+I_2+I_3+I_4. \end{aligned} \tag{3.4}$$

The next step is to let $M\to+\infty$ in (3.4). Since u is bounded we may easily pass to the limit in $I_{\delta,M}$, given in (3.3), I_1 and I_4 by simply using dominated convergence. As for I_2, we claim that it goes to zero as $M\to+\infty$.

To prove this claim, choose $M_0>a$ and let $M>M_0+\delta$. Then we can write, with the use of the fundamental theorem of calculus and Fubini's theorem:

$$\begin{aligned} \int_{M_0}^{M-\delta}\frac{(u(M)-u(y))^2}{(M-y)^{1+2s}}\mathrm{d}y &\leq 2\|u\|_{L^\infty(\mathbb{R}_+)}\int_{M_0}^{M-\delta}\frac{|u(M)-u(y)|}{(M-y)^{1+2s}}\mathrm{d}y\\ &\leq 2\|u\|_{L^\infty(\mathbb{R}_+)}\int_{M_0}^{M-\delta}\int_y^M\frac{|u'(\xi)|}{(M-y)^{1+2s}}d\xi\mathrm{d}y\\ &\leq 2\|u\|_{L^\infty(\mathbb{R}_+)}\int_{M_0}^{M-\delta}\int_{M_0}^M\frac{|u'(\xi)|}{(M-y)^{1+2s}}d\xi\mathrm{d}y\\ &= 2\|u\|_{L^\infty(\mathbb{R}_+)}\int_{M_0}^{M}\int_{M_0}^{M-\delta}\frac{|u'(\xi)|}{(M-y)^{1+2s}}\mathrm{d}yd\xi\\ &\leq \frac{\|u\|_{L^\infty(\mathbb{R}_+)}}{s\delta^{2s}}\int_{M_0}^{+\infty}|u'(\xi)|d\xi. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{-M}^{M_0}\frac{(u(M)-u(y))^2}{(M-y)^{1+2s}}\mathrm{d}y &\leq 4\|u\|^2_{L^\infty(\mathbb{R}_+)}\int_{-M}^{M_0}\frac{\mathrm{d}y}{(M-y)^{1+2s}}\mathrm{d}y\\ &= \frac{2}{s}\|u\|^2_{L^\infty(\mathbb{R}_+)}(M-M_0)^{-2s}. \end{aligned}$$

Hence

$$I_2\leq\frac{\|u\|_{L^\infty(\mathbb{R}_+)}}{s\delta^{2s}}\int_{M_0}^{+\infty}|u'(\xi)|d\xi+\frac{2}{s}\|u\|^2_{L^\infty(\mathbb{R}_+)}(M-M_0)^{-2s}.$$

Letting $M \to +\infty$ and then $M_0 \to +\infty$, we see that the integral goes to zero, as required. Passing to the limit in (3.4) and using dominated convergence we see that

$$\int_a^{+\infty} u'(x) \int_{|y-x|\geq\delta} \frac{u(x)-u(y)}{|x-y|^{1+2s}} \mathrm{d}y\mathrm{d}x = -\frac{1}{2}\int_{|y-a|\geq\delta} \frac{(u(a)-u(y))^2}{|a-y|^{1+2s}}\mathrm{d}y - \frac{1}{2}\int_{a-\delta}^{a} \frac{(u(x+\delta)-u(x))^2}{\delta^{1+2s}}\mathrm{d}x + \frac{1+2s}{2}\iint_{A_\delta} \frac{(u(x)-u(y))^2}{(x-y)^{2+2s}}\mathrm{d}y\mathrm{d}x, \tag{3.5}$$

where $A_\delta = ([a, +\infty) \times (-\infty, a]) \cap \{(x, y) \in \mathbb{R}^2 : y \leq x - \delta\}$.

The final step will be to pass to the limit as $\delta \to 0$ in (3.5). Observe that, since $u \in C^1(0, +\infty)$, we have for y close to a

$$\frac{(u(a)-u(y))^2}{|a-y|^{1+2s}} \leq C|a-y|^{1-2s} \in L^1_{\mathrm{loc}}(\mathbb{R}),$$

so the passing to the limit is justified in the first integral in the right-hand side of (3.5) by dominated convergence. As for the second integral, we see that, also because of the regularity of u:

$$\int_{a-\delta}^{a} \frac{(u(x+\delta)-u(x))^2}{\delta^{1+2s}}\mathrm{d}x \leq C\delta^{2-2s} \to 0$$

as $\delta \to 0^+$. As for the double integral, it also follows that

$$\frac{(u(x)-u(y))^2}{|x-y|^{2+2s}} \leq C|x-y|^{-2s} \in L^1_{\mathrm{loc}}(\mathbb{R}^2),$$

for x and y close to a. Therefore, we are allowed to pass to the limit in the right-hand side of (3.5).

However, the left-hand side of (3.5) has to be treated with a little more care, although in a standard way. By dominated convergence, it suffices to show that

$$\left| \int_{|y-x|\geq\delta} \frac{u(x)-u(y)}{|x-y|^{1+2s}}\mathrm{d}y \right| \leq C \tag{3.6}$$

for some positive constant C and every $x > a$. First, notice that for $\delta < \frac{a}{2}$:

$$\int_{|y-x|\geq\delta} \frac{u(x)-u(y)}{|x-y|^{1+2s}}\mathrm{d}y = \frac{1}{2}\int_{|z|\geq\delta} \frac{2u(x)-u(x+z)-u(x-z)}{|z|^{1+2s}}\mathrm{d}z = \frac{1}{2}\left(\int_{\delta\leq|z|\leq\frac{a}{2}} + \int_{|z|>\frac{a}{2}}\right) \frac{2u(x)-u(x+z)-u(x-z)}{|z|^{1+2s}}\mathrm{d}z.$$

The absolute value of the second of these integrals can be estimated by

$$2\|u\|_{L^\infty(\mathbb{R}_+)} \int_{|z|>\frac{a}{2}} \frac{\mathrm{d}z}{|z|^{1+2s}}.$$

To estimate the first integral, we recall our hypothesis that $\|u\|_{C^{2s+\beta}[b,+\infty)}$ is finite for some $\beta \in (0, 1)$ and every $b > 0$. Since $x > a$, it follows that

$$\int_{\delta\leq|z|\leq\frac{a}{2}} \left| \frac{2u(x)-u(x+z)-u(x-z)}{|z|^{1+2s}} \right| \mathrm{d}z \leq C\|u\|_{C^{2s+\beta}[\frac{a}{2},+\infty)} \int_{|z|\leq\frac{a}{2}} |z|^{\beta-1}\mathrm{d}z,$$

for some (explicit) $C > 0$. Thus (3.6) follows.

To summarize, we may pass to the limit as $\delta \to 0^+$ in (3.5), and (3.1) follows just multiplying by $c(1, s)$.

To conclude the proof of the lemma, let u be a positive bounded solution of (P_1). By Lemma 3.1, $u' > 0$ so that $u' \in L^1(1, +\infty)$, say. On the other hand, by standard regularity we obtain that $u \in C^1(0, +\infty)$ and that the $C^{2s+\beta}$ norm of u in any interval of the form $[b, +\infty)$ is bounded for every $\beta \in (0, 1)$. Thus the first part of the proof applies and we obtain (3.2) by just noticing that

$$\int_a^{+\infty} u'(x)(-\Delta)^s u(x)\mathrm{d}x = F(\rho) - F(u(a)),$$

where $\rho = \lim_{x\to+\infty} u(x)$. □

Our next result is obtained by letting $a \to 0^+$ in (3.2). We use ideas in Theorem 7.5 of [25].

Lemma 3.3 *Let u be a positive bounded solution of (*P_1*). Then*

$$F(\rho) = \mathcal{K}(s)\ell_0^2, \tag{3.7}$$

where $\rho = \lim_{x\to+\infty} u(x)$, ℓ_0 *is given in (1.5) and*

$$\begin{aligned}\mathcal{K}(s) = \frac{c(1,s)}{2}\Bigg(&-\frac{1}{2s} - \int_{-1}^{1} \frac{((t+1)^s - 1)^2}{|t|^{1+2s}}dt + \int_1^{+\infty} \frac{t^{2s} - ((t+1)^s - 1)^2}{t^{1+2s}}dt \\ &+(1+2s)\int_1^{+\infty}\int_0^1 \frac{(t^s - \tau^s)^2}{(t-\tau)^{2+2s}}d\tau dt\Bigg).\end{aligned} \tag{3.8}$$

Proof All the integrals in (3.8) can be seen to be convergent (but see the proof of Lemma 3.4 below).

We first remark that, by boundary regularity, the function $\frac{u(x)}{x^s}$ is in $C^1[0, +\infty)$ (cf. Theorem 7.4, part (iii) in [28]). Thus in particular, the value ℓ_0 given in (1.5) is well defined.

Let $a > 0$. Since $u = 0$ in $(-\infty, 0)$, we can write

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{(u(a) - u(y))^2}{|a-y|^{1+2s}}\mathrm{d}y &= \int_{-\infty}^{0} \frac{u(a)^2}{|a-y|^{1+2s}}\mathrm{d}y + \int_0^{+\infty} \frac{(u(a) - u(y))^2}{|a-y|^{1+2s}}\mathrm{d}y \\ &= \frac{1}{2s}\frac{u(a)^2}{a^{2s}} + \int_{-a}^{+\infty} \frac{(u(a) - u(z+a))^2}{|z|^{1+2s}}\mathrm{d}z.\end{aligned}$$

Similarly

$$\begin{aligned}\int_a^{+\infty}\int_{-\infty}^{a} \frac{(u(x) - u(y))^2}{|x-y|^{2+2s}}\mathrm{d}y\mathrm{d}x = &\frac{1}{1+2s}\int_a^{+\infty} \frac{u(x)^2}{x^{1+2s}}\mathrm{d}x \\ &+ \int_a^{+\infty}\int_0^{a} \frac{(u(x) - u(y))^2}{|x-y|^{2+2s}}\mathrm{d}y\mathrm{d}x.\end{aligned}$$

Thus by (3.2) we see that

$$\begin{aligned}F(\rho) = F(u(a)) &- \frac{c(1,s)}{4s}\frac{u(a)^2}{a^{2s}} - \frac{c(1,s)}{2}\int_{-a}^{+\infty} \frac{(u(a) - u(z+a))^2}{|z|^{1+2s}}\mathrm{d}z \\ &+ \frac{c(1,s)}{2}\int_a^{+\infty} \frac{u(x)^2}{x^{1+2s}}\mathrm{d}x + \frac{c(1,s)(1+2s)}{2}\int_a^{+\infty}\int_0^{a} \frac{(u(x) - u(y))^2}{|x-y|^{2+2s}}\mathrm{d}y\mathrm{d}x.\end{aligned} \tag{3.9}$$

Our intention is to pass to the limit in this equality as $a \to 0^+$. For this sake, it is clear that only the integrals need to be taken into account.

We first claim that

$$\lim_{a\to 0} \int_a^{+\infty} \int_0^a \frac{(u(x)-u(y))^2}{|x-y|^{2+2s}} \mathrm{d}y\mathrm{d}x = \ell_0^2 \int_1^{+\infty} \int_0^1 \frac{(t^s-\tau^s)^2}{(t-\tau)^{2+2s}} \mathrm{d}\tau\mathrm{d}t. \tag{3.10}$$

To prove (3.10), fix $\eta > 0$ and take $a < \frac{\eta}{2}$. Then for $x > \eta$ and $0 < y < a$ we have $x - y \geq \frac{x}{2}$. Therefore

$$\begin{aligned} \int_\eta^{+\infty} \int_0^a \frac{(u(x)-u(y))^2}{|x-y|^{2+2s}} \mathrm{d}y\mathrm{d}x &\leq 4\|u\|^2_{L^\infty(\mathbb{R})} \int_\eta^{+\infty} \int_0^a \frac{\mathrm{d}y}{|x-y|^{2+2s}} \mathrm{d}x \\ &\leq 2^{4+2s} \|u\|^2_{L^\infty(\mathbb{R})} a \int_\eta^{+\infty} x^{-2-2s} \mathrm{d}x \\ &= \frac{2^{4+2s}\|u\|^2_{L^\infty(\mathbb{R})}}{1+2s} \eta^{-1-2s} a. \end{aligned} \tag{3.11}$$

To analyze the same integral when x varies in the interval $[a, \eta]$, observe that the regularity of $u(x)/x^s$ implies

$$\lim_{x\to 0} \frac{u'(x)}{x^{s-1}} = s\ell_0.$$

Therefore, if we fix $\varepsilon > 0$, for small enough η we can guarantee that $u'(x) \leq s(\ell_0 + \varepsilon)x^{s-1}$ if $x < \eta$. Hence for $y < a < x < \eta$ we have

$$0 < u(x) - u(y) = \int_y^x u'(\xi)d\xi \leq (\ell_0 + \varepsilon)(x^s - y^s),$$

so that

$$\int_a^\eta \int_0^a \frac{(u(x)-u(y))^2}{|x-y|^{2+2s}} \mathrm{d}y\mathrm{d}x \leq (\ell_0+\varepsilon)^2 \int_a^\eta \int_0^a \frac{(x^s-y^s)^2}{|x-y|^{2+2s}} \mathrm{d}y\mathrm{d}x.$$

In the last integral, we change variables by $x = at$, $y = a\tau$ and recall (3.11) to obtain, for some $C > 0$,

$$\int_a^{+\infty} \int_0^a \frac{(u(x)-u(y))^2}{|x-y|^{2+2s}} \mathrm{d}y\mathrm{d}x \leq C\eta^{-1-2s} a + (\ell_0+\varepsilon)^2 \int_1^{\frac{\eta}{a}} \int_0^1 \frac{(t^s-\tau^s)^2}{(t-\tau)^{2+2s}} \mathrm{d}\tau\mathrm{d}t.$$

Letting $a \to 0^+$ and then $\varepsilon \to 0^+$ we have

$$\limsup_{a\to 0} \int_a^{+\infty} \int_0^a \frac{(u(x)-u(y))^2}{|x-y|^{2+2s}} \mathrm{d}y\mathrm{d}x \leq \ell_0^2 \int_1^{+\infty} \int_0^1 \frac{(t^s-\tau^s)^2}{(t-\tau)^{2+2s}} \mathrm{d}\tau\mathrm{d}t.$$

The opposite inequality for the inferior limit is shown similarly, and this establishes (3.10).

We finally deal with the remaining two integrals in (3.9). We write

$$\begin{aligned} -\int_{-a}^{+\infty} \frac{(u(z+a)-u(a))^2}{|z|^{1+2s}} \mathrm{d}z + \int_a^{+\infty} \frac{u(z)^2}{z^{1+2s}} \mathrm{d}z = &-\int_{-a}^{a} \frac{(u(z+a)-u(a))^2}{|z|^{1+2s}} \mathrm{d}z \\ &+ \int_a^{+\infty} \frac{u(z)^2 - (u(z+a)-u(a))^2}{|z|^{1+2s}} \mathrm{d}z. \end{aligned}$$

Reasoning exactly as with (3.10) it can be shown that

$$\lim_{a\to 0}\int_{-a}^{a}\frac{(u(z+a)-u(a))^2}{|z|^{1+2s}}\mathrm{d}z=\ell_0^2\int_{-1}^{1}\frac{((t+1)^s-1)^2}{|t|^{1+2s}}\mathrm{d}t.$$

On the other hand, using the C^1 regularity of $u(x)/x^s$ up to $x=0$ we can ensure that

$$u(x)=\ell_0 x^s+O(x^{s+1}),\quad \text{as } x\to 0, \tag{3.12}$$

where $O(x)$ is as usual a function which verifies $|O(x)|\le Cx$ for small x and some $C>0$. It follows from (3.12) that for small $\eta>0$, if $a<z<\eta$,

$$(u(z+a)-u(a))^2=\ell_0^2((z+a)^s-a^s)^2+O(z^{s+1}).$$

Thus if $\eta>0$ is small enough and $a<\eta$:

$$\begin{aligned}\int_a^{\eta}\frac{u(z)^2-(u(z+a)-u(a))^2}{z^{1+2s}}\mathrm{d}z&=\ell_0^2\int_a^{\eta}\frac{z^{2s}-((z+a)^s-a^s)^2+O(z^{s+1})}{z^{1+2s}}\mathrm{d}z\\&=\ell_0^2\int_a^{\eta}\frac{z^{2s}-((z+a)^s-a^s)^2}{z^{1+2s}}\mathrm{d}z+\int_a^{\eta}O(z^{-s})\mathrm{d}z\\&=\ell_0^2\int_1^{\frac{\eta}{a}}\frac{t^{2s}-((t+1)^s-1)^2}{t^{1+2s}}\mathrm{d}t+O(\eta^{1-s}).\end{aligned}$$

Moreover, by dominated convergence:

$$\lim_{a\to 0^+}\int_{\eta}^{+\infty}\frac{u(z)^2-(u(z+a)-u(a))^2}{z^{1+2s}}\mathrm{d}z=0.$$

Hence we deduce

$$\lim_{a\to 0^+}\int_a^{+\infty}\frac{u(z)^2-(u(z+a)-u(a))^2}{z^{1+2s}}\mathrm{d}z=\ell_0^2\int_1^{+\infty}\frac{t^{2s}-((t+1)^s-1)^2}{t^{1+2s}}\mathrm{d}t.$$

Finally, we can pass to the limit in (3.9) to conclude the proof of the lemma. □

Our last step is to obtain an alternative expression for the constant in (3.7). To do it, we take advantage of some of the results in [30], complemented with an additional analysis of the properties of $\mathcal{K}(s)$.

Lemma 3.4 *For $s\in(0,1)$, we have*

$$\mathcal{K}(s)=\frac{\Gamma(1+s)^2}{2}, \tag{3.13}$$

where $\mathcal{K}(s)$ is given in (3.8).

Proof Let us begin by proving (3.13) for $s>\frac{1}{2}$. This will follow by establishing (3.7) for a particular problem in two different ways. For $\lambda>0$ to be chosen later, consider the problem

$$\begin{cases}(-\Delta)^s u=\lambda(1-u) & \text{in } \mathbb{R}_+,\\ u=0 & \text{in } \mathbb{R}\backslash\mathbb{R}_+.\end{cases} \tag{3.14}$$

By Theorem 2.1, problem (3.14) admits a maximal solution relative to $\overline{u}=1$, which will be denoted by u. The function u is strictly increasing and verifies $\lim_{x\to+\infty}u(x)=1$.

We claim that $f(u) := \lambda(1-u) \in L^1(0,+\infty)$. To prove this we will construct a suitable subsolution of (3.14). Choose a nondecreasing function $v \in C^\infty(\mathbb{R})$ verifying

$$v(x) = \begin{cases} 0 & \text{in } (-\infty, 0], \\ 1 - x^{-2s} & \text{if } x \geq 2. \end{cases}$$

Then, for $x \geq 4$:

$$\begin{aligned}(-\Delta)^s v(x) &= c(1,s)\left(\int_{-\infty}^{0} \frac{1-x^{-2s}}{|x-y|^{1+2s}}\mathrm{d}y + \int_0^2 \frac{(1-x^{-2s})-v(y)}{|x-y|^{1+2s}}\,\mathrm{d}y\right.\\ &\quad \left. - \int_2^{+\infty} \frac{x^{-2s}-y^{-2s}}{|x-y|^{1+2s}}\,\mathrm{d}y\right)\\ &= c(1,s)x^{-2s}\left((1-x^{-2s})\int_{-\infty}^{0} \frac{\mathrm{d}\tau}{|1-\tau|^{1+2s}} + \int_0^{2/x} \frac{(1-x^{-2s})-v(\tau x)}{|1-\tau|^{1+2s}}\mathrm{d}\tau\right.\\ &\quad \left. - x^{-2s}\int_{2/x}^{+\infty} \frac{\tau^{-2s}-1}{|1-\tau|^{1+2s}}\,\mathrm{d}\tau\right)\\ &\leq c(1,s)x^{-2s}\left(\int_{-\infty}^{1/2} \frac{\mathrm{d}\tau}{|1-\tau|^{1+2s}} - x^{-2s}\int_{1/2}^{+\infty} \frac{\tau^{-2s}-1}{|1-\tau|^{1+2s}}\,\mathrm{d}\tau\right), \end{aligned} \tag{3.15}$$

where we have made the change of variables $\tau = y/x$ in the first three integrals above. Observe that the last integral converges, since it is to be understood in the principal value sense, as always. It follows from (3.15) that for some $C > 0$

$$(-\Delta)^s v(x) \leq Cx^{-2s}, \qquad \text{for } x \geq 4.$$

Since v is a smooth function, the same inequality holds for $x \geq 2$, by enlarging the constant if necessary. Therefore, if λ is large enough, we see that

$$(-\Delta)^s v(x) \leq \lambda(1-v(x)), \qquad \text{for } x \geq 2.$$

On the other hand, the monotonicity of v implies that v is bounded away from 1 in the interval $[0, 2]$, hence the same inequality can be achieved there by taking a larger value of λ.

Thus we have shown that v is a subsolution of (3.14) if λ is large enough. It follows by the maximality of u that $v \leq u$ in $\mathbb{R}$, therefore, if $x \geq 2$:

$$1 - u(x) \leq 1 - v(x) = x^{-2s} \in L^1(2, +\infty),$$

since $s > \frac{1}{2}$, which completes the proof of the claim.

We now apply Lemma 3.3 to problem (3.14) to obtain

$$F(1) = \frac{\lambda}{2} = \mathcal{K}(s)\ell_0^2, \tag{3.16}$$

where $\ell_0 = \lim_{x\to 0} u(x)/x^s$.

On the other hand, we now make the crucial observation that some of the results in [30] can be applied to solutions u of problems posed in unbounded domains Ω as long as $f(u) \in L^1(\Omega)$, which is precisely the situation in (3.14). More precisely, see the proof of Proposition 1.6 and (2.7) there. In particular by Theorem 1.9 in [30], we see that

$$\frac{\lambda}{2} = \frac{\Gamma(1+s)^2}{2}\ell_0^2. \tag{3.17}$$

Combining (3.16) and (3.17), we see that (3.13) holds for $s > \frac{1}{2}$.

Unfortunately, this procedure does not seem to be generalized to cover the whole range $s \in (0, 1)$. Indeed, we expect the maximal solution u of (3.14) to behave exactly like $1 - x^{-2s}$ as $x \to +\infty$, so that $f(u) \notin L^1(0, +\infty)$ if $s \leq \frac{1}{2}$.

Therefore we will prove (3.13) by showing that $\mathcal{K}(s)$ can be seen as an analytic function of the complex variable s in the strip $0 < \mathrm{Re}(s) < 1$. Since it coincides with $\Gamma(s+1)^2/2$ in the real segment $(\frac{1}{2}, 1)$, the well-known identity principle will imply that both functions coincide throughout the strip, therefore in the segment $(0, 1)$.

First of all, we write the function $\mathcal{K}(s)$ as follows

$$\mathcal{K}(s) = \frac{c(1,s)}{2}\left(-\frac{1}{2s} - F_1(s) + F_2(s) + (1+2s)F_3(s)\right), \tag{3.18}$$

where

$$\begin{aligned} F_1(s) &:= \int_{-1}^{1} \frac{((t+1)^s - 1)^2}{|t|^{1+2s}} \mathrm{d}t \\ F_2(s) &:= \int_{1}^{+\infty} \frac{t^{2s} - ((t+1)^s - 1)^2}{t^{1+2s}} \mathrm{d}t \\ F_3(s) &:= \int_{1}^{+\infty} \int_0^1 \frac{(t^s - \tau^s)^2}{(t-\tau)^{2+2s}} \mathrm{d}\tau \mathrm{d}t. \end{aligned} \tag{3.19}$$

Therefore, it suffices to verify that F_1, F_2 and F_3 are analytic in the strip $0 < \mathrm{Re}(s) < 1$. We will achieve this by showing that each of the integrals in (3.19) converges absolutely and uniformly in rectangles of the form $U_{\sigma_1,\sigma_2,K} = \{s \in \mathbb{C} : \sigma_1 \leq \mathrm{Re}(s) \leq \sigma_2, -K \leq \mathrm{Im}(s) \leq K\}$, where $0 < \sigma_1 < \sigma_2 < 1$ and $K > 0$.

We use the notation $s = \sigma + i\omega$, where $\sigma_1 \leq \sigma \leq \sigma_2$ and $|\omega| \leq K$. It is important to stress that the complex power functions appearing in (3.19) have to be understood in the sense

$$x^s = x^\sigma e^{i\omega \log x}, \quad x \in \mathbb{R}_+.$$

Thus in particular $|x^s| = x^\sigma$ for every $x > 0$.

We begin with the integral defining F_1. It is enough to prove the uniform convergence of the integral in $[-\frac{1}{2}, 1]$. Observe that, for $s \in U_{\sigma_1,\sigma_2,K}$, $t \in [-\frac{1}{2}, 1]$:

$$\begin{aligned} |(t+1)^s - 1|^2 &= ((t+1)^\sigma - \cos(\omega \log(t+1)))^2 + \sin^2(\omega \log(t+1)) \\ &\leq ((2^{1-\sigma_2}\sigma_2 + 2K)^2 + K^2)t^2 = Ct^2. \end{aligned} \tag{3.20}$$

Therefore

$$\int_{-\frac{1}{2}}^{1} \left| \frac{((t+1)^s - 1)^2}{|t|^{1+2s}} \right| \mathrm{d}t = \int_{-\frac{1}{2}}^{1} \frac{|(t+1)^s - 1|^2}{|t|^{1+2\sigma}} \mathrm{d}t \leq C \int_{-\frac{1}{2}}^{1} |t|^{1-2\sigma_2} \mathrm{d}t,$$

which shows the absolute and uniform convergence of the integral, therefore the analyticity of F_1. As for F_2, we have

$$|t^{2s} - ((t+1)^s - 1)^2| \leq |t^s - (t+1)^s + 1||t^s + (t+1)^s - 1| \leq |st^{s-1} - 1|t^\sigma \leq Ct^\sigma,$$

thus

$$\int_1^{+\infty} \left| \frac{t^{2s} - ((t+1)^s - 1)^2}{t^{1+2s}} \right| \mathrm{d}t \leq C \int_1^{+\infty} \frac{\mathrm{d}t}{t^{1+\sigma_1}},$$

which shows that F_2 is analytic as well.

Finally, we consider the integral defining F_3. We split it as follows:

$$\int_1^{+\infty}\int_0^1 \frac{(t^s-\tau^s)^2}{(t-\tau)^{2+2s}}\mathrm{d}\tau\mathrm{d}t = \int_1^2\int_0^{\frac{1}{2}} \frac{(t^s-\tau^s)^2}{(t-\tau)^{2+2s}}\mathrm{d}\tau\mathrm{d}t + \int_1^2\int_{\frac{1}{2}}^1 \frac{(t^s-\tau^s)^2}{(t-\tau)^{2+2s}}\mathrm{d}\tau\mathrm{d}t$$
$$+\int_2^{+\infty}\int_0^1 \frac{(t^s-\tau^s)^2}{(t-\tau)^{2+2s}}\mathrm{d}\tau\mathrm{d}t =: I_1(s)+I_2(s)+I_3(s).$$

Notice that I_1 defines an analytic function since it is a proper integral. Thus we only have to show the uniform convergence of I_2 and I_3. Regarding I_2, observe that for $s \in U_{\sigma_1,\sigma_2,K}$, $t \in [1,2]$ and $\tau \in [\frac{1}{2},1]$, we have, reasoning as in (3.20):

$$|t^s-\tau^s|^2 = (t^\sigma - \tau^\sigma\cos(\omega(\log t-\log\tau)))^2 + \sin^2(\omega(\log t-\log\tau))$$
$$\le C(t-\tau)^2\tau^{\sigma-2} \le C(t-\tau)^2,$$

for some $C>0$. Therefore:

$$\int_1^2\int_{\frac{1}{2}}^1 \left|\frac{(t^s-\tau^s)^2}{(t-\tau)^{2+2s}}\right|\mathrm{d}\tau\mathrm{d}t \le C\int_1^2\int_{\frac{1}{2}}^1 \frac{\mathrm{d}\tau}{(t-\tau)^{2\sigma}}\mathrm{d}t \le C\int_1^2\int_{\frac{1}{2}}^1 \frac{\mathrm{d}\tau}{(t-\tau)^{2\sigma_1}}\mathrm{d}t.$$

Finally, for the remaining integral I_3 we have

$$\int_2^{+\infty}\int_0^1 \left|\frac{(t^s-\tau^s)^2}{(t-\tau)^{2+2s}}\right|\mathrm{d}\tau\mathrm{d}t \le \int_2^{+\infty}\int_0^1 \frac{(t^\sigma+\tau^\sigma)^2}{(t-\tau)^{2+2\sigma}}\mathrm{d}\tau\mathrm{d}t$$
$$\le \int_2^{+\infty}\frac{(t^\sigma+1)^2}{(t-1)^{2+2\sigma}}\mathrm{d}t \le 36\int_2^{+\infty}\frac{\mathrm{d}t}{t^2},$$

thereby showing the analyticity of F_3. To summarize, we have shown that F_1, F_2 and F_3 define analytic functions in the strip $0<\mathrm{Re}(s)<1$. As we have already remarked, this concludes the proof of (3.13). □

Proof of Theorem 3.1 It is immediate taking into account Lemmas 3.1, 3.2, 3.3 and 3.4. □

3.2 Uniqueness of one-dimensional solutions

We finally come to the principal result of this section which is the uniqueness of positive solutions of (P_1).

Theorem 3.2 *Assume f is locally Lipschitz and $\rho>0$ is such that $f(\rho)=0$ and (F) holds. Then the problem*

$$\begin{cases}(-\Delta)^s u = f(u) & \text{in } \mathbb{R}_+,\\ u=0 & \text{in } \mathbb{R}\backslash\mathbb{R}_+,\end{cases} \tag{P_1}$$

admits at most a positive solution u verifying

$$\|u\|_{L^\infty(\mathbb{R})} = \rho, \tag{3.21}$$

that we will denote by u_ρ.

Proof Let u be a positive solution of (P_1) verifying (3.21) and denote by $\widetilde{u}$ the maximal solution relative to ρ given by Theorem 2.1. Then $u\le\widetilde{u}$ in $\mathbb{R}$. Since $(-\Delta)^s(\widetilde{u}-u)\ge -L(\widetilde{u}-u)$ in $(0,+\infty)$, where L is the Lipschitz constant of f, we deduce by the strong maximum principle that either $u \equiv \widetilde{u}$ in $\mathbb{R}$ or $u < \widetilde{u}$ in $(0, +\infty)$. Let us rule out the second possibility.

Indeed, assume $u < \widetilde{u}$ in $(0, +\infty)$. By Hopf's lemma (see Lemma 1.2 in [24]) we have

$$\lim_{x \to 0^+} \frac{\widetilde{u}(x) - u(x)}{x^s} > 0. \tag{3.22}$$

On the other hand, we may apply Theorem 3.1 to have

$$\lim_{x \to 0^+} \frac{u(x)}{x^s} = \frac{(2F(\rho))^{\frac{1}{2}}}{\Gamma(1+s)}, \tag{3.23}$$

and the same equality holds for $\widetilde{u}$. Hence we deduce

$$\lim_{x \to 0^+} \frac{\widetilde{u}(x) - u(x)}{x^s} = 0,$$

which is a contradiction with (3.22).

Thus we necessarily have $u \equiv \widetilde{u}$ in $\mathbb{R}$, thereby showing that the maximal solution is the only one verifying (3.21). The proof is concluded. □

4 Proof of the main results

This section is dedicated to prove the main results in the paper. We begin with the proof of the features of problem (P_1).

Proof of Theorem 1.1 Let $\rho > 0$ such that $f(\rho) = 0$ and (F) is satisfied. By Theorem 2.1, there exists a positive solution u_ρ of (P_1) verifying $\|u_\rho\|_{L^\infty(\mathbb{R})} = \rho$. Moreover, by Theorem 3.2, this is the only solution with this property, and u_ρ is strictly increasing and verifies (1.6).

Thus to conclude the proof, we need to show that, given any bounded, positive solution u of (P_1) and setting $\rho = \|u\|_{L^\infty(\mathbb{R})}$ we necessarily have $f(\rho) = 0$ and f verifies (F).

To show the first assertion, consider the functions

$$u_n(x) = u(x+n), \quad x \in \mathbb{R}.$$

It is clear that u_n is a solution of (P_1) but posed in the interval $(-n, +\infty)$. Since the sequence $\{u_n\}$ is uniformly bounded, we can use interior regularity as in the proof of Theorem 2.1 to obtain local C^α bounds, which permit to conclude that, passing to a subsequence, $u_n \to v$ locally uniformly, where v is a viscosity solution of

$$(-\Delta)^s v = f(v) \qquad \text{in } \mathbb{R}.$$

On the other hand, by Lemma 3.1, the function u is monotone. It follows that $v \equiv \rho$ in $\mathbb{R}$, and therefore $f(\rho) = 0$.

Finally, let us show that $F(s) < F(\rho)$ for $s \in [0, \rho)$, and the proof of the theorem will be concluded. Suppose this is not true. Then there exists a first point $\rho_0 \in (0, \rho)$ such that

$$F(\rho_0) = \max_{t \in [0,\rho]} F(t).$$

Thus, in particular, $f(\rho_0) = 0$ and (F) holds with ρ_0 in place of ρ. By Theorem 2.1, we get a positive solution v of (P_1) which is increasing and verifies $\|v\|_{L^\infty(\mathbb{R})} = \rho_0$.

Now we use Theorem A.2 and Remark A.1 in the Appendix with v as a subsolution and ρ as a supersolution and obtain a positive solution w verifying $v \le w \le \rho$ in $\mathbb{R}$, which is maximal relative to ρ, and verifies in particular $\lim_{x\to+\infty} w(x) = \rho$. Using the Lipschitz condition on f we have

$$(-\Delta)^s(w-v) \geq -L(w-v) \quad \text{in } \mathbb{R}_+,$$

for some $L > 0$. By Hopf's Lemma:

$$\lim_{x\to 0^+} \frac{w(x)-v(x)}{x^s} > 0.$$

But, on the other hand, by Theorem 3.1

$$\lim_{x\to 0^+} \frac{w(x)-v(x)}{x^s} = \frac{\sqrt{2}}{\Gamma(1+s)}\left(F(\rho)^{\frac{1}{2}} - F(\rho_0)^{\frac{1}{2}}\right) \leq 0,$$

which is a contradiction. The claim follows. □

Proof of Proposition 1.1 The proof of this result is a consequence of a more general fact: if v is a function defined in $\mathbb{R}$ and vanishing in $\mathbb{R}\backslash\mathbb{R}_+$ and we set $u(x) = v(x_N)$ for $x \in \mathbb{R}^N$, then

$$(-\Delta)^s u(x) = (-\Delta)^s v(x_N) \quad \text{in } \mathbb{R}^N, \tag{4.1}$$

where the first $s-$laplacian is meant to be in $\mathbb{R}^N$ and the second one in $\mathbb{R}$. A similar result for fully nonlinear integro-differential operators can be found in Lemma 2.1 of [31].

To prove (4.1), we observe that by its very definition and Fubini's theorem, we have for $x \in \mathbb{R}^N$:

$$\begin{aligned}(-\Delta)^s u(x) &= c(N,s)\int_{\mathbb{R}^N} \frac{v(x_N)-v(y_N)}{|x-y|^{N+2s}}\mathrm{d}y\\ &= c(N,s)\int_{-\infty}^{+\infty}(v(x_N)-v(y_N))\int_{\mathbb{R}^{N-1}} \frac{\mathrm{d}y'}{(|x'-y'|^2+(x_N-y_N)^2)^{\frac{N+2s}{2}}}\mathrm{d}y_N\\ &= c(N,s)\int_{\mathbb{R}^{N-1}} \frac{\mathrm{d}z'}{(|z'|^2+1)^{\frac{N+2s}{2}}}\int_{-\infty}^{+\infty}\frac{v(x_N)-v(y_N)}{|x_N-y_N|^{1+2s}}\mathrm{d}y_N,\end{aligned}$$

where we have performed the change of variables $y' = x' + |x_N - y_N|z'$ in the integral taken in $\mathbb{R}^{N-1}$ in the second line above. Thus the proof of the theorem reduces to show that

$$c(N,s)\int_{\mathbb{R}^{N-1}} \frac{\mathrm{d}z'}{(|z'|^2+1)^{\frac{N+2s}{2}}} = c(1,s). \tag{4.2}$$

With regard to the integral in (4.2), we have

$$\int_{\mathbb{R}^{N-1}} \frac{\mathrm{d}z'}{(|z'|^2+1)^{\frac{N+2s}{2}}} = (N-1)\omega_{N-1}\int_0^{+\infty}\frac{r^{N-2}}{(r^2+1)^{\frac{N+2s}{2}}}\mathrm{d}r,$$

where we denote as usual by ω_{N-1} the measure of the unit ball in $\mathbb{R}^{N-1}$. In the last integral obtained, we perform the change of variables $r = \tan t$ to obtain

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} \frac{\mathrm{d}z'}{(|z'|^2+1)^{\frac{N+2s}{2}}} &= (N-1)\omega_{N-1}\int_0^{\frac{\pi}{2}}(\sin t)^{N-2}(\cos t)^{2s}\mathrm{d}t\\ &= \frac{(N-1)\omega_{N-1}}{2}B\left(\frac{N-1}{2}, s+\frac{1}{2}\right)\\ &= \frac{(N-1)\omega_{N-1}}{2}\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(s+\frac{1}{2}\right)}{\Gamma\left(s+\frac{N}{2}\right)},\end{aligned}$$

where $B(x, y)$ is the Beta function. Next, we use a well-known expression for ω_{N-1} (cf. for instance page 9 in [23]) to obtain that

$$\int_{\mathbb{R}^{N-1}} \frac{dz'}{(|z'|^2+1)^{\frac{N+2s}{2}}} = \frac{\pi^{\frac{N-1}{2}}\Gamma\left(s+\frac{1}{2}\right)}{\Gamma\left(s+\frac{N}{2}\right)}. \tag{4.3}$$

Finally, with the use of (1.2) and (4.3) we see that

$$c(N,s)\int_{\mathbb{R}^{N-1}} \frac{dz'}{(|z'|^2+1)^{\frac{N+2s}{2}}} = 4^s s(1-s)\frac{\pi^{-\frac{1}{2}}\Gamma\left(s+\frac{1}{2}\right)}{\Gamma(2-s)} = c(1,s),$$

as was to be shown. This concludes the proof of the theorem. □

Proof of Theorem 1.2 Since we have proved the uniqueness of solutions of (P_1) with the same supremum ρ (see Theorem 1.1) the proof of Theorem 1.2 will follow by showing the existence of two one-dimensional solutions $\underline{u}$, $\overline{u}$ of (P_N) verifying $\underline{u} \le u \le \overline{u}$ in $\mathbb{R}^N$ and

$$\lim_{x_N\to+\infty} \underline{u}(x) = \lim_{x_N\to+\infty} \overline{u}(x) = \rho.$$

Step 1. There exists a one-dimensional solution $\overline{u}$ of (P_N) with $u \le \overline{u} \le \rho$.

Indeed, let $\overline{u}$ be the maximal solution of (P_N) relative to ρ given by Theorem A.2 in the Appendix. Then by maximality it is clear that $\overline{u}$ is one dimensional and $u \le \overline{u} \le \rho$.

Observe that this implies that f verifies (F) by Theorem 1.1.

Step 2. For every $R > 0$ and every $\varepsilon > 0$ small enough, there exists $x_0 \in \mathbb{R}^N_+$ such that $B_R(x_0) \subset\subset \mathbb{R}^N_+$ and $u \ge \rho - \varepsilon$ in $B_R(x_0)$.

To prove this assertion take $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N_+$ such that $u(x_n) \to \rho$ as $n \to +\infty$. We claim that $x_{n,N} \to +\infty$ (observe that if $f \in C^1(\mathbb{R})$, this would follow at once from the monotonicity of u in the x_N direction given by Theorem 1 in [3]).

Arguing as in the proof of Theorem A.2 in the Appendix, we obtain that

$$u(x) \le A\varphi(x) \quad \text{in } \{x \in \mathbb{R}^N_+ : 0 < x_N < 1\}, \tag{4.4}$$

where $A > 0$ and φ is given by (A.3). Since $\varphi = 0$ on $\partial\mathbb{R}^N_+$, this actually shows that $x_{n,N}$ is bounded away from zero, so extracting a subsequence we may assume that either $x_{n,N} \to \mu$ for some $\mu > 0$ or $x_{n,N} \to +\infty$. Define

$$u_n(x) = u(x + x_n) \quad x \in \mathbb{R}^N.$$

Proceeding as in previous situations, we obtain that, passing to a subsequence $u_n \to v$ locally uniformly in $\mathbb{R}^N$, where v is a solution of

$$(-\Delta)^s v = f(v) \qquad \text{in } D.$$

Here $D = \{x \in \mathbb{R}^N : x_{n,N} > -\mu\}$ in case $x_{n,N} \to \mu$ or $D = \mathbb{R}^N$ when $x_{n,N} \to +\infty$. In either case, and using that $f(\rho) = 0$ and the Lipschitz condition on f, the strong maximum principle implies $v \equiv \rho$. However, from (4.4) we have in the former case

$$v(x) \le A\varphi(x + \mu e_N) \quad \text{in } \{x \in \mathbb{R}^N_+ : -\mu < x_N < 0\}$$

which would yield that v vanishes on ∂D, impossible. Hence the latter possibility holds and this shows $x_{n,N} \to +\infty$ and $u(x + x_n) \to \rho$ locally uniformly in $\mathbb{R}^N$.

Finally, let $\varepsilon > 0$ and $R > 0$ be arbitrary. We have $u(x + x_n) \ge \rho - \varepsilon$ in B_R if n is larger than some $n_0 = n_0(\varepsilon, R)$. Then $u(x) \ge \rho - \varepsilon$ in $B_R(x_n)$ for those values of n, as was to be shown.

Step 3. For every $\eta > 0$, there exists $c(\eta) > 0$ such that

$$u(x) \geq c(\eta) \text{ when } x_N \geq \eta. \tag{4.5}$$

Indeed, let $\varepsilon > 0$. When $f(0) < 0$, choose a small positive δ such that (2.9) is verified, otherwise set $\delta = 0$. Recall that by Step 1 f verifies (F). Thus we may apply Lemma 2.3: there exists $R_0 = R_0(\varepsilon, \delta)$ such that the maximal solution $u_{R_0,\delta}$ of (2.10) verifies

$$\|u_{R_0,\delta}\|_{L^\infty(B_{R_0})} = \rho - \varepsilon. \tag{4.6}$$

Let $x_0 \in \mathbb{R}^N_+$ be given in Step 2 above, for these particular values of ε and R_0. Then by (4.6)

$$u_{R_0,\delta}(z - x_0) \leq u(z), \qquad z \in B_{R_0}(x_0).$$

Since $u \geq 0$ in $\mathbb{R}^N$ and $u_{R_0,\delta}(\cdot - x_0) = -\delta \leq 0$ outside $B_{R_0}(x_0)$, we also have

$$u_{R_0,\delta}(z - x_0) \leq u(z), \qquad z \in \mathbb{R}^N. \tag{4.7}$$

On the other hand, recall that by Lemma 2.3, $u_{R_0,\delta}$ is radially symmetric and radially decreasing. Hence there exists $R_1 \in (0, R_0]$ such that the set of points where $u_{R_0,\delta} > 0$ is precisely B_{R_1}. Denote $\Theta_{R_1} := \{x \in \mathbb{R}^N : x_N > R_1\}$, and consider the set

$$\Omega_{R_1} := \left\{x \in \Theta_{R_1} : u_{R_0,\delta}(z - x) < u(z),\ z \in \mathbb{R}^N_+\right\}.$$

It follows by (4.7) and the strong maximum principle that $x_0 \in \Omega_{R_1}$, hence this set is nonempty. We now claim that Ω_{R_1} is both open and closed relative to Θ_{R_1}, therefore

$$\Omega_{R_1} = \Theta_{R_1}. \tag{4.8}$$

Indeed it is clear from the continuity of all functions involved that Ω_{R_1} is open. As for the closedness, if $\{\xi_n\} \subset \Omega_{R_1}$ verifies $\xi_n \to \xi \in \Theta_{R_1}$, then $u_{R_0,\delta}(z - \xi_n) \leq u(z)$ in $\mathbb{R}^N$, and by the strong maximum principle and the positivity of u, this inequality is strict in $\mathbb{R}^N_+$, hence $\xi \in \Omega_{R_1}$. We deduce that (4.7) holds for every x with $x_N \geq R_1$.

Finally, let $\eta > 0$ and take $0 < \varepsilon < \min\{\eta, R_1\}$ fixed but arbitrary. If $z \in \mathbb{R}^N_+$ is such that $z_N \geq \eta$, it easily follows that $z \in B_{R_1-\varepsilon}(x_z)$, where $x_z := (z', R_1 + z_N - \varepsilon) \in \Theta_{R_1}$. Therefore, by (4.7), we see that

$$u(z) \geq c(\eta) := \inf\left\{u_{R_0,\delta}(x) \colon x \in B_{R_1-\varepsilon}\right\} > 0,$$

which concludes the proof of Step 3.

Step 4. For every $M > 2R_0$ and $\nu < M - 2R_0$, there exists a maximal solution $u_{\nu,M}$ of the problem

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \Sigma_{\nu,M} := \{x \in \mathbb{R}^N \colon \nu < x_N < M\}, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Sigma_{\nu,M}, \end{cases} \tag{4.9}$$

relative to u, which only depends on x_N and verifies $\|u_{\nu,M}\|_{L^\infty(\mathbb{R}^N)} \geq \rho - \varepsilon$.

Consider the maximal solution $\widetilde{u}_{R_0,\delta}$ of problem (2.10). If we choose, say, $x_0 = (0, \frac{M}{2})$, then the function $\widetilde{u}_{R_0,\delta}(x - x_0)$ is a subsolution of

$$\begin{cases} (-\Delta)^s u = f_\delta(u) & \text{in } \Sigma_{\nu,M} := \{x \in \mathbb{R}^N \colon \nu < x_N < M\}, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Sigma_{\nu,M}, \end{cases} \tag{4.10}$$

while u is a supersolution, and they are ordered because of Step 3. The existence of a maximal solution $u_{\nu,M,\delta}$ of (4.10) relative to u then follows directly by Theorem A.2 in the Appendix (cf. also Remark A.1). It is clear that $\|u_{\nu,M,\delta}\|_{L^\infty(\mathbb{R}^N)} \geq \rho - \varepsilon$. Proceeding as in the proof of Theorem 2.1, passing to the limit when $\delta \to 0^+$, we get the existence of a maximal solution $u_{\nu,M}$ of (4.9).

Thus only the one-dimensional symmetry of $u_{\nu,M}$ remains to be shown. For this aim we will first show that for every unitary vector $\theta \in \mathbb{R}^{N-1}$ and $\lambda > 0$

$$u_{\nu,M}(x' + \lambda\theta, x_N) \leq u(x) \quad x \in \mathbb{R}^N. \tag{4.11}$$

The proof of this statement is a consequence again of the sliding method. However, we should warn that it is not completely standard since now we are sliding with solutions which do not have a compact support as in most previous situations (see for instance [8]).

Fix a unitary vector $\theta \in \mathbb{R}^{N-1}$. We will see that (4.11) holds for small λ. If it were not true, then there would exist sequences $\lambda_n \to 0^+$ and $\{x_n\}_{n\in\mathbb{N}} \subseteq \Sigma_{\nu,M}$ such that

$$u_{\nu,M}(x_n' + \lambda_n\theta, x_{n,N}) \geq u(x_n) \quad n \in \mathbb{N}. \tag{4.12}$$

We may assume with no loss of generality that $x_{0,N} \to x_0 \in [\nu, M]$. If we now define the translated functions

$$u_{\nu,M,n}(x) := u_{\nu,M}(x' + x_n', x_N), \quad u_n(x) := u(x' + x_n', x_N), \quad x \in \mathbb{R}^N,$$

we can proceed similarly as in previous situations to obtain that, up to extraction of a subsequence, $u_{\nu,M,n} \to U_{\nu,M}$ and $u_n \to U$ uniformly on compact sets of $\mathbb{R}^N$ as $n \to +\infty$ where $U_{\nu,M}$ and U are solutions of (4.9) and (P_N), respectively. On the other hand since, by construction, $u_{\nu,M}(x) \leq u(x)$, for any $x \in \mathbb{R}^N$, we have

$$U_{\nu,M}(x) \leq U(x) \quad x \in \mathbb{R}^N.$$

Then, by (4.12) we deduce

$$U_{\nu,M}(0, x_0) = U(0, x_0). \tag{4.13}$$

Observe that, by (4.5), we have

$$U \geq c(\nu) > 0 \text{ on } \partial\Sigma_{\nu,M} \text{ while } U_{\nu,M} = 0 \text{ there.} \tag{4.14}$$

Therefore $(0, x_0) \in \Sigma_{\nu,M}$ and by (4.13) and the strong maximum principle, we can conclude that $U_{\nu,M} = U$ in $\mathbb{R}^N$. However, this is impossible by (4.14). Therefore (4.11) is true for small enough $\lambda > 0$.

Next, define

$$\lambda^* := \sup\{\mu > 0\colon (4.11) \text{ holds for every } \lambda \in (0, \mu)\},$$

and assume $\lambda^* < +\infty$. By continuity we have $u_{\nu,M}(x' + \lambda^*\theta, x_N) \leq u(x)$ for any $x \in \mathbb{R}^N$, and we reach a contradiction arguing exactly as before. The contradiction shows that $\lambda^* = +\infty$, that is, (4.11) holds for every $\lambda > 0$ and every unitary $\theta \in \mathbb{R}^{N-1}$.

Finally, since $u_{\nu,M}(x' + \lambda^*\theta, x_N)$ is a solution of problem (4.9) which lies below u, we see by maximality that

$$u_{\nu,M}(x' + \lambda^*\theta, x_N) \leq u_{\nu,M}(x) \qquad x \in \mathbb{R}^N.$$

Since $\lambda > 0$ and $\theta \in \mathbb{R}^{N-1}$ are arbitrary, this shows that $u_{\nu,M}$ depends only on x_N.

Step 5. There exists a one-dimensional solution $\underline{u}$ of (P_N) verifying $\|\underline{u}\|_{L^\infty(\mathbb{R}^N)} = \rho$ and $\underline{u} \leq u$ in $\mathbb{R}^N$.

By a similar argument as in Remark 2.1, we see that $u_{\nu,M}$ is decreasing in ν and increasing in M. Proceeding as in the proof of Theorem 2.1, we see that

$$\underline{u}_\varepsilon(x) := \lim_{\nu\to 0}\lim_{M\to+\infty} u_{\nu,M}(x_N), \quad x \in \mathbb{R}^N$$

is a nonnegative one-dimensional solution of (P_N), which verifies $\underline{u}_\varepsilon \leq u$ in $\mathbb{R}^N$ and $\|\underline{u}_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \geq \rho - \varepsilon$. Moreover, it can be checked that $\underline{u}_\varepsilon$ is increasing in ε as $\varepsilon \to 0^+$. Therefore

$$\underline{u} := \lim_{\varepsilon\to 0^+} \underline{u}_\varepsilon(x), \quad x \in \mathbb{R}^N$$

is a nonnegative one-dimensional solution of (P_N), which verifies $\underline{u} \leq u$ in $\mathbb{R}^N$ and $\|\underline{u}\|_{L^\infty(\mathbb{R}^N)} = \rho$.

Completion of the proof. By Theorem 1.1, we have that $\underline{u} = \overline{u} = u_\rho$. Then by Theorem 1.1 and Proposition 1.1 u coincides with u_ρ. □

Proof of Theorem 1.3 The first step is to show that $\|u\|_{L^\infty(\mathbb{R}^N)} \leq \rho$. Assume on the contrary that the set $D := \{x \in \mathbb{R}^N_+ : u(x) > \rho\}$ is nonempty. Then the function $v = \rho - u$ verifies

$$\begin{cases} (-\Delta)^s v = -f(u) \geq 0 & \text{in } D, \\ v \geq 0 & \text{in } \mathbb{R}^N \setminus D. \end{cases}$$

We can use Lemma 4 in [3] to deduce that $v \geq 0$ in D, that is $u \leq \rho$ in D, which is a contradiction (notice that the requirement in [3] that D is connected, which we can not ensure in our situation, is not really necessary). The contradiction shows that $u \leq \rho$.

The rest of the proof is entirely similar to that of Theorem 1.2. Indeed, the existence of a one-dimensional solution $\overline{u}$ of (P_N) verifying $u \leq \overline{u}$ in $\mathbb{R}^N$ follows exactly the same way.

As for the existence of a one-dimensional solution $\underline{u}$ of (P_N) verifying $\underline{u} \leq u$ in $\mathbb{R}^N$, we notice that Step 2 is no longer needed and Step 3 can be directly proved with the use of the sliding method, as in [7,36]. Indeed we claim that for every $\eta > 0$ there exists $c(\eta) > 0$ such that

$$u(x) \geq c(\eta) \quad \text{if } x_N \geq \eta. \tag{4.15}$$

To see this, we use hypothesis (1.8): there exist $c, \nu > 0$ such that $f(t) \geq ct$ if $t \in [0, \nu]$. Choose $R > 0$ so that the first eigenvalue of $(-\Delta)^s$ in B_R verifies $\lambda_1(B_R) \leq c$, and let ϕ be an associated positive eigenfunction normalized by $\|\phi\|_{L^\infty(B_R)} = 1$. Then it is clear that for every x_0 such that $x_{0,N} > R$ the function

$$\underline{u}(x) = \delta\phi(x - x_0), \quad x \in \mathbb{R}^N,$$

is a subsolution of (P_N) when $0 < \delta \leq \nu$. Moreover, if we fix such an x_0 it is possible to choose a small enough δ to have in addition $\underline{u} \leq u$ in $\mathbb{R}^N$. Indeed, this inequality is trivially satisfied outside $B_R(x_0)$, while in $B_R(x_0)$ the inequality is also true for small δ because u is bounded away from zero there.

We can now 'slide' the ball around $\mathbb{R}^N_+$ just like in Step 3 in the proof of Theorem 1.2 to obtain (4.15). Arguing as in Step 4 there, we can now construct a one-dimensional solution $\underline{u}$ of (P_N) verifying $\underline{u} \leq u$ in $\mathbb{R}^N$. Finally, observe that by Theorem 1.1, problem (P_N) admits a unique one-dimensional solution given by u_ρ. Therefore, $u = u_\rho$, as we wanted to show. □

The proof of our last result is just a direct consequence of Theorem 1.3.

Proof of Corollary 1 Assume there exists a positive bounded solution of (P_N) and let $\rho_0 := \|u\|_{L^\infty(\mathbb{R}^N)}$. We choose $\rho > \rho_0$ and modify f in the interval (ρ_0, ρ) in such a way that f remains positive in $(0, \rho)$ and $f(\rho) = 0$. It is clear that (F) is verified for the value of ρ so chosen. Hence, by Theorem 1.3, problem (P_N) admits a unique solution v which is one dimensional and verifies

$$\lim_{x\to+\infty} v(x) = \rho.$$

By uniqueness, we should have $u \equiv v$ in $\mathbb{R}^N$, but this is impossible as $\|u\|_{L^\infty(\mathbb{R}^N)} = \rho_0 < \rho$. This contradiction shows that problem (P_N) does not admit any positive bounded solution, as we wanted to show. □

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5 Appendix A. A solution between a sub- and a supersolution

In this Appendix, we collect a couple of results which deal with the existence of maximal solutions for some problems related to the ones considered in the paper. To begin with, let Ω be a bounded domain and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. We consider

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{A.1}$$

where $g \in C(\mathbb{R}^N)$.

For convenience, we only deal in this Appendix with subsolutions, supersolutions and solutions in the viscosity sense. In some cases, however, it is known that with some requirements on f and g the concepts of viscosity and classical solutions of (A.1) coincide (see [13,14,35]).

We say that a function $u \in C(\mathbb{R}^N)$ is a viscosity subsolution of (A.1) if $u \le g$ in $\mathbb{R}^N \setminus \Omega$ and verifies the following: for any $x_0 \in \Omega$ and any function ϕ which is C^2 in a neighborhood U of x_0 and such that $u(x_0) = \phi(x_0)$ and $u \le \phi$ in U we have $(-\Delta)^s v(x_0) \le f(x_0, v(x_0))$, where

$$v(x) := \begin{cases} \phi(x) & \text{if } x \in U, \\ u(x) & \text{if } x \in \mathbb{R}^N \setminus U. \end{cases}$$

Supersolutions are defined by reversing the above inequalities. A function u is a viscosity solution of (A.1) it is both a viscosity sub- and supersolution of (A.1). We remark that the continuity assumption on both the sub and supersolution can be relaxed to an appropriate lower semicontinuity, but we are only interested in this work in continuous sub- and supersolutions.

The existence of a solution between a sub- and a supersolution is well known in several instances, mainly when an iteration procedure is available. However, that a maximal solution can be obtained in general is perhaps less known, so we will include a sketch of the main proofs. Given a viscosity supersolution $\overline{u}$ we say that u is the maximal solution relative to $\overline{u}$ if for every other viscosity solution v of (A.1) verifying $v \le \overline{u}$ in $\mathbb{R}^N$ we have $v \le u$ in $\mathbb{R}^N$.

Then we have:

Theorem A.1 *Let Ω be a bounded Lipschitz domain satisfying the exterior sphere condition. Assume $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ is continuous and that $g \in C(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$. If there exist viscosity sub- and supersolution $\underline{u}, \overline{u} \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ of (A.1) with $\underline{u} \le \overline{u}$ in $\mathbb{R}^N$, then there exists a maximal viscosity solution $\widetilde{u}$ of (A.1) relative to $\overline{u}$.*

Sketch of proof Let us begin by observing that the problem can be reduced to $g = 0$: if w is the unique $s-$harmonic function in Ω which coincides with g outside Ω and we let $v = u - w$, then v is a solution of (A.1) with right-hand side $h(x, v) = f(x, v + w(x))$ and vanishing outside Ω. Thus, we may assume in what follows that $g = 0$.

The existence of a solution in the interval $[\underline{u}, \overline{u}]$ follows exactly as in Theorem 1 in [16] (we notice that only regularity theory and the maximum principle are needed; see also [1]). Thus we only show the existence of a maximal solution in this interval. We define the nonempty set

$$\mathcal{F} := \left\{u \in C(\mathbb{R}^N) : u \text{ is a viscosity solution of (A.1) such that } \underline{u} \le u \le \overline{u} \text{ in } \mathbb{R}^N\right\}$$

and

$$\widetilde{u}(x) := \sup\{u(x) : u \in \mathcal{F}\}.$$

We observe that for every $u \in \mathcal{F}$ we have $\|u\|_{L^\infty(\mathbb{R}^N)} \le C$, $\|f(\cdot, u)\|_{L^\infty(\Omega)} \le C$, for some positive constant C which does not depend on u. Thus by regularity theory (cf. for instance Proposition 1.1 in [29]), we obtain

$$\|u\|_{C^s(\overline{\Omega})} \le C.$$

This means that the set $\mathcal{F}$ is equicontinuous, thus $\widetilde{u}$ is continuous in $\mathbb{R}^N$ and vanishes outside Ω. Moreover, it is well known that $\widetilde{u}$ is a subsolution of (A.1) in the viscosity sense.

Thus there exists a solution of (A.1) in the interval $[\widetilde{u}, \overline{u}]$. By its very definition it follows that this solution is indeed $\widetilde{u}$, which is clearly the maximal solution in the interval $[\underline{u}, \overline{u}]$. Let us mention in passing that the existence of the maximal solution could also be shown by following the approach in [26].

We finally show that the maximal solution $\widetilde{u}$ just obtained does not depend on $\underline{u}$. Indeed, assume $\underline{u}_1$ and $\underline{u}_2$ are subsolutions of (A.1) which lie below the supersolution $\overline{u}$ in $\mathbb{R}^N$. Let $\widetilde{u}_i$ be the maximal solution in the interval $[\underline{u}_i, \overline{u}]$, $i = 1, 2$ and set $\underline{u}^+ = \max\{\widetilde{u}_1, \widetilde{u}_2\}$. Then $\underline{u}^+$ is a subsolution of (A.1) below $\overline{u}$. Thus there exists a solution w verifying $\underline{u}^+ \le w \le \overline{u}$ in $\mathbb{R}^N$. In particular $\underline{u}_i \le \widetilde{u}_i \le w \le \overline{u}$ in $\mathbb{R}^N$, $i = 1, 2$ and by maximality of $\widetilde{u}_i$ we deduce $\widetilde{u}_1 = \widetilde{u}_2 = w$ in $\mathbb{R}^N$. □

Theorem A.1 can be generalized to deal with unbounded domains Ω. Only some minor points in the proof above need to be especially treated. For simplicity, we will restrict our attention next to the case $\Omega = \mathbb{R}^N_+$ and f not depending on x, which is the main concern in this paper:

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \mathbb{R}^N_+. \end{cases} \tag{A.2}$$

We have also set $g = 0$. In this context, we have a result which is completely analogue to Theorem A.1.

Theorem A.2 *Assume $f : \mathbb{R} \to \mathbb{R}$ is continuous and there exist viscosity sub- and supersolution $\underline{u}, \overline{u} \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ of (A.2) with $\underline{u} \le \overline{u}$ in $\mathbb{R}^N$. Then there exists a maximal viscosity solution of (A.2) relative to $\overline{u}$.*

Sketch of proof First of all we truncate f outside $[\inf \underline{u}, \sup \overline{u}]$ to make it bounded. Let φ be the solution of the one-dimensional problem

$$\begin{cases} (-\Delta)^s \varphi = 1 & \text{in } (0,1), \\ \varphi = 0 & \text{in } (-\infty, 0), \\ \varphi = 1 & \text{in } (1, +\infty). \end{cases} \tag{A.3}$$

Then φ solves the same problem in $\Sigma_1 = \{x \in \mathbb{R}^N_+ : 0 < x_N < 1\}$ (cf. the proof of Proposition 1.1 in Sect. 4). We notice that for large enough $c > 0$ the function $-c\varphi$ (resp. $c\varphi$) is a subsolution (resp. supersolution) of (A.2). Therefore $\underline{v} := \max\{\underline{u}, -c\varphi\}$ and $\overline{v} := \min\{\overline{u}, c\varphi\}$ are well ordered sub- and supersolution of A.2 satisfying $\underline{v} = \overline{v} = 0$ in $\mathbb{R}^N \setminus \mathbb{R}^N_+$.

We choose now any smooth function w defined in $\mathbb{R}^N$ and verifying $w = 0$ in $\mathbb{R}^N \setminus \mathbb{R}^N_+$, $\underline{v} \le w \le \overline{v}$ in $\mathbb{R}^N$. For $R > 0$, let $B_R^+ = \{x \in \mathbb{R}^N_+ : |x| < R\}$ and consider the problem

$$\begin{cases} (-\Delta)^s u = f(x, u) & \text{in } B_R^+, \\ u = w & \text{in } \mathbb{R}^N \setminus B_R^+. \end{cases} \tag{A.4}$$

By Theorem A.1, there exists a solution u_R of (A.4) verifying $\underline{v} \le u_R \le \overline{v}$ in $\mathbb{R}^N$. Moreover, the family $\{u_R\}_{R>0}$ is uniformly bounded and by standard interior regularity we also have

$$\|u_R\|_{C^s(K)} \le C,$$

for every compact set $K \subset \mathbb{R}^N_+$. Thus $\{u_R\}_{R>0}$ is also equicontinuous and we can select a sequence $R_n \to +\infty$ such that $u_{R_n} \to v$ locally uniformly in $\mathbb{R}^N_+$ for some function $v \in C(\mathbb{R}^N)$ which verifies $\underline{v} \le v \le \overline{v}$ in $\mathbb{R}^N$, therefore vanishes in $\mathbb{R}^N \setminus \mathbb{R}^N_+$.

Passing to the limit in (A.4), we obtain that v is a solution of (A.2) which verifies $\underline{u} \le v \le \overline{u}$ in $\mathbb{R}^N$. The existence of a maximal solution relative to $\overline{u}$ is shown exactly as in the proof of Theorem A.1, with the only prevention that the barrier φ constructed above has to be used instead of the boundary regularity for bounded domains. □

Remark A.1 (a) Of course the same result is true when $N = 1$, in particular for problem (P_1).

(b) With a minor variation in the proof of Theorem A.2 it can be seen that the same statements hold when problem (A.2) is posed in a strip $\Sigma_{\nu,M} := \{x \in \mathbb{R}^N : \nu < x_N < M\}$, where $M > \nu > 0$.

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