History states of systems and operators

A. Boette[1] and R. Rossignoli[1,2]

[1]*Departamento de Física-IFLP/CONICET, Universidad Nacional de La Plata, C.C. 67, La Plata (1900), Argentina*
[2]*Comisión de Investigaciones Científicas (CIC), La Plata (1900), Argentina*

We discuss some fundamental properties of discrete system-time history states. Such states arise for a quantum reference clock of finite dimension and lead to a unitary evolution of system states when satisfying a static discrete Wheeler-DeWitt-type equation. We consider the general case where system-clock pairs can interact, analyzing first their different representations and showing there is always a special clock basis for which the evolution for a given initial state can be described by a constant Hamiltonian H. It is also shown, however, that when the evolution operators form a complete orthogonal set, the history state is maximally entangled for any initial state, as opposed to the case of a constant H, and can be generated through a simple double-clock setting. We then examine the quadratic system-time entanglement entropy, providing an analytic evaluation and showing it satisfies strict upper and lower bounds determined by the energy spread and the geodesic evolution connecting the initial and final states. We finally show that the unitary operator that generates the history state can itself be considered as an operator history state, whose quadratic entanglement entropy determines its entangling power. Simple measurements on the clock enable one to efficiently determine overlaps between system states and also evolution operators at any two times.

I. INTRODUCTION

The incorporation of time in a fully quantum framework [1] has recently attracted wide attention [2–11]. On the one hand, it is relevant as a fundamental problem and a key issue in the search for a coherent theory of quantum gravity [12–18]. On the other hand, a quantum description of time enables one to exploit the quantum features of superposition and entanglement in the development of new models of parallel-in-time simulation [6,7].

The concept of time is related to the quantification of evolution through a reference physical system called clock. Historically, the readings of this clock provided an external classical parameter called time. Nonetheless, if we aim to introduce time into a fully quantum framework, the clock has to be a quantum system itself. This is even more important in attempts to quantize gravity where time has to be described by a dynamical entity [13–18].

Here we describe the system and the reference clock through a discrete system-time history state which enforces a discrete unitary evolution on the system states. We consider the general case where the system-clock pairs can interact. This scenario provides a more general starting point, more adequate for some quantum gravity or cosmological models where interactions between an internal relational clock and evolving degrees of freedom cannot be excluded [17,18].

We first discuss different representations of the history state, showing that for a fixed initial state, there is always an adequate selection of clock basis for which the resultant evolution corresponds to a constant Hamiltonian, with the history state satisfying a discrete counterpart of a standard Wheeler-DeWitt-type equation [12]. The general interacting formalism opens, however, new possibilities.

The entanglement of the history state is a measure of the number of orthogonal states visited by the system at orthogonal times [7], and for a constant Hamiltonian clearly depends on the seed system state. This dependence becomes, however, attenuated when the Hamiltonian is not constant in time, and in the case where the evolution operators form a complete orthogonal set, it is in fact always *maximum*, irrespective of the initial state. The corresponding history state admits, nonetheless, a simple generation through a two-clock scenario, where the clocks are linked to conjugate system variables.

We then analyze the quadratic entanglement entropy of history states, which, as opposed to the standard entropy, can be explicitly evaluated in the general case, enabling one to characterize the system evolution and also to connect the entanglement of states and operators. For a general constant Hamiltonian, it can be analytically determined for any number of steps. Moreover, we show that it is upper bounded by the quadratic entropy of the energy spread of the initial state and lower bounded by that of the geodesic evolution connecting the initial and final states according to the Fubini-Study metric [19]. And its average over all initial system states is directly proportional to the quadratic operator entanglement entropy [20–23] of the unitary gate that generates the history state. Through the channel-state duality [24–28], it is also shown that the pure state which represents the latter is itself an operator history state, whose quadratic entanglement entropy determines its entangling power.

Finally, we show that through measurements on the clock, it is possible to use both system and operator history states to efficiently determine the overlap between system states and also the trace of the evolution operator between any two times. The latter reduces to the trace of a unitary operator

(result of the deterministic quantum computation with one quantum bit (DQC1) circuit [29]) for the simple case of a qubit clock. The properties of general discrete history states and their entanglement are discussed in Sec. II, whereas the entanglement and history states of unitary operators are discussed in Sec. III. Conclusions are finally given in Sec. IV.

II. DISCRETE HISTORY STATES

We consider a system S and a reference clock system T in a joint pure state $|\Psi\rangle \in \mathcal{H}_S \otimes \mathcal{H}_T$, with $\mathcal{H}_T$ of finite dimension N. Any such state can be written as

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_t |S_t\rangle |t\rangle, \tag{1}$$

where $|t\rangle$, $t = 0, \ldots, N-1$, are orthogonal states of T ($\langle t|t'\rangle = \delta_{tt'}$) and $|S_t\rangle$ are states of S, not necessarily orthogonal or normalized, yet satisfying $\sum_t \langle S_t|S_t\rangle/N = \langle\Psi|\Psi\rangle = 1$. Consider now a unitary operator $\mathcal{U}$ for the whole system of the form

$$\mathcal{U} = \sum_{t=1}^{N} U_{t,t-1} \otimes |t\rangle\langle t-1|, \tag{2}$$

where $t = N$ is identified with $t = 0$ and $U_{t,t-1}$ are arbitrary unitary operators on S satisfying $U_{0,N-1} \ldots U_{1,0} = \mathbb{1}$. If $|\Psi\rangle$ fulfills the eigenvalue equation

$$\mathcal{U}|\Psi\rangle = |\Psi\rangle, \tag{3}$$

the states $|S_t\rangle$ will undergo a *unitary* evolution with t:

$$
\begin{aligned}
|S_t\rangle = \sqrt{N}\langle t|\Psi\rangle &= \sqrt{N}\langle t|\mathcal{U}|\Psi\rangle \\
&= U_{t,t-1}|S_{t-1}\rangle = U_t|S_0\rangle,
\end{aligned} \tag{4}
$$

where $U_t = U_{t,t-1} \ldots U_{1,0}$, with $U_0 = \mathbb{1}$. The states $|S_t\rangle$ will then have a unit norm if $|\Psi\rangle$ is normalized.

Thus, the state (1) is a discrete finite-dimensional version of the history state of the Page-Wootters formalism [1,3]. Moreover, writing $\mathcal{U} = \exp[-i\mathcal{J}]$, with $\mathcal{J}$ Hermitian (and spectrum $\subset [0, 2\pi)$), Eq. (3) is equivalent to

$$\mathcal{J}|\Psi\rangle = 0, \tag{5}$$

which is a discrete cyclic version of a Wheeler-DeWitt-type equation [12]. Note, however, that $\mathcal{J}$ will contain $S - T$ interaction terms in the general case where $U_{t,t-1}$ depends on t.

A unitary evolution of the states $|S_t\rangle$ actually occurs if $|\Psi\rangle$ is *any* eigenstate of $\mathcal{U}$: Its eigenvalues are $e^{-i2\pi k/N}$, $k = 0, \ldots, N-1$, and its eigenstates all have the form (1) with $|S_t\rangle$ satisfying a *shifted* unitary evolution: $|S_t\rangle = e^{i2\pi k/N} U_{t,t-1}|S_{t-1}\rangle = e^{i2\pi kt/N} U_t|S_0\rangle$. Each eigenvalue has degeneracy equal to the dimension $d_S = \dim \mathcal{H}_S$ of the system space, with its eigenspace spanned by orthogonal history states $|\Psi_k^l\rangle$ generated by d_S orthogonal initial states $|S_0^l\rangle$: $\langle\Psi^l|\Psi^{l'}\rangle = \langle S_0^l|S_0^{l'}\rangle = \delta^{ll'}$ [7].

If $U_{t,t-1}$ is independent of t $\forall t = 1, \ldots, N$, then

$$U_{t,t-1} = \exp[-iH_S], \tag{6}$$

with H_S a fixed Hermitian Hamiltonian for system S with eigenvalues $2\pi k/N$, k *integer*. The operator (2) then becomes

separable: $\mathcal{U} = \exp[-iH_S] \otimes \exp[-iP_T]$, implying

$$\mathcal{J} = H_S \otimes \mathbb{1} + \mathbb{1} \otimes P_T, \tag{7}$$

which contains no interaction terms. Here, P_T is the generator of time translations, satisfying $e^{-iP_T}|t-1\rangle = |t\rangle$ $\forall t$ and $P_T|k\rangle_T = \frac{2\pi k}{N}|k\rangle_T$, with $|k\rangle_T$ the discrete Fourier transform (DFT) of the states $|t\rangle$:

$$|k\rangle_T = \frac{1}{\sqrt{N}} \sum_t e^{i2\pi kt/N}|t\rangle, \quad k = 0, \ldots, N-1. \tag{8}$$

Equations (5)–(7) then become an exact discrete version of the usual static Wheeler-DeWitt equation [3]. The ensuing condition $\langle t|\mathcal{J}|\Psi\rangle = 0$ implies

$$-\langle t|P_T|\Psi\rangle = H_S|S_t\rangle, \tag{9}$$

which is a discrete version of Schrödinger's equation: As $-\langle t|P_T|t'\rangle = i\frac{\partial}{\partial t}\frac{1}{N}\sum_k e^{i2\pi k(t-t')/N}$, for $N \to \infty$ $-\langle t|P_T|t'\rangle \to i\delta'(t-t')$ and $-\langle t|P_T|\Psi\rangle \to i\frac{\partial}{\partial t}|S_t\rangle$.

A. Representations and entanglement of the history state

By considering an arbitrary orthogonal basis $\{|q\rangle\}$ of $\mathcal{H}_S$, we may first rewrite $|\Psi\rangle$ as

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{q,t} \psi(q,t)|qt\rangle, \tag{10}$$

where $|qt\rangle = |q\rangle|t\rangle$ and $\psi(q,t) = \langle q|S_t\rangle = \sqrt{N}\langle qt|\Psi\rangle$ is a "wave function" satisfying a unitary evolution with t: $\psi(q,t) = \sum_{q'}\langle q|U_{t,t-1}|q'\rangle\psi(q',t-1)$.

We may then obtain the Schmidt decomposition of $|\Psi\rangle$, which we will here write as

$$|\Psi\rangle = \sum_k \lambda_k |k\rangle_S |-k\rangle_T, \tag{11}$$

where $\lambda_k > 0$ are the singular values of the matrix $\psi(q,t)/\sqrt{N}$ and $|k\rangle_{S(T)}$ orthonormal states of S (T) derived from the singular value decomposition of $\psi(q,t)$, with $|-k\rangle \equiv |N-k\rangle$. They are eigenstates of the reduced states $\rho_{S(T)} = \text{Tr}_{T(S)}|\Psi\rangle\langle\Psi|$, with λ_k^2 their nonzero eigenvalues. While the states $|S_t\rangle \propto \langle t|\Psi\rangle$ are not necessarily orthogonal but are equally probable, the states $|k\rangle_S \propto {}_T\langle -k|\Psi\rangle$ are all orthogonal but not equally probable, with λ_k^2 representing a "permanence" probability.

In the constant case (6) and (7), the Schmidt states $|k\rangle_S$ and $|k\rangle_T$ are just the eigenstates of H_S and P_T,

$$H_S|k\rangle_S = \frac{2\pi k}{N}|k\rangle_S, \quad P_T|k\rangle_T = \frac{2\pi k}{N}|k\rangle_T, \tag{12}$$

since $|S_t\rangle = e^{-iH_S t}|S_0\rangle = \sum_k \lambda_k e^{-i2\pi kt/N}|k\rangle_S$ with $\lambda_k = {}_S\langle k|S_0\rangle$, and hence $|\Psi\rangle = \frac{1}{\sqrt{N}}\sum_{k,t}\lambda_k e^{-i2\pi kt/N}|k\rangle_S|t\rangle$ becomes Eq. (11), with $|k\rangle_T$ the strictly orthogonal states (8). The Schmidt coefficients λ_k represent in this case the distribution of $|S_0\rangle$ over distinct energy eigenstates [in the case of degeneracy, $\lambda_k|k\rangle_S$ denotes the projection of $|S_0\rangle$ onto the eigenspace of energy $2\pi k/N$ (mod 2π), with λ_k^2 the total probability of measuring this energy in $|S_0\rangle$]. It is then apparent from Eqs. (7) and (11) that $|\Psi\rangle$ satisfies Eq. (5), which becomes a zero "total momentum" condition: $k_S + k_T = 0$ (mod N).
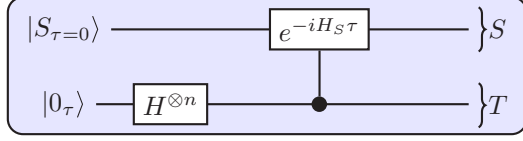
FIG. 1. Schematic circuit representing the generation of the history state (14) in the special time basis, where the system evolves according to a constant Hamiltonian H_S. Here, $H^{\otimes n}$ denotes the Hadamard operator over n qubits, with $2^n = N$.

In the case of arbitrary unitary operators $U_{t,t-1}$ in (2), for any given initial state $|S_0\rangle$ there is always, however, a special orthogonal basis of $\mathcal{H}_T$ for which the corresponding states of S evolve according to a constant Hamiltonian H_S satisfying (12). It is just necessary to use the inverse DFT of the Schmidt states $|k\rangle_T$ of (11),

$$|\tau\rangle = \frac{1}{\sqrt{N}} \sum_k e^{-i2\pi k\tau/N} |k\rangle_T, \tag{13}$$

with $k, \tau = 0, \ldots, N-1$ [if the Schmidt rank is less than N, the states $|k\rangle_T$ of (11) can be completed with orthogonal states], which will not coincide in general with the original states $|t\rangle$. The state (11) then becomes

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{\tau,k} \lambda_k e^{-i2\pi k\tau/N} |k\rangle_S |\tau\rangle = \frac{1}{\sqrt{N}} \sum_\tau |S_\tau\rangle |\tau\rangle, \tag{14}$$

where $|S_\tau\rangle = \sum_k e^{-i2\pi k\tau/N} \lambda_k |k\rangle_S$ satisfies

$$|S_\tau\rangle = \sqrt{N}\langle\tau|\Psi\rangle = \exp[-i\tau H_S]|S_{\tau=0}\rangle, \tag{15}$$

with $|S_{\tau=0}\rangle = \sum_k \lambda_k |k\rangle_S$ and H_S defined over the Schmidt states $|k\rangle_S$ by Eq. (12). The Schmidt coefficients λ_k can then be interpreted as the distribution of $|S_{\tau=0}\rangle$ over these energy eigenstates. In terms of the operators H_S and P_T defined by (12), $|\Psi\rangle$ satisfies Eq. (5) also for an effective noninteracting $\mathcal{J}$ of the form (7), and can be generated from $|S_{\tau=0}\rangle|0_\tau\rangle$ with the circuit of Fig. 1.

Assuming now $d_S = N$ (the Schmidt decomposition selects in any case subspaces of equal dimension on S and T), we can also consider the inverse DFT of the system Schmidt states, $|\xi\rangle = \frac{1}{\sqrt{N}} \sum_k e^{-i2\pi k\xi/N} |k\rangle_S$, which satisfy $e^{-iH_S}|\xi\rangle = |\xi + 1\rangle$ and are the special system states analogous to $|\tau\rangle$. We can then also rewrite $|\Psi\rangle$ as

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{\xi,\tau} \Lambda_{\xi-\tau} |\xi\tau\rangle = \sum_\xi \Lambda_\xi |\Psi_\xi\rangle, \tag{16}$$

where $\sqrt{N}\langle\xi\tau|\Psi\rangle = \Lambda_{\xi-\tau}$ depends just on $\xi - \tau$, and

$$\Lambda_\xi = \frac{1}{\sqrt{N}} \sum_k e^{i2\pi k\xi/N} \lambda_k \tag{17}$$

is the DFT of the Schmidt coefficients λ_k, with $|\Psi_\xi\rangle = \frac{1}{\sqrt{N}} \sum_\tau |\xi + \tau\rangle|\tau\rangle$ orthogonal maximally entangled history states: $\langle\Psi_\xi|\Psi_{\xi'}\rangle = \delta_{\xi\xi'}$ $(|\xi + \tau\rangle \equiv |\xi + \tau - N\rangle$ if $\xi + \tau \geqslant N)$.

The representation (16) is then "conjugate" to (11), expressing $|\Psi\rangle$ as a superposition of maximally entangled orthogonal history states. Like (11), it is *symmetric* in $S-T$: States $|S_\tau\rangle = \sqrt{N}\langle\tau|\Psi\rangle = \sum_\xi \Lambda_{\xi-\tau}|\xi\rangle$ evolve unitarily

with τ [Eq. (15)], while clock states $|T_\xi\rangle = \sqrt{N}\langle\xi|\Psi\rangle = \sum_\tau \Lambda_{\xi-\tau}|\tau\rangle$ evolve unitarily with ξ:

$$|T_\xi\rangle = \sqrt{N}\langle\xi|\Psi\rangle = \exp[-i\xi P_T]|T_{\xi=0}\rangle, \tag{18}$$

where $|T_{\xi=0}\rangle = \sum_k \lambda_k| - k\rangle_T$, complementing Eq. (15). Both ξ and τ always run from 0 to $N-1$ with uniform weight, irrespective of the seed state.

From the Schmidt decomposition (11), we can evaluate the system-time entanglement entropy [7],

$$E(S, T) = S(\rho_S) = S(\rho_T) = -\sum_k \lambda_k^2 \log_2 \lambda_k^2, \tag{19}$$

where $S(\rho) = -\mathrm{Tr}\,\rho \log_2 \rho$. If $|S_0\rangle$ happens to be a common eigenstate of all $U_{t,t-1}$, such that $|S_t\rangle = e^{-i\phi_t}|S_0\rangle \forall t$, then $|\Psi\rangle \propto |S_0\rangle \sum_t e^{-i\phi_t}|t\rangle$ becomes separable and $E(S, T) = 0$ (stationary state), whereas if all $|S_t\rangle$ are orthogonal (i.e., fully distinguishable), $|\Psi\rangle$ becomes maximally entangled, with (1) already the Schmidt decomposition and $E(S, T) = \log_2 N$ maximum. Thus, $2^{E(S,T)}$ measures the actual system evolution time, in the sense of counting the number of effective equally probable orthogonal states the system visits at orthogonal times. For constant $U_{t,t-1}$ [Eq. (6)], $E(S, T)$ is just a measure of the *energy spread* (mod 2π) of the initial state, as $\lambda_k = {}_S\langle k|S_0\rangle$. A similar interpretation holds for the general case in terms of the effective H_S defined by (12).

On the other hand, the entropy determined by the conjugate distribution $|\Lambda_\xi|^2$,

$$\tilde{E}(S, T) = -\sum_\xi |\Lambda_\xi|^2 \log_2 |\Lambda_\xi|^2, \tag{20}$$

measures the spread of $|\Psi\rangle$ over maximally entangled evolutions or, equivalently, the spread of system states $|\xi\rangle$ for a given clock state $|\tau\rangle$ (or vice versa), and is a measure of *time uncertainty*. It vanishes when $|\Psi\rangle$ is maximally entangled ($\Lambda_\xi = \delta_{\xi,0}$ if $\lambda_k = \frac{1}{\sqrt{N}} \forall k$), in which case there is complete synchronization between the special system and clock basis states ($|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_\tau |\tau\rangle|\tau\rangle$), and becomes maximum for a product state ($\Lambda_\xi = \frac{1}{\sqrt{N}} \forall \xi$ if $\lambda_k = \delta_{k,0}$), in which case system and clock states are completely uncorrelated, as seen from (16). These two entropies satisfy the entropic uncertainty relation [7] (see also [11,30–32])

$$E(S, T) + \tilde{E}(S, T) \geqslant \log_2 N, \tag{21}$$

which is saturated in the previous limits.

B. The case of a complete set of evolution operators

While for a constant Hamiltonian the system-time entanglement (19) clearly depends on the seed state $|S_0\rangle$, such dependence becomes softened in the more general case where the operators $U_{t,t-1}$ depend on t and do not commute among themselves, i.e., when the 'Hamiltonian' $H_t \propto \ln U_{t,t-1}$ is time dependent and $[H_t, H_{t'}] \neq 0$ for some pairs $t \neq t'$. If they have no common eigenstate, $|\Psi\rangle$ will be entangled for *any* $|S_0\rangle$. The extreme case is that where the U_t's of (4) form a *complete* set of *orthogonal* unitaries on S, such that

$$\mathrm{Tr}\,[U_t^\dagger U_{t'}] = d_S \delta_{tt'}, \quad t, t' = 0, \ldots, d_S^2 - 1, \tag{22}$$

implying $N = d_S^2$. In this case, the history state (1) becomes *maximally entangled* for *any* initial state $|S_0\rangle$,

$$E(S, T) = \log_2 d_S, \tag{23}$$

such that $|\Psi\rangle = \frac{1}{\sqrt{d_S}} \sum_k |k\rangle_S |-k\rangle_T \ \forall \ |S_0\rangle$.

Proof. We may view Eq. (22) as the scalar product between column vectors $\frac{1}{\sqrt{d_S}} U_t$ of a $d_S^2 \times d_S^2$ unitary matrix U of elements $U_{ij,t} = \frac{1}{\sqrt{d_S}} \langle i|U_t|j\rangle$, with $\{|i\rangle\}$ any orthonormal basis of S, such that (22) is equivalent to $U^\dagger U = \mathbb{1}_{d_S^2}$. This matrix then satisfies as well $UU^\dagger = \mathbb{1}_{d_S^2}$, i.e., $\sum_t \langle i|U_t|j\rangle \langle l|U_t^\dagger|k\rangle = d_S \delta_{ik}\delta_{jl}$, which implies $\sum_t U_t|j\rangle\langle l|U_t^\dagger = d_S \delta_{jl}\mathbb{1}_S$ and hence

$$\sum_t U_t|S_0\rangle\langle S_0'|U_t^\dagger = d_S \langle S_0'|S_0\rangle \mathbb{1}_S, \tag{24}$$

for any two states $|S_0\rangle$, $|S_0'\rangle$ of S. In particular, for $|S_0\rangle = |S_0'\rangle$, Eq. (24) implies a *maximally mixed* reduced state $\rho_S = \mathrm{Tr}_T|\Psi\rangle\langle\Psi|$ for *any* seed state $|S_0\rangle$,

$$\rho_S = \frac{1}{d_S^2} \sum_t U_t|S_0\rangle\langle S_0|U_t^\dagger = \frac{1}{d_S}\mathbb{1}_S. \tag{25}$$

Equation (25) then leads to Eq. (23). ∎

Therefore, a complete orthogonal set of U_t's ensures that the system will visit d_S orthogonal states irrespective of the initial state $|S_0\rangle$. The Schmidt decomposition (11) will then select a subspace of $\mathcal{H}_T$ of dimension d_S connected with S through $|\Psi\rangle$. Due to the d_S-fold degeneracy $\lambda_k = \frac{1}{\sqrt{d_S}} \ \forall \ k$, any orthogonal basis $\{|k\rangle_T\}$ of this subspace can be used in (11), with all states $|k\rangle_S = \sqrt{d_S} \ _T\langle -k|\Psi\rangle$ directly orthogonal.

A convenient choice of complete orthogonal set is provided by the Weyl operators [33–35]

$$U_t \equiv U_{pq} = \exp[i2\pi p Q/d_S]\exp[-i2\pi q P/d_S], \tag{26}$$

where $p, q = 0, \ldots, d_S - 1$, $t = qd_S + p$, $Q|q\rangle = q|q\rangle$, $P|p\rangle = p|p\rangle$, and $\{|q\rangle\}$, $\{|p\rangle\}$ are orthogonal bases of S related through a DFT: $|p\rangle = \frac{1}{\sqrt{d_S}} \sum_q e^{i2\pi pq/d_S}|q\rangle$. They satisfy, for any eigenstate $|q_0\rangle$ of Q,

$$U_{pq}|q_0\rangle = e^{i2\pi p(q_0+q)/d_S}|q_0 + q\rangle, \tag{27}$$

which implies Eq. (22), i.e., $\mathrm{Tr}\, U_{p'q'}^\dagger U_{pq} = d_S \delta_{q'q}\delta_{p'p}$.

The discrete evolution under these operators can then be achieved by application of just *two* different unitaries $U_{t,t-1}$ to the preceding state (here, $m \geqslant 1$, integer):

$$U_{t,t-1} = \begin{cases} e^{i2\pi Q/d_S}, & t \neq md_S \\ e^{-i2\pi P/d_S}e^{i2\pi Q/d_S}, & t = md_S. \end{cases} \tag{28}$$

For instance, if S is a qubit ($d_S = 2$), we may take $Q = (\mathbb{1} - \sigma_z)/2$, $P = (\mathbb{1} - \sigma_x)/2$, with $e^{i2\pi Q/d_S} = \sigma_z$, $e^{-i2\pi P/d_S} = \sigma_x$. Hence, $|\Psi\rangle = \frac{1}{2}[|S_0\rangle|0\rangle + \sigma_z|S_0\rangle|1\rangle + \sigma_x|S_0\rangle|2\rangle + i\sigma_y|S_0\rangle|3\rangle]$ is maximally entangled $\forall \ |S_0\rangle$ [$E(S, T) = 1$], with $|S_1\rangle = \sigma_z|S_0\rangle$, $|S_2\rangle = \sigma_x|S_0\rangle = -i\sigma_y|S_1\rangle$, $|S_3\rangle = i\sigma_y|S_0\rangle = \sigma_z|S_2\rangle$, and $|S_0\rangle = -i\sigma_y|S_3\rangle$.

In the general case, it is natural here to view system T as formed by two clocks with identical Hilbert space dimension d_S, which govern *time-independent* Hamiltonians $H_1 = -2\pi Q/d_S$ and $H_2 = 2\pi P/d_S$ associated with conjugate operators Q, P on S. Then we may write the history state
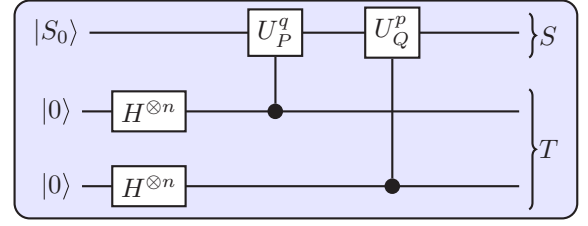

FIG. 2. Schematic circuit representing the generation of a maximally entangled history state $|\Psi\rangle$, for any initial system state $|S_0\rangle$. Here, $U_P = e^{-i2\pi P/d_S}$, $U_Q = e^{i2\pi Q/d_S}$, with P, Q conjugate operators on S and $2^n = d_S$.

(1) for the operators (26) as

$$|\Psi\rangle = \frac{1}{d_S^2} \sum_{p,q} U_{pq}|S_0\rangle|p\rangle_{T_1}|q\rangle_{T_2}, \tag{29}$$

which represents a history state of history states. It can then be implemented with the circuit of Fig. 2.

C. The quadratic $S - T$ entanglement entropy: Analytic evaluation and bounds

The analytic evaluation of the entropy (19) in the general case requires the determination of the singular values λ_k, i.e., the eigenvalues λ_k^2 of ρ_S or ρ_T, which is difficult in most cases. It is then convenient to use the quadratic (also called linear) entropy $S_2(\rho) = 2\mathrm{Tr}[\rho(\mathbb{1} - \rho)] = 2(1 - \mathrm{Tr}\,\rho^2)$, which does not require explicit knowledge of the eigenvalues and is a linear function of the purity $\mathrm{Tr}\,\rho^2$. Like $S(\rho)$, it vanishes iff ρ is pure and is maximum iff ρ is maximally mixed [with $S_2(\rho) = 1$ for a maximally mixed single-qubit state], satisfying the majorization relation $S_2(\rho') \geqslant S_2(\rho)$ if $\rho' \prec \rho$ [36,37]. The associated $S - T$ entanglement entropy is

$$E_2(S, T) = S_2(\rho_S) = S_2(\rho_T) = 2\left(1 - \sum_k \lambda_k^4\right) \tag{30}$$

$$= 2\left(1 - \frac{1}{N^2}\sum_{t,t'}|\langle S_t|S_{t'}\rangle|^2\right), \tag{31}$$

and can be determined just from the overlaps between the evolved states. For the complete orthogonal set (22), it is easily verified that $\sum_{t,t'}|\langle S_t|S_{t'}\rangle|^2 = d_S^3$, so that $E_2(S, T) = 2(1 - \frac{1}{d_S})$ becomes maximum.

The overlaps $\langle S_t|S_{t'}\rangle$ are also experimentally accessible through a measurement at the clock T of the nondiagonal operators $|t'\rangle\langle t|$ ($t \neq t'$):

$$\frac{1}{N}\langle S_{t'}|S_t\rangle = \langle\Psi|\mathbb{1}_S \otimes |t'\rangle\langle t||\Psi\rangle = \langle\sigma_{t't}^x\rangle + i\langle\sigma_{t't}^y\rangle, \tag{32}$$

where $\sigma_{t't}^x = |t'\rangle\langle t| + |t\rangle\langle t'|$, $\sigma_{t't}^y = (|t'\rangle\langle t| - |t\rangle\langle t'|)/i$ are Hermitian Pauli operators for the pair $t \neq t'$.

Let us now consider the evolution for a general constant Hamiltonian H of arbitrary spectrum for system S, such that $U_t = e^{-iHt} \ \forall \ t$. In contrast with (19), Eq. (31) can in this case be explicitly evaluated. Writing

$$|S_0\rangle = \sum_k c_k|E_k\rangle, \quad H|E_k\rangle = E_k|E_k\rangle, \tag{33}$$

with $E_k \neq E_{k'}$ if $k \neq k'$ (in the case of degenerate states $|k_l\rangle$, $c_k|E_k\rangle = \sum_l c_{kl}|k_l\rangle$, with $|c_k|^2 = \sum_l |c_{kl}|^2$), then $|S_t\rangle = \sum_k e^{-iE_k t} c_k |E_k\rangle$ and Eq. (31) becomes, for equally spaced times $t = t_f \frac{j}{N-1}$, $j = 0, \ldots, N-1$,

$$E_2(S, T) = 2\left(1 - \frac{1}{N^2}\sum_{t,t'}\left|\sum_k |c_k|^2 e^{-iE_k(t-t')}\right|^2\right) \quad (34)$$

$$= 2\sum_{k \neq k'}|c_k c_{k'}|^2\left[1 - \frac{\sin^2\frac{(E_k - E_{k'})t_f N}{2(N-1)}}{N^2\sin^2\frac{(E_k-E_{k'})t_f}{2(N-1)}}\right]. \quad (35)$$

The exact result for a continuous evolution can also be obtained from (35), by taking the limit $N \to \infty$,

$$E_2(S, T) \underset{N\to\infty}{\to} 2\sum_{k\neq k'}|c_k c_{k'}|^2\left[1 - \frac{\sin^2\left(\frac{(E_k-E_{k'})t_f}{2}\right)}{\left(\frac{(E_k-E_{k'})t_f}{2}\right)^2}\right]. \quad (36)$$

Equation (36) provides a good approximation to (35) if $\frac{|E_k-E_{k'}|t_f}{N-1} \ll 1 \ \forall \ k \neq k'$ with finite weight $|c_k c_{k'}|^2 > 0$.

Equations (35) and (36) are essentially measures of the *spread* of $|S_0\rangle$ over distinct energy eigenstates. For small t_f such that $|E_k - E_{k'}|t_f \ll 1 \ \forall \ k, k'$, a second-order expansion shows they are proportional to the *energy fluctuation* in $|S_0\rangle$: $|\langle S_t|S_{t'}\rangle|^2 \approx 1 - \langle(\Delta H)^2\rangle(t-t')^2$, with $\Delta H = H - \langle H\rangle$ and $\langle O\rangle = \langle S_0|O|S_0\rangle$, implying

$$E_2(S, T) \approx \frac{N+1}{3(N-1)}\langle(\Delta H)^2\rangle t_f^2 \underset{N\to\infty}{\longrightarrow} \frac{1}{3}\langle(\Delta H)^2\rangle t_f^2. \quad (37)$$

It then becomes proportional to the square of the speed $\sqrt{\langle(\Delta H)^2\rangle}$ of the continuous quantum evolution according to the Fubini-Study metric [19,38].

It is also apparent from (35) that $E_2(S, T)$ is upper bounded by the quadratic entropy of the energy distribution $|c_k|^2$,

$$E_2(S, T) \leqslant 2\sum_{k\neq k'}|c_k c_{k'}|^2 = 2\left(1 - \sum_k |c_k|^4\right). \quad (38)$$

The maximum (38) for a fixed distribution $|c_k|^2$ is reached for an equally spaced spectrum of the form

$$E_k = \frac{N-1}{t_f}\frac{2\pi k}{N} + C, \quad (39)$$

with k integer $\in [0, N-1]$ since in this case the bracket in (35) takes its maximum value $1 \ \forall \ k \neq k'$.

The spectrum (39) is just Eq. (12) for the scaled Hamiltonian $H_S = \frac{t_f}{N-1}(H - C)$ (for which $t = 0, \ldots, N-1$), so that the energy states $|E_k\rangle$ become the Schmidt states $|k\rangle_S$ of (11) and $|c_k|$ the Schmidt coefficients λ_k. For other spectra, the states $|\tilde{k}\rangle_T = \frac{1}{\sqrt{N}}\sum_t e^{-iE_k t}|t\rangle$ in

$$|\Psi\rangle = \frac{1}{\sqrt{N}}\sum_{k,t} c_k e^{-iE_k t}|E_k\rangle|t\rangle = \sum_k c_k|E_k\rangle|\tilde{k}\rangle_T \quad (40)$$

are not necessarily all orthogonal, so that $E(S, T)$ will become normally smaller [7]. Nonetheless, for large N and not too small t_f, the states $|\tilde{k}\rangle_T$ will typically be almost orthogonal, so that the deviation from the upper bound (38) will not be large, becoming significant only in the presence of quasidegeneracies in the spectrum: The bracket in (36) vanishes just for $E_k \to E_{k'}$, becoming close to 1 for $|E_k - E_{k'}|t_f/2 > \pi$, while that in (35), which is a periodic function of $E_k - E_{k'}$ with period $\Delta_N = 2\pi\frac{N-1}{t_f}$, vanishes for $E_k \to E_{k'} + m\Delta_N$, $m = 0$ or integer, becoming close to 1 whenever $|E_k - E_{k'} - m\Delta_N|t_f/2 > \pi$.

On the other hand, Eq. (36) also admits a *lower bound* for fixed initial and final states $|S_0\rangle$ and $|S_{t_f}\rangle = \sum_k c_k e^{-iE_k t_f}|E_k\rangle$, reached when the evolution (over N equally spaced times $t = t_f\frac{j}{N-1}$ under a constant H) remains in the subspace spanned by $|S_0\rangle$ and $|S_{t_f}\rangle$:

$$E_2(S, T) \geqslant E_2^{\min}(S, T) = 1 - \frac{\sin^2\frac{N\phi}{N-1}}{N^2\sin^2\frac{\phi}{N-1}}, \quad (41)$$

where $\phi \in [0, \pi/2]$ is determined by the overlap between the initial and final states,

$$\cos\phi = |\langle S_0|S_{t_f}\rangle| = \left|\sum_k |c_k|^2 e^{-iE_k t_f}\right|. \quad (42)$$

Writing the final state as

$$|S_{t_f}\rangle = e^{-i\gamma}(\cos\phi|S_0\rangle + \sin\phi|S_0^\perp\rangle), \quad (43)$$

where $\langle S_0^\perp|S_0\rangle = 0$, $E_2^{\min}(S, T)$ is the result of Eq. (35) for an evolution under a two-level Hamiltonian

$$H^{\min} = \frac{\phi}{t_f}\sigma_y + \frac{\gamma}{t_f}, \quad \sigma_y = -i(|S_0\rangle\langle S_0^\perp| - |S_0^\perp\rangle\langle S_0|), \quad (44)$$

such that

$$|S_t^{\min}\rangle \equiv \exp[-iH^{\min}t]|S_0\rangle$$
$$= e^{-i\gamma t/t_f}\left(\cos\frac{\phi t}{t_f}|S_0\rangle + \sin\frac{\phi t}{t_f}|S_0^\perp\rangle\right), \quad (45)$$

with $|S_{t_f}^{\min}\rangle = |S_{t_f}\rangle$.

The demonstration of (41) is given in the Appendix, but the result is physically clear: The $S - T$ entanglement is a measure of the distinguishability between the evolved states, and the minimum value is then obtained for an evolution within the subspace containing the initial and final states, where all intermediate states will be closer than in a general evolution. Such evolution, given by Eq. (45), proceeds precisely along the geodesic determined by the Fubini-Study metric [19,38], saturating the Mandelstam-Tamm bound [39] $\Delta t\Delta E \geqslant \cos^{-1}(|\langle S_0|S_{t_f}\rangle|) = \phi$ ($\Delta t = t_f$, $\Delta E = \sqrt{\langle(\Delta H^{\min})^2\rangle} = \phi/t_f$).

As a check, for small t_f such that $|E_k - E_{k'}|t_f \ll 1 \ \forall \ k \neq k'$, a fourth-order expansion of (35) and (41) leads to

$$E_2(S, T) - E_2^{\min}(S, T) \approx \kappa[\langle(\Delta H)^4\rangle - \langle(\Delta H)^2\rangle^2]t_f^4 \geqslant 0, \quad (46)$$

where $\kappa = \frac{(N+1)(N-2)(N-4/3)}{60(N-1)^3} > 0 \ \forall \ N > 2$. Hence, the difference (46) is verified to be non-negative and of fourth order in t_f, being proportional to the fluctuation of $(\Delta H)^2$. The latter vanishes just for the geodesic evolution, where $\Delta H = \Delta H^{\min} = \frac{\phi}{t_f}\sigma_y$ and hence $\langle(\Delta H^{\min})^4\rangle = \langle(\Delta H^{\min})^2\rangle^2 = \phi^4/t_f^4$, implying $E_2(S, T) = E_2^{\min}(S, T)$. Such fluctuation represents a curvature coefficient which measures the deviation from the geodesic [38,40].

For $\phi \in [0, \pi/2]$, the bound (41) is, of course, an increasing function of ϕ for $N \geqslant 2$, i.e., of the Wootters distance [41] $s(|S_0\rangle, |S_{t_f}\rangle) = 2\arccos(|\langle S_0|S_{t_f}\rangle|) = 2\phi$, and hence a decreasing function of the overlap $|\langle S_{t_f}|S_0\rangle|$. It is also a *decreasing* function of $N \geqslant 2$ for $\phi \in (0, \pi/2]$. The minimum value is thus achieved in the continuous limit $N \rightarrow \infty$, where $E_2^{\min}(S, T) \rightarrow 1 - (\sin^2 \phi)/\phi^2$. Then, we may also write, for any $N \geqslant 2$,

$$E_2(S, T) \geqslant 1 - \frac{\sin^2 \phi}{\phi^2}. \tag{47}$$

III. ENTANGLEMENT AND HISTORY STATES OF EVOLUTION OPERATORS

We now examine the application of the previous formalism to the evolution operators themselves. The aim is to link properties of previous history states with those of the operators that generate it. For this purpose, the pure state representation of operators [24–28] provides a convenient approach, enabling a direct derivation of their entanglement properties [20–23].

A. Entanglement of operators and pure state representation

We first briefly review the concept of operator entanglement and its pure state representation. Any operator $\mathcal{W}$ for a bipartite system $A+B$ can be expanded as

$$\mathcal{W} = \sum_{i,j} M_{ij} C_i \otimes D_j, \tag{48}$$

where C_i and D_j are orthogonal operators for A and B, respectively, satisfying

$$\text{Tr }C_i^\dagger C_j = \delta_{ij} d_A, \quad \text{Tr }D_i^\dagger D_j = \delta_{ij} d_B. \tag{49}$$

Hence, $M_{ij} = \frac{1}{d_A d_B} \text{Tr }[C_i^\dagger \otimes D_j^\dagger \mathcal{W}]$. We can use, for instance, the Weyl operators (26) for the sets $\{C_i\}$, $\{D_j\}$.

Equations (49) imply $\text{Tr }[\mathcal{W}^\dagger \mathcal{W}] = d_A d_B \text{Tr }[M^\dagger M]$. If $\mathcal{W}$ is unitary, then $\text{Tr }[M^\dagger M] = 1$, entailing that the numbers $\{|M_{ij}|^2\}$ are in this case standard probabilities. By means of the singular value decomposition, we can write the $d_A^2 \times d_B^2$ matrix M as $M = UDV^\dagger$, where U and V are unitary matrices and D a diagonal matrix with non-negative entries $\lambda_k^{\mathcal{W}}$ satisfying $\sum_k (\lambda_k^{\mathcal{W}})^2 = \text{Tr }M^\dagger M = 1$. We can then rewrite $\mathcal{W}$ in the Schmidt form,

$$\mathcal{W} = \sum_k \lambda_k^{\mathcal{W}} A_k \otimes B_k, \tag{50}$$

where $A_k \equiv \sum_i U_{ik} C_i$ and $B_k \equiv \sum_j V_{jk}^* D_j$ are again orthogonal operator bases for A and B satisfying $\text{Tr }A_k^\dagger A_l = d_A \delta_{kl}$, $\text{Tr }B_k^\dagger B_l = d_B \delta_{kl}$. The von Neumann entanglement entropy of $\mathcal{W}$ can then be defined as

$$E(\mathcal{W}) = - \sum_k (\lambda_k^{\mathcal{W}})^2 \log_2 (\lambda_k^{\mathcal{W}})^2. \tag{51}$$

Similarly, $E_2(\mathcal{W}) = 2\sum_k [1 - (\lambda_k^{\mathcal{W}})^4]$. These entropies vanish when $\mathcal{W}$ is a product of local unitaries and are maximum when $\mathcal{W}$ is a uniform sum of d^2 products $A_k \otimes B_k$, with $d = \text{Min}[d_A, d_B]$.

The previous analogy between operators and states can be manifestly described through the Choi isomorphism [24–28].

Any operator O in a system with Hilbert space $\mathcal{H}$ of dimension d can be associated with a pure state $|O\rangle \in \mathcal{H} \otimes \mathcal{H}$, given by

$$|O\rangle = (O \otimes \mathbb{1})|\mathbb{1}\rangle = \frac{1}{\sqrt{d}} \sum_q (O|q\rangle)|q\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{q,q'} \langle q'|O|q\rangle|q'\rangle|q\rangle, \tag{52}$$

where $|\mathbb{1}\rangle = \frac{1}{\sqrt{d}} \sum_q |q\rangle|q\rangle$ is a maximally entangled state in $\mathcal{H} \otimes \mathcal{H}$ and $\{|q\rangle\}$ an orthonormal set. In this way,

$$\langle O|O'\rangle = \frac{1}{d} \text{Tr }[O^\dagger O']. \tag{53}$$

Therefore, orthogonal operators satisfying $\text{Tr }[O_i^\dagger O_j] = d\delta_{ij}$ correspond to orthonormal states $\langle O_i|O_j\rangle = \delta_{ij}$. And unitary operators U correspond to normalized states $|U\rangle$.

The operator (48) can then be associated with the pure state (note that $|\mathbb{1}_{AB}\rangle = |\mathbb{1}_A\rangle|\mathbb{1}_B\rangle$)

$$|\mathcal{W}\rangle = (\mathcal{W} \otimes \mathbb{1}_{A'B'})|\mathbb{1}_A\rangle|\mathbb{1}_B\rangle = \sum_{ij} M_{ij}|C_i\rangle|D_j\rangle, \tag{54}$$

where $|C_i\rangle = (C_i \otimes \mathbb{1}_{A'})|\mathbb{1}_A\rangle$, $|D_j\rangle = (D_j \otimes \mathbb{1}_{B'})|\mathbb{1}_B\rangle$ form orthogonal sets: $\langle C_k|C_i\rangle = \delta_{ki}$, $\langle D_k|D_j\rangle = \delta_{kj}$. Thus, $M_{ij} = \langle C_i, D_j|\mathcal{W}\rangle$, with $\langle \mathcal{W}|\mathcal{W}\rangle = \text{Tr }[M^\dagger M]$.

The state representation of the Schmidt form (50) acquires then the standard appearance

$$|\mathcal{W}\rangle = \sum_k \lambda_k^{\mathcal{W}}|A_k\rangle|B_k\rangle, \tag{55}$$

with $\langle A_k|A_l\rangle = \delta_{kl} = \langle B_k|B_l\rangle$, and the entanglement entropy (51) of a unitary $\mathcal{W}$ can also be expressed as

$$E(\mathcal{W}) = S(\rho_A^{\mathcal{W}}) = S(\rho_B^{\mathcal{W}}), \quad \rho_{A(B)}^{\mathcal{W}} = \text{Tr}_{B(A)}|\mathcal{W}\rangle\langle\mathcal{W}|, \tag{56}$$

with $S(\rho) = -\text{Tr}\rho \log_2 \rho$. Similarly, $E_2(\mathcal{W}) = S_2(\rho_A^{\mathcal{W}}) = S_2(\rho_B^{\mathcal{W}})$, with $S_2(\rho) = 2(1 - \text{Tr }\rho^2)$.

B. Generating operators and operator history states

The history state (1) can be generated from an initial product state $|S_0\rangle|0\rangle$ as

$$|\Psi\rangle = \mathcal{W}(I \otimes H^{\otimes n})|S_0\rangle|0\rangle, \tag{57}$$

where $H^{\otimes n}$ denotes the Hadamard operator acting on the clock ($H^{\otimes n}|0\rangle = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} |t\rangle$, with $N = 2^n$) and

$$\mathcal{W} = \sum_t U_t \otimes |t\rangle\langle t| \tag{58}$$

the control-U_t operator. By expanding U_t in an orthogonal basis of operators C_i, we have

$$\mathcal{W} = \sum_{t,i} M_{ti} C_i \otimes |t\rangle\langle t|, \quad M_{ti} = \frac{1}{d_S} \text{Tr }C_i^\dagger U_t, \tag{59}$$

where the coefficients M_{tj} satisfy $\sum_j |M_{tj}|^2 = \frac{1}{d_S} \text{Tr }U_t^\dagger U_t = 1$, and are hence standard probabilities at fixed t. Since the projectors $|t\rangle\langle t|$ are also orthogonal and have unit trace, the
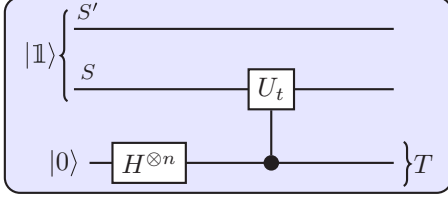
FIG. 3. Schematic circuit representing the generation of the operator history state (60).

Schmidt coefficients $\lambda_k^{\mathcal{W}}$ of (50) are here just the singular values of the matrix $M/\sqrt{N}$. The ensuing entanglement entropy (51) is the same as that of $\mathcal{W}(I \otimes H^{\otimes n})$, as they differ just by a local unitary.

The pure state (54) associated with the operator (58) is itself an *operator history state*,

$$|\mathcal{W}\rangle = \frac{1}{\sqrt{N}} \sum_t |U_t\rangle |T_t\rangle, \tag{60}$$

where $|U_t\rangle = (U_t \otimes \mathbb{1}_{S'})|\mathbb{1}_S\rangle = \frac{1}{\sqrt{d_S}} \sum_q U_t |q\rangle |q\rangle$ and $|T_t\rangle = (T_t \otimes \mathbb{1}_{T'})|\mathbb{1}_T\rangle = |tt\rangle$, with $T_t = \sqrt{N}|t\rangle\langle t|$ and $\langle T_t|T_{t'}\rangle = \delta_{tt'}$. Writing $|tt\rangle$ simply as $|t\rangle$, Eq. (60) is the standard history state (1) for a maximally entangled initial state $|\mathbb{1}_S\rangle = \frac{1}{\sqrt{d_s}} \sum_q |q\rangle |q\rangle$ of a bipartite system under a local evolution $U_t \otimes \mathbb{1}_{S'}$, so that it can be generated with the circuit depicted in Fig. 3.

The entanglement of the history state (60) is the operator entanglement (51) of $\mathcal{W}$, which is then a measure of the distinguishability of the operator states $|U_t\rangle$. Its quadratic entanglement can be directly evaluated with Eq. (31), where now $\langle U_t|U_{t'}\rangle = \frac{1}{d_S}\text{Tr}\,[U_t^\dagger U_{t'}]$:

$$E_2(\mathcal{W}) = 2\left(1 - \frac{1}{N^2} \sum_{t,t'} |\langle U_t|U_{t'}\rangle|^2\right). \tag{61}$$

It is now immediate to see that if $N = d_S^2$ and the operators $\{U_t\}$ form a *complete orthogonal set* [Eq. (22)], the operator history state (60) is *maximally entangled*,

$$E(\mathcal{W}) = \log_2 d_S^2 = 2 \log_2 d_S, \tag{62}$$

while $E_2(\mathcal{W}) = 2(1 - \frac{1}{d_S^2})$, since all states $|U_t\rangle$ become orthogonal: $\langle U_t|U_{t'}\rangle = \delta_{tt'}$. The form (60) is then already the Schmidt decomposition of $|\mathcal{W}\rangle$. Since in this case the original history state (1) has maximum entanglement $E(S, T) = \log_2 d_S$ for any initial state $|S_0\rangle$, this result indicates a close relation between the entangling power of $\mathcal{W}$ and its operator entanglement, which will be discussed below. It is also apparent that if the d_S^2 operators U_t are not all orthogonal, then $E(\mathcal{U}) < 2 \log_2 d_S$.

For a smaller number $N < d_S^2$ of times, $E(\mathcal{W})$ will be maximum if all N states $|U_t\rangle$ are orthogonal. In the case of a *constant* Hamiltonian with energies E_k, such that $U_t = e^{-iHt}$ $\forall\, t$, then

$$\langle U_t|U_{t'}\rangle = \frac{1}{d_S} \sum_k e^{-iE_k(t-t')}. \tag{63}$$

For $N = d_S$, an equally spaced spectrum $E_k = 2\pi k/N + C$, $k = 0, \ldots, N-1$ [i.e., Eq. (39) if $t \to \frac{t_f}{N-1}j$], ensures

that all $|U_t\rangle$ are strictly orthogonal: $\langle U_t|U_{t'}\rangle = \delta_{tt'}$ $\forall\, t, t'$ [the ensuing operators U_t are in fact the first d_S operators of the Weyl set (26)]. Hence, $E(\mathcal{W})$ will reach for this spectrum the maximum value

$$E(\mathcal{W}) = \log_2 d_S, \tag{64}$$

compatible with a fixed H and $N = d_S$ times. The same holds for $E_2(\mathcal{W})$. This result correlates with the extremal properties of this spectrum discussed in Sec. II C.

On the other hand, since $U_{t,t-1} = U_t U_{t-1}^\dagger$, the operator $\mathcal{U}$ of Eq. (2) is related with $\mathcal{W}$ by

$$\mathcal{U} = \mathcal{W}(I \otimes \exp[-iP_T])\mathcal{W}^\dagger, \tag{65}$$

where $\exp[-iP_T] = \sum_t |t\rangle\langle t-1|$. The associated pure state is also a history state,

$$|\mathcal{U}\rangle = \frac{1}{\sqrt{N}} \sum_t |U_{t,t-1}\rangle |T_{t,t-1}\rangle, \tag{66}$$

where $|T_{t,t-1}\rangle = \sqrt{N}(|t\rangle\langle t-1| \otimes \mathbb{1}_{T'})|\mathbb{1}_T\rangle = |t, t-1\rangle$ are again orthogonal states. Its entanglement is then a measure of the distinguishability of the step evolution operator states $|U_{t,t-1}\rangle$, and depends on the *order* of the operators U_t, in contrast with $E(\mathcal{W})$. It vanishes in the constant case (6) and (7).

C. Operator entanglement and entangling power

We have seen that there is a relation between the entanglement of the operator $\mathcal{W}$ and that of the history states it generates, $|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_t U_t |S_0\rangle |t\rangle$. Here we will prove that the quadratic operator entanglement entropy $E_2(U, T) \equiv E_2(\mathcal{W})$, given by Eq. (61), is proportional to the *entangling power* of $\mathcal{W}$, defined as the average quadratic entanglement it generates when applied [Eq. (57)] to initial product states $|S_0\rangle |0\rangle$:

$$\langle E_2(S, T)\rangle = \frac{d_S}{d_S + 1} E_2(\mathcal{W}), \tag{67}$$

where

$$\langle E_2(S, T)\rangle = \int_{\mathcal{H}} 2(1 - \text{Tr}\,\rho_S^2)\, dS_0 \tag{68}$$

is the average over all initial states $|S_0\rangle$ of the quadratic entanglement entropy $E_2(S, T)$ of the history state: The integral runs over the whole set of initial states $|S_0\rangle$ with the Haar measure dS_0 (the only normalized unitarily invariant measure over the Hilbert space) and ρ_S is the reduced state of S in $|\Psi\rangle$.

Proof. Since $\rho_S = \frac{1}{N} \sum_t U_t |S_0\rangle\langle S_0|U_t^\dagger$, we obtain

$$\langle\text{Tr}\,\rho_S^2\rangle = \frac{1}{N^2} \sum_{t,t'} \int_{\mathcal{H}} \langle S_0|U_t^\dagger U_{t'}|S_0\rangle\langle S_0|U_{t'}^\dagger U_t|S_0\rangle\, dS_0. \tag{69}$$

Here we can define $O = U_t^\dagger U_{t'}$ and $P = U_{t'}^\dagger U_t = O^\dagger$ to use the relation [42]

$$\int_{\mathcal{H}} \langle S_0|O|S_0\rangle\langle S_0|P|S_0\rangle\, dS_0 = \frac{\text{Tr}[O]\text{Tr}[P] + \text{Tr}[OP]}{d_S(d_S + 1)}. \tag{70}$$

Since in this case $OP = \mathbb{1}_S$, we obtain

$$\langle\text{Tr}\,\rho_S^2\rangle = \frac{\frac{1}{N^2} \sum_{t,t'} |\text{Tr}\,[U_t^\dagger U_{t'}]|^2 + d_S}{d_S(d_S + 1)}. \tag{71}$$
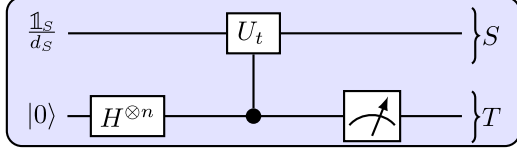
FIG. 4. Schematic circuit representing the measurement of the operator overlaps (79).

On the other hand, $E_2(\mathcal{W}) = 2(1 - \mathrm{Tr}\,\rho_U^2)$, with $\rho_U^2 = \frac{1}{N^2}\sum_{t,t'}|U_t\rangle\langle U_t^\dagger|U_{t'}\rangle\langle U_{t'}^\dagger|$. Thus,

$$\mathrm{Tr}\,\rho_U^2 = \frac{1}{N^2}\sum_{t,t'}|\langle U_t^\dagger|U_{t'}\rangle|^2 = \frac{1}{(d_S N)^2}\sum_{t,t'}|\mathrm{Tr}[U_t^\dagger U_{t'}]|^2. \tag{72}$$

Replacing (72) in (71) leads to $\langle \mathrm{Tr}\,\rho_S^2\rangle = \frac{d_S \mathrm{Tr}(\rho_U^2)+1}{d_S+1}$ and hence to Eq. (67). ∎

Therefore, the average over all initial system states of the quadratic $S - T$ entanglement is just that of the generating unitary operator times $\frac{d_S}{d_S+1}$. It is first verified that if the operators U_t form a complete orthogonal set, $E_2(\mathcal{W}) = 2(1 - d_S^{-2})$ is maximum, and Eq. (67) yields $\langle E_2(S, T)\rangle = 2(1 - d_S^{-1})$, the maximum attainable value in a d_S-dimensional space, then entailing it is always maximum, irrespective of the initial state (Sec. II B).

In general, for a reduced set of d orthogonal unitaries U_t, with $N = d \leqslant d_S^2$, $E_2(\mathcal{W}) = 2(1 - d^{-1})$ and hence

$$\langle E_2(S, T)\rangle = 2\frac{d_S(d-1)}{d(d_S+1)}. \tag{73}$$

In order to visualize this relation, we define the effective average number of orthogonal states the system visits as

$$\overline{d}_{S,T} = \frac{1}{1 - \frac{1}{2}\langle E_2(S, T)\rangle} = \frac{d(d_S+1)}{d_S+d}, \tag{74}$$

such that $\langle E_2(S, T)\rangle = 2(1 - \frac{1}{\overline{d}_{S,T}})$. If $d = d_S^2$, then $\overline{d}_{S,T} = d_S$ becomes maximum, while if $d = d_S$, which is, for instance, the case of a constant Hamiltonian with spectrum $2\pi k/N$ (d_S orthogonal operators $U_t = \exp[-iHt]$), then Eq. (74) leads to $\overline{d}_{S,T} = (d_S + 1)/2$, i.e., just *half* the maximum value for large d_S. For any other spectrum and $N = d_S$, $\overline{d}_{S,T} \leqslant (d_S + 1)/2$, i.e.,

$$\langle E_2(S, T)\rangle \leqslant 2\frac{d_S - 1}{d_S + 1} \quad (U_t = e^{-iHt}, \ N = d_S). \tag{75}$$

Noticeably, it is sufficient to have $d \propto d_S$ ($\ll d_S^2$ for large d_S) to reach a high $\overline{d}_{S,T}$, i.e., $\overline{d}_{S,T} = \frac{m}{m+1}(d_S + 1)$ if $d = md_S$ (and $m \leqslant d_S$), as seen from (74).

D. Measuring operator overlaps

The overlaps $\langle U_t|U_{t'}'\rangle$, which are the operator fidelities defined in [43] and are involved in the quadratic entanglement (61) of the generating operator $\mathcal{W}$, can be experimentally obtained by measuring $|T_t\rangle\langle T_{t'}|$ in the time part T (Fig. 4). Remarkably, it is sufficient to start with the system in a *maximally mixed* state: If we trace out system S' in the operator

history state (60), we obtain

$$\rho_{ST} = \frac{1}{Nd_S}\sum_{t,t'}U_t U_{t'}^\dagger \otimes |t\rangle\langle t'|, \tag{76}$$

where we have written $|T_t\rangle$ as $|t\rangle$. Hence, tracing over S,

$$\rho_T = \frac{1}{N}\sum_{t,t'}\langle U_{t'}|U_t\rangle|t\rangle\langle t'|. \tag{77}$$

Thus, setting $U_{t,t'} = U_t U_{t'}^\dagger$,

$$\langle|t'\rangle\langle t|\rangle = \frac{1}{N}\langle U_{t'}|U_t\rangle = \frac{1}{Nd_S}\mathrm{Tr}[U_{t'}^\dagger U_t] = \frac{1}{Nd_S}\mathrm{Tr}[U_{t,t'}]. \tag{78}$$

Using again $\sigma_{t't}^x = |t'\rangle\langle t| + |t\rangle\langle t'|$, $\sigma_{t't}^y = -i(|t'\rangle\langle t| - |t\rangle\langle t'|)$, the trace of the evolution operator between any two times can then be obtained by measuring the averages of $\sigma_{tt'}^x$ and $\sigma_{t't}^y$, which provide the real and imaginary parts:

$$\langle\sigma_{t't}^x\rangle = \frac{2}{N}\mathrm{Re}[\langle U_t|U_{t'}\rangle], \quad \langle\sigma_{t't}^y\rangle = \frac{2}{N}\mathrm{Im}[\langle U_t|U_{t'}\rangle]. \tag{79}$$

Of course, the state (76) can be generated just by preparing system S in the maximally mixed state, as the purifying system S' of the original operator state is traced out. Note also that $U_0 = \mathbb{1}$, so that the averages $\langle\sigma_{0t}^\mu\rangle$ determine $\mathrm{Tr}\,U_t$.

In the special case $N = 2$, T is a single qubit and we recover the standard DQC1 scheme for measuring the trace of an operator [29]. The ensuing operator history state is $|\mathcal{W}\rangle = \frac{1}{\sqrt{2}}(|U_0\rangle|T_0\rangle + |U_1\rangle|T_1\rangle)$, and its quadratic entanglement is

$$E_2(\mathcal{W}) = 1 - |\langle U_0|U_1\rangle|^2 = 1 - |\mathrm{Tr}\,U|^2/d_S^2. \tag{80}$$

Its square root is just the entangling power of the DQC1 circuit defined in [44].

IV. CONCLUSIONS

Quantum mechanics has mostly considered time as an external classical parameter. In this work, we have determined some fundamental properties concerning the generation and entanglement of discrete history states within a parallel-in-time discrete model of quantum evolution, based on a finite-dimensional quantum clock [7]. It was first shown that a general unitary evolution for the system states follows from a static eigenvalue equation, which can be recast as a generalized discrete version of a Wheeler-DeWitt equation. The ensuing system-time entanglement is a measure of the actual number of distinguishable states visited by the system at distinguishable times and satisfies an entropic energy-time uncertainty relation. Its dependence on the initial system state becomes attenuated for nonconstant noncommuting Hamiltonians, and in particular we have presented a simple two-clock scheme which generates a *maximally entangled* history state irrespective of the seed state. Thus, history states essentially independent of initial conditions can be generated. On the other hand, for any fixed seed system state, there is always a special clock basis selection for which the evolution corresponds to a constant Hamiltonian.

We have also shown that the quadratic entropy provides a convenient measure of the system-time entanglement entropy.

It can be evaluated analytically and satisfies strict and physical upper and lower bounds, the former connected with the energy spread of the initial state and the latter determined by the evolution along the geodesic path between the initial and final states. Hence, such path, which provides the minimum evolution time [39], minimizes as well the quadratic $S - T$ entanglement entropy.

Finally, by means of the channel-state duality, we have shown that the unitary operator generating the history state corresponds to an operator history state, with its quadratic entanglement entropy representing its entangling power. We have also provided a simple scheme which allows one to efficiently obtain the overlaps between system states and the traces of the evolution operator between any two times through measurements on the clock.

The present formalism is interesting as a fundamental aspect of quantum theory, where there are some possible scenarios to explore further in connection with quantum gravity, such as interaction between relational clocks [17,18] and emergence of causality [45,46]. The incorporation of time in a discrete quantum clock system also enables the development of new models of parallel-in-time simulation, taking advantage of the quantum features of superposition and entanglement. This description of time could also be suitable for application in Floquet systems, and in particular Floquet time crystals [47].

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APPENDIX

Proof of the lower bound of Eq. (41). We first assume a sufficiently short final time t_f such that $|\frac{(E_k - E_{k'})t_f}{2}| \leqslant \pi \ \forall \ k \neq k'$. Note that the overlap $|\langle S_0|S_{t_f}\rangle|$, given by Eq. (42), is unaffected by any translation $E_k \rightarrow E_k + 2j\pi/t_f \ \forall j \in \mathbb{Z}$, for a given k. The angle $\phi \in [0, \pi/2]$ determined by this overlap

can also be rewritten as

$$\phi = \arcsin \sqrt{1 - \left|\langle S_0|S_{t_f}\rangle\right|^2} \tag{A1}$$

$$= \arcsin \sqrt{2 \sum_{k \neq k'} |c_k c_{k'}|^2 \sin^2 \frac{(E_k - E_{k'})t_f}{2}}. \tag{A2}$$

It is now expected that the overlap between any pair of intermediate states will be smaller than those between states $|S_t^{\min}\rangle = e^{-iH_{\min}t}|S_0\rangle$ along the geodesic, such that [Eq. (45)] $|\langle S_t|S_{t'}\rangle| \leqslant |\langle S_t^{\min}|S_{t'}^{\min}\rangle| = |\cos[\phi \frac{t - t'}{t_f}]|$. This inequality is verified since the function

$$F(s) = \arcsin \sqrt{2 \sum_{k \neq k'} |c_k c_{k'}|^2 \sin^2 \frac{(E_k - E_{k'})t_f s}{2}} - \phi s, \tag{A3}$$

where $s = |\frac{t - t'}{t_f}| \leqslant 1$, is a concave function of s for $s \in [0, 1]$ and satisfies $F(0) = F(1) = 0$, so that $F(s) \geqslant 0 \ \forall \ s \in [0, 1]$. Hence, for short times t_f such that all relative phases have not yet completed one period ($|E_k - E_{k'}|t_f < 2\pi \ \forall \ k, k'$), all intermediate overlaps of the actual evolution are smaller than those along the geodesic, and hence the actual $E_2(S, T)$ entropy is larger than that along the geodesic path.

For larger times t_f, the inequality (41) also holds but for a different reason: If $|\frac{(E_k - E_{k'})t_f}{2}| > \pi$ for some pairs k, k', $F(s)$ may not be concave and can also be negative for some values of s. However, the relevant term of the exact expression for $E_2(S, T)$ satisfies

$$\frac{\sin^2 \frac{\gamma N}{N-1}}{N^2 \sin^2 \frac{\gamma}{N-1}} \leqslant \frac{\sin^2 \frac{(\gamma - j\pi)N}{N-1}}{N^2 \sin^2 \frac{(\gamma - j\pi)}{N-1}}, \tag{A4}$$

where $\gamma = \frac{(E_k - E_{k'})t_f}{2}$ and j is such that $|\gamma - j\pi| \in [0, \pi/2]$. This translation of the energy difference does not affect the overlap [Eq. (A2)], but shows that the actual entropy $E_2(S, T)$ for large times will not become lower than the bound previously obtained. In this case, some relative phases may have completed one or more periods, but the final effect will be to decrease the average overlap and hence to increase $E_2(S, T)$.

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