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Natural frequencies of thin, rectangular plates with holes or orthotropic "patches" carrying an elastically mounted mass

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Abstract

The approximate solution for the title problem is obtained in the case of simply supported and clamped rectangular plates made of isotropic or orthotropic materials. A variational approach (the well known Rayleigh–Ritz method) is used, where the displacement amplitude is expressed in terms of beam functions. This means that each coordinate function satisfies identically all the boundary conditions at the outer edge of the plate. Free vibration analysis has been performed on various different cases; solid isotropic and orthotropic plates, orthotropic plates with a hole and isotropic plates with an orthotropic inclusion or "patch", carrying an elastically mounted concentrated mass. It is important to point out that the case of an orthotropic patch is interesting from a technological viewpoint since it constitutes a model of a repair implemented on the virgin structural element when it has suffered damage. This approach has been implemented by the aeronautical industry in some instances. The obtained results are in very good agreement with those of particular cases of simply supported plates available in the literature.

Keywords: Vibration; Mass; Plate; Elastic mounting; Orthotropy

1. Introduction

The present study deals with the analysis of transverse vibrations of simply supported and clamped plates carrying a concentrated mass elastically mounted.

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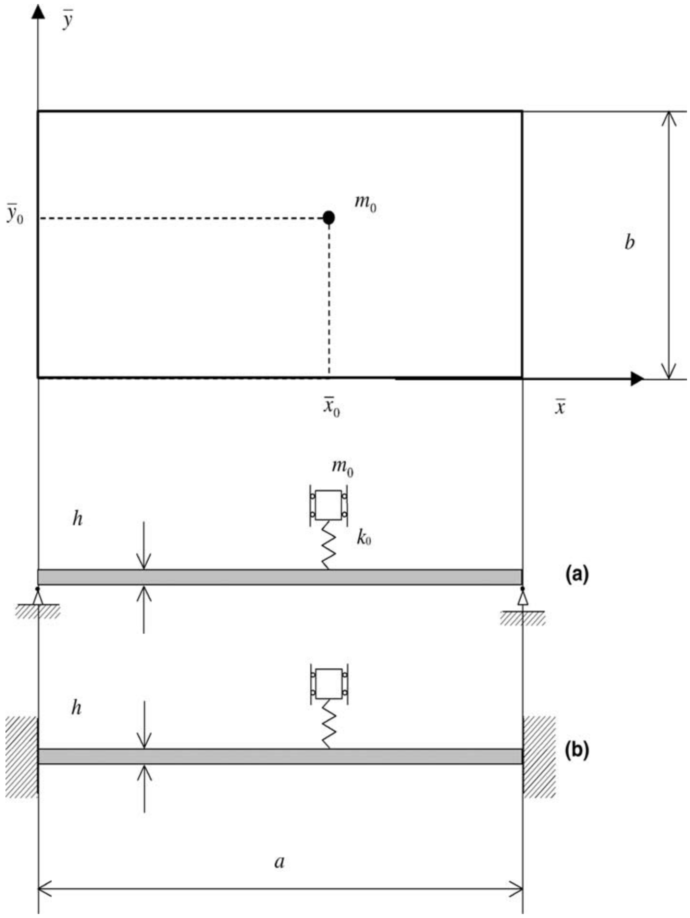

Fig. 1. Rectangular plate carrying an elastically mounted mass: (a) simply supported and (b) clamped.

The plates have different characteristics, simply supported or clamped edges, isotropic or orthotropic properties and holes or patches of orthotropic materials. Lekhnitskii's notation will be used (Lekhnitskii, 1968).

The mass is supposed elastically attached to the plate to take into account the presence of a motor or engine that is mounted on an elastic foundation or when the connection of the mass to the plate possesses elastic properties. The present paper extends considerably previous works on the subject (Laura et al., 1977; Ercoli et al., 1992; Ávalos et al., 1993; Ávalos et al., 1997; Larrondo et al., 1998; Ávalos et al., 1999) where, in general, simply supported edges were considered. The use of beam functions allows for a very complete combination of classical boundary conditions.

In a rather limited number of situations i.e. hinged ends and simply supported edges the differential system is ameniable to solution by an exact approach (Laura et al., 1977). In other cases an approximate approach is necessary (Laura et al., 1977, who made use of the Galerkin method and Rossi et al., 1993, employed the finite element methodology when making use of a discretized approach).

The problem is of basic interest in several fields of technology since it appears in a large variety of engineering situations; e.g. slabs supporting engines or motors, printed circuit boards with electronic elements attached to them, etc. A plate-like chassis or a printed circuit can be approximated as flat rectangular plates carrying concentrated masses, with holes or patches of another material and subjected to vibration.

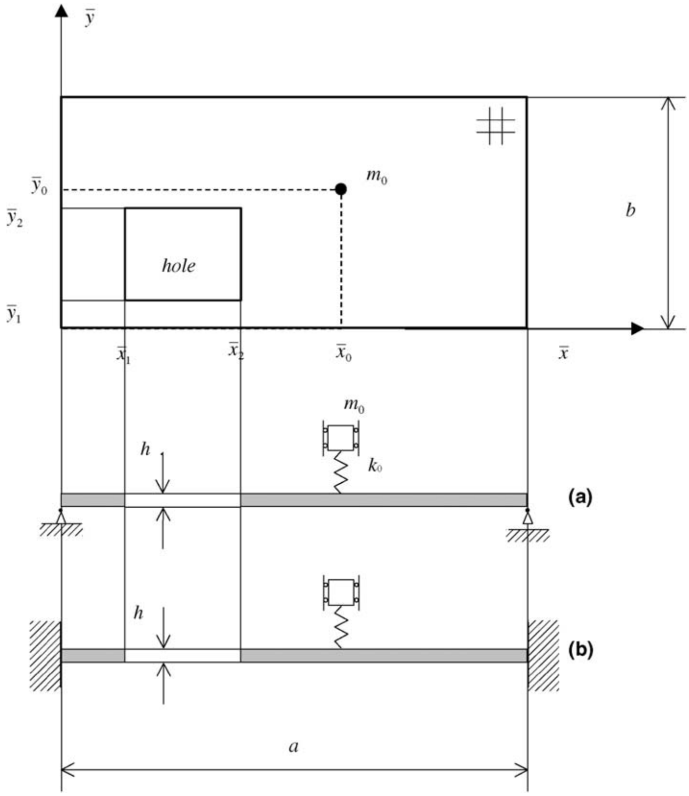

Fig. 2. Rectangular plate with a hole and an elastically mounted mass: (a) simply supported and (b) clamped.

The natural frequencies of the modified structural element are calculated based on an approximation in terms of a summation of beam eigenfunctions.

The numerical results presented in this study correspond to the natural frequencies of the vibrating models depicted in Figs. 1–3.

2. Approximate analytical solution

According to the classical thin plate theory the energy functional corresponding to the vibrating system presented in Fig. 1 is given by:

$$J(W,Z) = \frac{1}{2}\int\int (D_1 W_{\bar{x}\bar{x}}^2 + 2\nu_2 D_1 W_{\bar{x}\bar{x}} W_{\bar{y}\bar{y}} + D_2 W_{\bar{y}\bar{y}}^2 + 4D_k W_{\bar{x}\bar{y}}^2)\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y} - \frac{1}{2}\rho\omega^2 h\int\int W^2\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y}$$
$$- \frac{1}{2} m_0\omega^2 (W(\bar{x}_0,\bar{y}_0) + Z)^2 + \frac{1}{2}k_0 Z^2 \tag{1}$$

where $W = W(\bar{x},\bar{y})$ is the deflection amplitude of the middle plane of the plate; Z is the mass displacement amplitude relative to the middle plane of the plate; ν_2 is one of the Poisson's ratios and D_1, D_2, D_k are bending and twisting rigidities for principle directions of elasticity for orthotropic plates

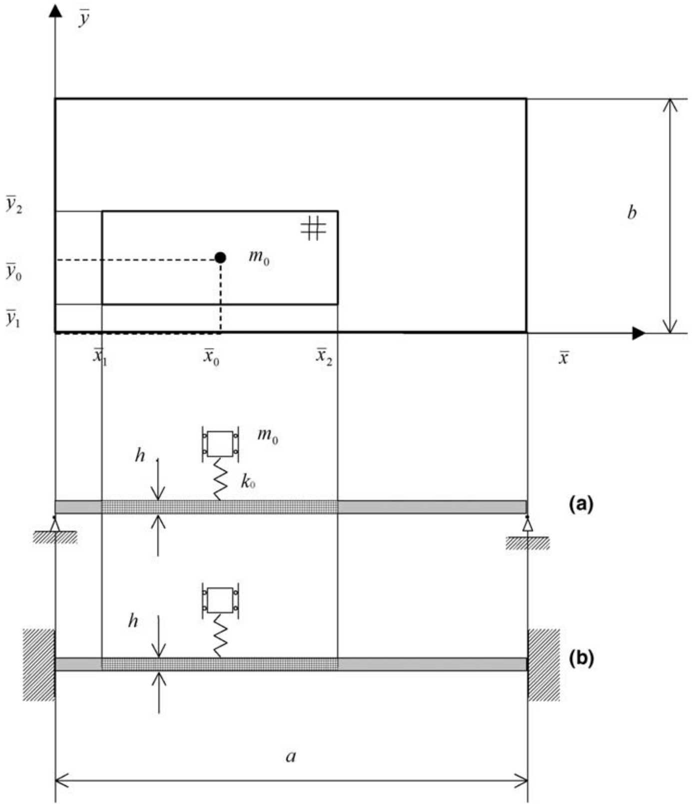

Fig. 3. Rectangular plate with an orthotropic patch and an elastically mounted mass on it: (a) simply supported and (b) clamped.

$$D_1 = \frac{E_1 h^3}{12(1 - \nu_1 \nu_2)}, \quad D_2 = \frac{E_2 h^3}{12(1 - \nu_1 \nu_2)}, \quad D_k = \frac{G h^3}{12}$$

E_1, E_2: Young's moduli, G: shear modulus; ν_1 is the other Poisson's ratio. $W_0(\bar{x}_0, \bar{y}_0)$ is the plate displacement amplitude at the mass position $(\bar{x}_0, \bar{y}_0)$; k_0 is the spring constant; m_0 is the magnitude of the concentrated mass; ρ, h are the density and the thickness of the plate, respectively and ω is the natural circular frequency.

The rotatory inertia of the concentrated mass is not taken into account in the present analysis.

The above functional expression corresponds to orthotropic plates (Lekhnitskii, 1968). In the case of an isotropic plate there is a single Young's modulus $E_1 = E_2 = E$ and one Poisson's ratio $\nu_2 = \nu_1 = \nu$ and all the bending rigidities are reduced to one rigidity

$$D_1 = D_2 = D, \quad \text{while } D_k = \frac{D}{2}(1 - \nu)$$

The classic Rayleigh–Ritz method is used to determine the natural frequencies of the proposed mechanical systems.

As the length of the rectangular plate's sides are a and b in the $\bar{x}$ and $\bar{y}$ directions respectively, the coordinates are written in the dimensionless form:

$$x = \frac{\bar{x}}{a}; \quad y = \frac{\bar{y}}{b} \tag{2a, b}$$

$$x_0 = \frac{\bar{x}_0}{a}; \quad y_0 = \frac{\bar{y}_0}{b} \tag{3a, b}$$

and the aspect ratio of the plate:

$$\lambda = \frac{a}{b} \tag{4}$$

The expression of the deflection of the plate is approximated in the form of a series of beam functions $X_m(x)$ and $Y_n(y)$

$$W(x,y) \cong \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} X_m(x) Y_n(y) \tag{5}$$

If the edges of the plate are simply supported the beam functions are:

$$X_m(x) = \sin m\pi x \tag{6a}$$

$$Y_n(y) = \sin n\pi y \tag{6b}$$

and for the case of a fully clamped plate, the following beam functions are used, in x-direction:

$$X_m(x) = (\sin k_m x - \sinh k_m x) \frac{\cos k_m - \cosh k_m}{\sin k_m - \sinh k_m} - \cos k_m x + \cosh k_m x \tag{7a}$$

and in y-direction:

$$Y_n(y) = (\sin k_n y - \sinh k_n y) \frac{\cos k_n - \cos k_n}{\sin k_n - \sin k_n} - \cos k_n y + \cosh k_n y \tag{7b}$$

where the k_s' are the roots of the transcendental equation

$$\cos k \cosh k = 1$$

The Rayleigh–Ritz method requires the minimization of the functional (1) with respect to the A_{mn} coefficients and to the mass displacement Z:

$$\begin{aligned}
\frac{\partial J[W]}{\partial A_{ql}} = \frac{D_1}{ab} \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} \Bigg[& \lambda^{-2} \int_{A_n} \frac{d^2 X_m}{dx^2} Y_n \frac{d^2 X_q}{dx^2} Y_l \, dx\, dy \\
& + \nu_2 \int_{A_n} \left[\frac{d^2 X_m}{dx^2} X_q \frac{d^2 Y_l}{dy^2} Y_n + \frac{d^2 X_q}{dx^2} X_m Y_l \frac{d^2 Y_n}{dy^2} \right] dx\, dy + \lambda^2 \frac{D_2}{D_1} \int_{A_n} X_m \frac{d^2 Y_n}{dy^2} X_q \frac{d^2 Y_l}{dy^2} dx\, dy \\
& + \frac{4D_k}{D_1} \int_{A_n} \frac{dX_m}{dx} \frac{dY_n}{dy} \frac{dX_q}{dx} \frac{dY_l}{dy} dx\, dy - \frac{\rho h}{D_1} a^4 \omega^2 \lambda^{-2} \int_{A_n} X_m Y_n X_q Y_l \, dx\, dy \Bigg] \\
& - \rho hab\omega^2 \frac{m_0}{m_p} \left[\left(\sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} X_q(x_0) X_m(x_0) Y_l(y_0) Y_n(y_0) \right) + Z X_q(x_0) Y_l(y_0) \right] = 0
\end{aligned} \tag{8a}$$

$$\frac{\partial J[W,Z]}{\partial Z} = Zk_0 - \rho hab\omega^2 \frac{m_0}{m_p} \left(\left(\sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} X_m(x_0) Y_n(y_0) \right) + Z \right) = 0 \tag{8b}$$

The non-triviality solution of the set of Eqs. (8) yields a secular determinant in the natural frequency coefficients of the vibrating system. In matrix fashion it is:

$$|\mathbf{U}^* - \Omega^2 \mathbf{T}^*| = 0 \tag{9}$$

where

$$\Omega_i = \sqrt{\frac{\rho h}{D_1}}\omega_i a^2 \tag{10}$$

$$\mathbf{U}^* = \begin{bmatrix} U & 0 \\ 0 & \lambda^{-1}K_0 \end{bmatrix} \quad \text{with } \mathbf{U} = \lfloor u_{qlmn} \rfloor \tag{11a}$$

$$\mathbf{T}^* = \begin{bmatrix} \mathbf{T} & \lambda^{-2}\frac{m_0}{m_p}X_q(x_0)Y_l(y_0) \\ \lambda^{-2}\frac{m_0}{m_p}X_m(x_0)Y_n(y_0) & \lambda^{-2}\frac{m_0}{m_p} \end{bmatrix} \quad \text{with } \mathbf{T} = \lfloor t_{qlmn} \rfloor \tag{11b}$$

The sub-matrix **U** coefficients are:

$$\begin{aligned} u_{qlmn} = {} & \lambda^{-2}\int_{A_n}\left(\frac{\mathrm{d}^2X_q}{\mathrm{d}x^2}\frac{\mathrm{d}^2X_m}{\mathrm{d}x^2}\right)(Y_lY_n)\mathrm{d}x\,\mathrm{d}y + v_2 \\ & \times \int_{A_n}\left[\left(X_q\frac{\mathrm{d}^2X_m}{\mathrm{d}x^2}\right)\left(\frac{\mathrm{d}^2Y_l}{\mathrm{d}y^2}Y_n\right) + \left(\frac{\mathrm{d}^2X_q}{\mathrm{d}x^2}X_m\right)\left(Y_l\frac{\mathrm{d}^2Y_n}{\mathrm{d}y^2}\right)\right]\mathrm{d}x\,\mathrm{d}y + \lambda^2\frac{D_2}{D_1} \\ & \times \int_{A_n}(X_qX_m)\left(\frac{\mathrm{d}^2Y_l}{\mathrm{d}y^2}\frac{\mathrm{d}^2Y_n}{\mathrm{d}y^2}\right)\mathrm{d}x\,\mathrm{d}y + \frac{4D_k}{D_1}\int_{A_n}\left(\frac{\mathrm{d}X_q}{\mathrm{d}x}\frac{\mathrm{d}X_m}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}Y_l}{\mathrm{d}y}\frac{\mathrm{d}Y_n}{\mathrm{d}y}\right)\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{12a}$$

and the sub-matrix **T** coefficients are:

$$t_{qlmn} = \lambda^{-2}\int_{A_n}(X_qX_m)(Y_lY_n)\mathrm{d}x\,\mathrm{d}y \tag{12b}$$

These matrix coefficients correspond to the plate with no concentrated mass.

The dimensionless spring constant K_0 is assumed as:

$$K_0 = \frac{ab}{D_1}k_0 \tag{13}$$

and takes into account the relative rigidity of the spring to the rigidity of the plate D_1.

The magnitude m_0 of the concentrated mass is referred to the mass of the solid plate $m_p = \rho hab$.

3. Numerical results

The natural frequencies of the rectangular plates shown in Figs. 1–3 are analyzed. The plates are simply supported or clamped at their external edges and carry an elastically mounted concentrated mass m_0 at an arbitrary position (x_0, y_0).

Table 1 shows the values of the first natural frequency coefficients Ω_i for simply supported isotropic square plates. The mass is located at $x_0 = y_0 = 3/4$. For the isotropic material $v = 0.30$ is assumed.

It is important to notice that for some normal modes of vibration there exists an effective decoupling of the spring-mass system from the plate (Ávalos et al., 1993). It happens when a nodal line of the plate oscillation contains the position of the attached system. In that case the mass does not disturb neither the mode nor the frequency of vibration. In the case of simply supported bare, rectangular, isotropic plates ($m_0 = 0$) the frequency coefficients are given by $\Omega = \pi^2(m^2 + n^2\lambda^2)$.

For the isotropic square plate, Fig. 1a, with the spring-mass system at $x_0 = y_0 = 3/4$ the frequency coefficients $\Omega = \pi^2(m^2 + n^2)$ with "m" ≠ "n", are also roots of the frequency equation, because, in these cases, two modal shapes have the same frequency. These two modal shapes may exist simultaneously, their relative amplitudes depending upon the initial conditions.

Table 1
Comparison of results of the first four natural frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\frac{\rho h}{D}}$ in the case of a simply supported square isotropic plate with an elastically mounted mass m_0 at $x_0 = 3/4$, $y_0 = 3/4$

K_0/λ	m_0/m_p	Ω_i					
0.5	0.25	1.4128	19.752	49.368	78.969	98.696	Present study
		1.4123	19.752	49.368	78.969	98.696	Ávalos et al. (1993)
	0.50	0.9990	19.752	49.368	78.969	98.696	
		1.0000	19.752	49.368	78.969	98.696	
	1.00	0.7071	19.752	49.368	78.969	98.696	
		0.7062	19.752	49.368	78.869	98.696	
5	0.25	4.4084	19.869	49.550	79.083	98.696	
		4.4121	19.870	49.550	79.083	98.696	
	0.50	3.1180	19.866	49.549	79.083	98.696	
		3.1204	19.867	49.550	79.083	98.696	
	1.00	2.2050	19.865	49.549	79.083	98.696	
		2.2066	19.865	49.549	79.083	98.696	
∞	0.25	17.103	33.954	66.091	93.267	98.696	
		17.149	34.449	66.825	93.755	98.696	
	0.50	14.785	29.928	64.692	92.791	98.696	
		14.892	30.437	65.466	93.333	98.696	
	1.00	11.776	27.640	63.953	92.532	98.696	
		11.927	28.081	64.735	93.103	98.696	

The coefficients $\Omega_{mn} = \pi^2(m^2 + n^2\lambda^2)$ where $m \neq n$, are also roots of the frequency equation and are omitted in the table.

The combination of these “pure” modal shapes may give modal patterns with nodal lines, which coincide with the diagonals of the plate, and therefore contain the point $x_0 = y_0 = 3/4$.

Table 2
Comparison of results of the first four natural frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\frac{\rho h}{D}}$ in the case of a simply supported rectangular isotropic plate with an elastically mounted mass m_0 at $x_0 = 3/4$, $y_0 = 1/2$, $\lambda = 2$

K_0/λ	m_0/m_p	Ω_i					
0.5	0.25	1.9982	49.368	78.982	128.31	365.18	Present study
		1.9984	49.368	78.982	128.31	365.18	Ávalos et al. (1993)
	0.50	1.4131	49.368	78.982	128.31	365.18	
		1.4131	49.368	78.982	128.31	365.18	
	1.00	0.9995	49.368	78.982	128.31	365.18	
		1.0000	49.368	78.982	128.31	365.18	
5	0.25	6.2694	49.551	79.212	128.38	365.20	
		6.2729	49.551	79.212	128.38	365.20	
	0.50	4.4333	49.549	79.211	128.38	365.20	
		4.4358	49.549	79.211	128.38	365.20	
	1.00	3.1349	49.548	79.211	128.38	365.20	
		3.1367	49.548	79.211	128.38	365.20	
∞	0.25	36.741	63.902	118.12	229.60	371.38	
		36.967	64.194	118.84	284.83	375.38	
	0.50	29.395	61.726	116.48	219.88	371.21	
		29.713	62.025	117.39	272.01	375.04	
	1.00	22.243	60.599	115.49	214.45	371.14	
		22.567	60.888	116.51	264.63	374.87	

The coefficients $\Omega_{mn} = \pi^2(m^2 + n^2\lambda^2)$, where “$m$” is a multiple of 4 or “n” is a multiple of 2, are also roots of the frequency equation and are omitted in the table.

The results, shown in Table 1, have been calculated using $M \times N = 20 \times 20$ terms in Eq. (5).

Table 2 presents the natural frequency coefficients for isotropic rectangular plates.

The aspect ratio of the plate is chosen as $\lambda = a/b = 2$ and the concentrated mass is located at $x_0 = 3/4$, $y_0 = 1/2$. The results are compared with the results obtained by Ávalos et al. (1993).

Tables 3–5 show the lowest six natural frequency coefficients $\Omega_i = \sqrt{\frac{\rho h}{D_1}}\omega_i a^2$ of simply supported orthotropic square plates with different locations of the attached spring-mass system, and various mass relations $m_0/m_p = 0.1, 0.3, 0.5$ and 1. The first coefficient $\Omega_0 = \sqrt{\frac{K_0}{m_0/m_p}}$ corresponds to the spring-mass discrete system and the remaining coefficients $\Omega_i = \sqrt{\frac{\rho h}{D_1}}\omega_i a^2$, with $i = 1, 2, \ldots 6$, correspond to the vibrating plate model.

The given data for the orthotropic material are $D_2 = D_k = D_1/2$; $\nu_2 = 0{,}30$.

In the first row the natural frequencies of the bare plate ($m_0/m_p = 0$) are shown.

It is appreciated that when the relative rigidity of the spring is small ($K_0 = 0.1, 1, 10$), the first frequency coefficient Ω_1, practically coincides with the frequency of the one degree spring-mass system, with a negligible influence of the plate on it (see Fig. 4) (Rossit and Laura, 2001).

Table 3
Natural frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\frac{\rho h}{D_1}}$ of a simply supported square orthotropic plate carrying an elastically mounted mass m_0 at its center

m_0/m_p	K_0	Ω_0	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0	–	–	19.9844	43.4711	51.1889	79.5101	79.9377	101.085
0.1	0.1	1.0000	0.99945	19.9945	43.4711	51.1889	79.5126	79.9377
	1	3.1623	3.14369	20.0866	43.4711	51.1889	79.5352	79.9377
	10	10.000	9.34828	21.2070	43.4711	51.1889	79.7626	79.9377
	50	22.362	14.9875	28.6235	43.4711	51.1889	79.9377	80.7871
	100	31.623	15.9575	36.4461	43.4711	51.1889	79.9377	82.0685
	500	70.711	16.6265	43.4711	51.1889	57.0144	79.9377	88.7714
	∞	∞	16.7745	43.4711	51.1889	65.9014	79.9377	93.2890
0.3	0.1	0.5774	0.57706	19.9944	43.4711	51.1889	79.5126	79.9377
	1	1.8257	1.81519	20.0849	43.4711	51.1889	79.5352	79.9377
	10	5.7735	5.44529	21.0208	43.4711	51.1889	79.7600	79.9377
	50	12.9099	9.78889	25.3325	43.4711	51.1889	79.9377	80.7245
	100	18.2574	11.2604	29.9570	43.4711	51.1889	79.9377	81.8386
	500	40.825	12.7489	43.4711	45.4202	51.1889	79.9377	87.0718
	∞	∞	13.1483	43.4711	51.1889	56.6778	79.9377	91.5425
0.5	0.1	0.4472	0.44699	19.9944	43.4711	51.1889	79.5126	79.9377
	1	1.4142	1.40606	20.0846	43.4711	51.1889	79.5352	79.9377
	10	4.4721	4.22424	20.9896	43.4711	51.1889	79.7595	79.9377
	50	10.000	7.73974	24.8233	43.4711	51.1889	79.9377	80.7126
	100	14.142	9.08490	28.7855	43.4711	51.1889	79.9377	81.7971
	500	31.623	10.6575	42.5048	43.4711	51.1889	79.9377	86.7673
	∞	∞	11.1330	43.4711	51.1889	53.6720	79.9377	91.1510
1	0.1	0.3162	0.31607	19.9944	43.4711	51.1889	79.5126	79.9377
	1	1.0000	0.99423	20.0843	43.4711	51.1889	79.5352	79.9377
	10	3.1623	2.99020	20.9672	43.4711	51.1889	79.7591	79.9377
	50	7.0711	5.55105	24.4777	43.4711	51.1889	79.9377	80.7039
	100	10.000	6.61544	27.9697	43.4711	51.1889	79.9377	81.7669
	500	22.361	8.02094	40.2202	43.4711	51.1889	79.9377	86.5478
	∞	∞	8.50187	43.4711	51.0384	51.1889	79.9377	90.8507

Table 4
Natural frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\frac{\rho h}{D_1}}$ of a simply supported square orthotropic plate carrying an elastically mounted mass m_0 at $x_0 = 3/4$, $y_0 = 1/2$

m_0/m_p	K_0	Ω_0	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0	–	–	19.9844	43.4711	51.1889	79.5101	79.9377	101.085
0.1	0.1	1.0000	0.99965	19.9894	43.4711	51.1928	79.5113	79.9377
	1	3.1623	3.14979	20.0355	43.4711	51.2281	79.5227	79.9377
	10	10.000	9.56247	20.6036	43.4711	51.5901	79.6376	79.9377
	50	22.362	16.2611	25.4927	43.4711	53.3907	79.9377	80.1829
	100	31.623	17.3370	31.1552	43.4711	55.9506	79.9377	80.9413
	500	70.711	17.9548	40.7088	43.4711	68.5418	79.9377	87.4183
	∞	∞	18.0771	43.1087	43.4711	73.7628	79.9377	94.3297
0.3	0.1	0.5774	0.57715	19.9894	43.4711	51.1928	79.5113	79.9377
	1	1.8257	1.81863	20.0347	43.4711	51.2280	79.5226	79.9377
	10	5.7735	5.54917	20.5033	43.4711	51.5800	79.6362	79.9377
	50	12.9099	10.5703	22.7689	43.4711	53.1368	79.9377	80.1829
	100	18.2574	12.5454	25.3712	43.4711	55.0030	79.9377	80.9413
	500	40.825	14.5845	32.9244	43.4711	63.8345	79.9377	87.4183
	∞	∞	15.0948	36.0948	43.4711	70.5355	79.9377	94.3297
0.5	0.1	0.4472	0.44710	19.9894	43.4711	51.1928	79.5113	79.9377
	1	1.4142	1.40872	20.0345	43.4711	51.2280	79.5226	79.9377
	10	4.4721	4.30202	20.4868	43.4711	51.5781	79.6360	79.9377
	50	10.000	8.31794	22.4348	43.4711	53.0923	79.9377	80.1410
	100	14.142	10.1016	24.4900	43.4711	54.8500	79.9377	80.7687
	500	31.623	12.3840	30.7700	43.4711	62.9139	79.9377	84.8127
	∞	∞	13.0816	34.5129	43.4711	69.6126	79.9377	91.0271
1	0.1	0.3162	0.31623	19.9894	43.4711	51.1928	79.5113	79.9377
	1	1.0000	0.99614	20.0344	43.4711	51.2280	79.5226	79.9377
	10	3.1623	3.04383	20.4751	43.4711	51.5766	79.6358	79.9377
	50	7.0711	5.94236	22.2217	43.4711	53.0602	79.9377	80.1361
	100	10.000	7.32758	23.9308	43.4711	54.7418	79.9377	80.7499
	500	22.361	9.40840	29.1426	43.4711	62.2516	79.9377	84.5655
	∞	∞	10.1853	32.5952	43.4711	43.4711	79.9377	90.5299

The remaining five values show the frequency coefficients of the continuous system modified by the presence of the spring-mass system (obviously this happens if the spring-mass system is not located at a nodal line).

For higher values of K_0, the frequency coefficient of the discrete system is strongly modified by the influence of the plate.

In Fig. 4 the influence of the relative rigidity of the spring K_0 on the frequency coefficients Ω_0 and Ω_1 is shown for two mass relations $m_0/m_p = 0.10$ and 0.50. As it is mentioned above, it can be appreciated that both frequencies are practically the same for small values of K_0. In case (a) the difference between Ω_0 and Ω_1 cannot be appreciated for $K_0 \leqslant 10$, gradually increases from $K_0 = 10$ to 100 and increases rapidly when K_0 exceeds 100.

The effect of the spring's rigidity on higher frequencies of the vibrating system is shown in Fig. 5. In Fig. 6 the influence of the magnitude of the mass on the frequency coefficients is shown for different values of K_0 ($x_0 = y_0 = 1/2$).

Table 6 presents the natural frequency coefficients of a simply supported orthotropic plate of aspect ratio $\lambda = 3/2$, with the spring-mass system located at two different positions ($x_0 = y_0 = 1/2$ and $x_0 = y_0 = 4/5$). Some of these cases ($x_0 = y_0 = 4/5$; $m_0/m_p = 0.10$; $K_0 = 10$; $K_0 \to \infty$) have been obtained by the present

Table 5
Natural frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\frac{\rho h}{D_1}}$ of a simply supported square orthotropic plate carrying an elastically mounted mass m_0 at $x_0 = 3/4,\ y_0 = 3/4$

m_0/m_p	K_0	Ω_0	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0	–	–	19.9844	43.4711	51.1889	79.5101	79.9377	101.085
0.1	0.1	1.0000	0.99975	19.9869	43.4734	51.1909	79.5107	79.9402
	1	3.1623	3.15289	20.0100	43.4941	51.2085	79.5160	79.9631
	10	10.000	9.67813	20.2962	43.7031	51.3934	79.5487	80.2164
	50	22.362	17.2567	23.6480	44.6333	52.4059	79.5487	81.5323
	100	31.623	18.3459	28.5445	45.6393	54.0705	79.5884	83.3503
	500	70.711	18.8333	36.1346	47.7737	63.5411	79.5954	94.3552
	∞	∞	18.9187	37.8830	48.3689	68.6773	79.5973	98.6023
0.3	0.1	0.5774	0.57723	19.9869	43.4734	51.1909	79.5107	79.9402
	1	1.8257	1.82038	20.0095	43.4940	51.2085	79.5160	79.9631
	10	5.7735	5.60395	20.2440	43.6952	51.3882	79.5484	80.2134
	50	12.9099	11.0826	21.4293	44.4693	52.2496	79.5802	81.4493
	100	18.2574	13.5280	22.9796	45.1775	53.3983	79.5876	83.0210
	500	40.825	16.1119	28.3091	46.8083	59.4087	79.5944	91.8893
	∞	∞	16.6843	31.2410	47.4658	64.5803	79.5963	97.6492
0.5	0.1	0.4472	0.44710	19.9869	43.4734	51.1909	79.5107	79.9402
	1	1.4142	1.41007	20.0095	43.4940	51.2085	79.5160	79.9631
	10	4.4721	4.34290	20.2355	43.6937	51.3871	79.5484	80.2128
	50	10.000	8.67811	21.2276	44.4415	52.2228	79.5801	81.4336
	100	14.142	10.8270	22.3406	45.1040	53.2949	79.5874	82.9618
	500	31.623	13.8524	26.2137	46.6216	58.6783	79.5942	91.3992
	∞	∞	14.7768	28.8295	47.2650	63.5974	79.5961	97.3864
1	0.1	0.3162	0.31622	19.9869	43.4734	51.1909	79.5107	79.9402
	1	1.0000	0.99709	20.0094	43.4940	51.2085	79.5160	79.9631
	10	3.1623	3.07195	20.2295	43.6926	51.3864	79.5483	80.2123
	50	7.0711	6.17795	21.1058	44.4216	52.2035	79.5800	81.4221
	100	10.000	7.80615	21.9804	45.0523	53.2227	79.5873	82.9187
	500	22.361	10.5514	24.8171	46.4865	58.1674	79.5941	91.0389
	∞	∞	11.6730	26.9028	47.1138	62.8425	79.5959	97.1688

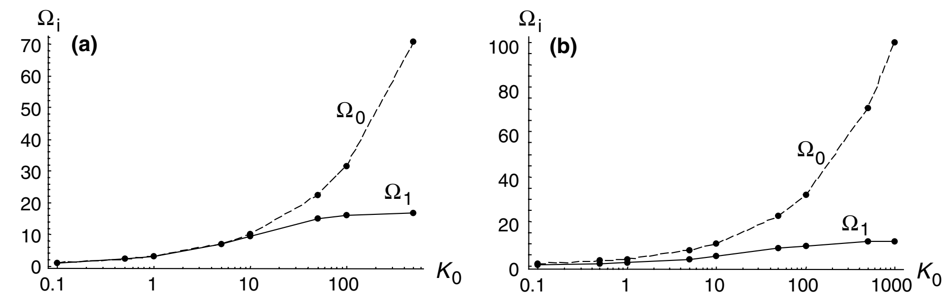

Fig. 4. The effect of the rigidity of the spring on the frequency coefficients Ω_0 and Ω_1: (a) $m_0/m_p = 0.10$ and (b) $m_0/m_p = 0.50$.

approach and by the finite element method proposed by Rossi (1997). The results' agreement is excellent.

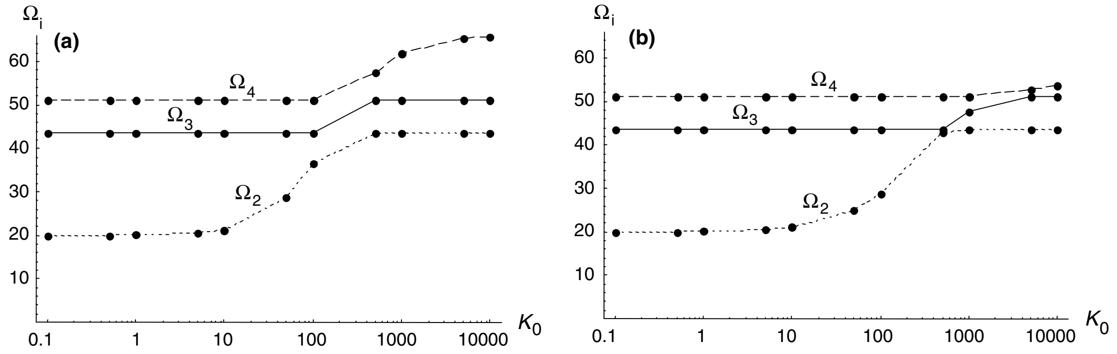

Fig. 5. The effect of the rigidity of the spring on the frequency coefficients Ω_2, Ω_3 and Ω_4: (a) $m_0/m_p = 0.10$ and (b) $m_0/m_p = 0.50$.

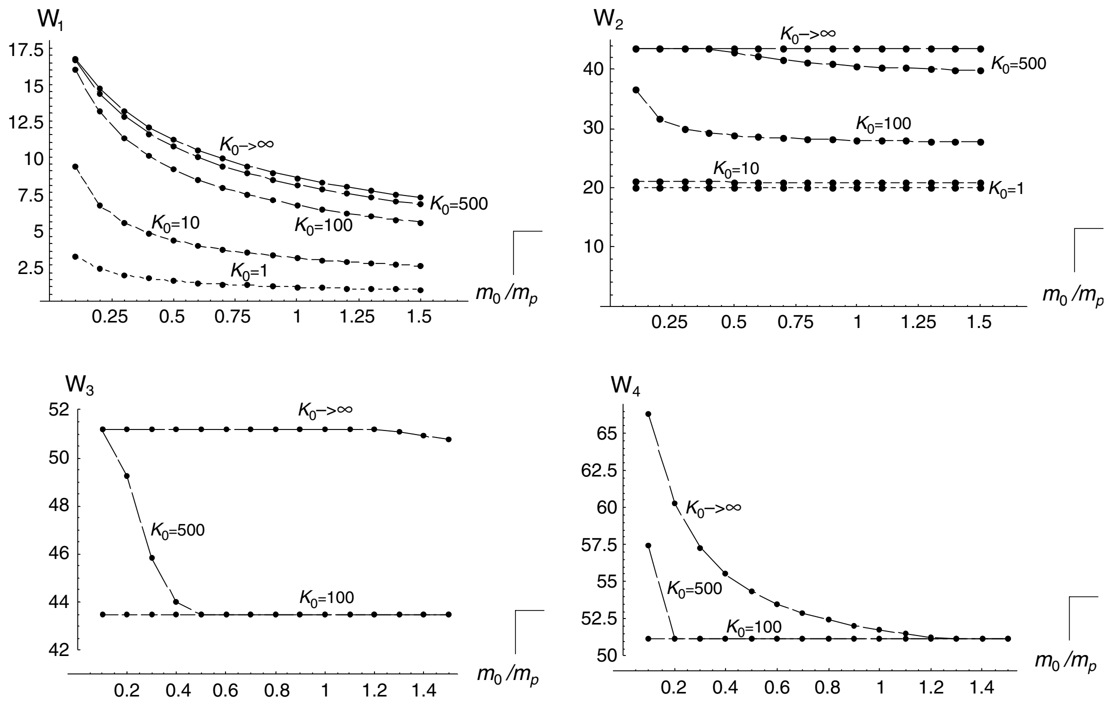

Fig. 6. The effect of the magnitude of the concentrated mass on the first four frequency coefficients.

Similar tables are presented for the clamped plate. Tables 7 and 8 contain the first four natural frequency coefficients of clamped orthotropic plates with the spring-mass system located at $x_0 = y_0 = 1/2$ and $x_0 = y_0 = 4/5$.

Comparing Tables 3 and 7, one can see the effect of the higher rigidity of the clamped plate in the coefficient Ω_1, for two values of K_0 (1, 10).

In Tables 9 and 10; the results of a simply supported orthotropic plate with a rectangular hole carrying an elastically mounted mass, see Fig. 2, are presented.

Table 6
First four natural frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\frac{\rho h}{D_1}}$ of a simply supported rectangular orthotropic plate carrying an elastically mounted mass m_0; $\lambda = 3/2$ (see Fig. 5)

x_0	y_0	m_0/m_p	K_0	Ω_1	Ω_2	Ω_3	Ω_4
0.5	0.5	0.1	1	4.716	30.381	63.910	79.510
			10	14.037	32.041	63.910	79.510
			∞	25.367	63.910	79.510	96.572
		0.3	1	2.723	30.379	63.910	79.510
			10	8.174	31.769	63.910	79.510
			∞	19.871	63.910	79.510	83.668
0.8	0.8	0.1	1	4.734	30.247	63.932	79.528
			10	14.673	30.453	64.135	79.691
				14.672[a]	30.453[a]	64.138[a]	79.695[a]
			∞	29.438	58.229	74.697	99.278
				29.438[a]	58.212[a]	74.684[a]	99.202[a]
		0.3	1	2.733	30.247	63.932	79.528
			10	8.484	30.415	64.127	79.687
			∞	27.479	47.346	71.301	91.730

[a] These results were obtained using the finite element method (Rossi, 1997).

Table 7
First four natural frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\frac{\rho h}{D_1}}$ of a clamped square orthotropic plate carrying an elastically mounted mass m_0, located at x_0, y_0

x_0	y_0	m_0/m_p	K_0	Ω_1	Ω_2	Ω_3	Ω_4
0.5	0.5	0.1	1	3.1522	33.632	61.558	73.592
			10	9.6741	34.474	61.558	73.592
			∞	26.199	61.558	73.592	65.881
		0.3	1	1.8206	33.631	61.558	73.592
			10	5.5940	34.422	61.558	73.592
			∞	19.284	61.558	73.592	76.523
0.8	0.8	0.1	1	3.1603	33.547	61.564	73.597
			10	9.9124	33.572	61.618	73.645
			∞	33.180	56.952	69.106	84.212
		0.3	1	1.8242	33.546	61.564	73.597
			10	5.723	33.571	61.617	73.645
				5.723[a]	33.574[a]	61.626[a]	73.652[a]
			∞	31.789	43.645	65.477	78.629

[a] Finite element method results.

The comparison with the results obtained by Ávalos et al. (1999) is shown, when it is available, in Table 9.

Table 10 presents results of the first four natural frequencies of the plate of Fig. 2a.

The mass position is $x_0 = y_0 = 4/5$. In the case of $m_0/m_p = 0.10$ and $K_0 = 10$, the results have been obtained using both methods, the proposed approach and the finite element method. As it is shown the agreement is quite good.

The last vibrating model that is analyzed in the present study corresponds to the vibrating plate shown in Fig. 3. The plate has a rectangular patch made of another material and the spring-mass system is located at the center of the patch.

Table 8
First four natural frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\frac{\rho h}{D_1}}$ of a clamped rectangular orthotropic plate carrying an elastically mounted mass m_0, located at x_0, y_0; $\lambda = 3/2$

x_0	y_0	m_0/m_p	K_0	Ω_1	Ω_2	Ω_3	Ω_4
0.5	0.5	0.1	1	4.7290	51.208	90.548	114.81
			10	14.523	52.447	90.548	114.81
			∞	39.874	90.548	114.81	125.78
		0.3	1	2.7301	51.207	90.548	114.81
			10	8.3972	52.372	90.548	114.81
			∞	29.318	90.548	112.91	114.81
0.8	0.8	0.1	1	4.7392	51.082	90.556	114.82
			10	14.870	51.121	90.637	114.88
				14.868[a]	51.124[a]	90.645[a]	114.89[a]
			∞	50.517	84.387	105.94	127.62
		0.3	1	2.7364	51.082	90.556	114.82
			10	8.5861	51.118	90.636	114.88
			∞	48.329	65.691	97.584	121.21

[a] Finite element method results.

Table 9
First two natural frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\frac{\rho h}{D_1}}$ of a simply supported square orthotropic plate with a square hole carrying an elastically mounted mass m_0. Comparison results (see Fig. 2a)

Hole		Sprig-mass system			Ω_1		Ω_2	
$x_1 = y_1$	$x_2 = y_2$	$x_0 = y_0$	m_0/m_p	K_0/λ	Present study	Ávalos et al. (1999)	Present study	Ávalos et al. (1999)
0.2	0.3	0.75	0.1	1	3.1528	3.152	19.938	19.93
				10	9.6766	9.676	20.225	20.22
				100	18.292	–	28.508	–
				∞	18.855	18.83	37.843	37.44
			0.3	1	1.8203	1.820	19.938	19.93
				10	5.6033	5.602	20.172	20.17
				100	13.510	–	22.913	–
				∞	16.637	16.50	31.176	30.37
0.1	0.4	0.75	0.1	1	3.1527	3.152	19.395	19.39
				10	9.6681	9.668	19.690	19.68
				100	17.870	–	28.432	–
				∞	18.365	18.34	38.220	37.79
			0.3	1	1.8202	1.820	19.394	19.39
				10	5.6000	5.600	19.633	19.63
				100	13.387	–	22.501	–
				∞	16.276	16.15	31.186	30.34

The patch may be a strengthened zone of the plate or a repaired portion, and its elastic properties are supposed to have orthotropic characteristics (Ercoli et al., 1992; Felix et al., 2003).

The natural compatibility conditions at the joints are not satisfied, but this is legitimate when using the Rayleigh methodology, since it requires the satisfaction only of the essential boundary conditions.

The essential boundary conditions at the joints are satisfied by the beam functions adopted in this approach (Tables 11 and 12).

In Tables 13–20 the effects of replacing a rectangular portion of an isotropic plate with a patch made of an orthotropic material are studied.

Table 10
First four natural frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\frac{\rho h}{D_1}}$ of a simply supported rectangular orthotropic plate with a rectangular hole and a spring-mass system; $\lambda = 3/2$

Hole		Spring-mass system			Ω_1	Ω_2	Ω_3	Ω_4
$x_1 = y_1$	$x_2 = y_2$	$x_0 = y_0$	m_0/m_p	K_0				
0.1	0.4	0.8	0.1	1	4.7336	29.294	63.770	80.380
				10	14.667	29.505	63.931	80.666
					14.665[a]	29.425[a]	63.783[a]	80.509[a]
				∞	28.540	58.568	72.834	106.54
			0.3	1	2.733	29.294	63.770	80.380
				10	8.482	29.463	63.924	80.658
				∞	26.745	47.274	69.228	99.467

[a] Finite element method results.

Table 11
First four natural frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\frac{\rho h}{D_1}}$ of a clamped square orthotropic plate with a square hole and a spring-mass system

Hole		Spring-mass system			Ω_1	Ω_2	Ω_3	Ω_4
$x_1 = y_1$	$x_2 = y_2$	$x_0 = y_0$	m_0/m_p	K_0				
0.1	0.4	0.8	0.1	1	3.1602	32.414	61.320	74.412
				10	9.9123	32.438	61.367	74.496
				∞	–	32.097	56.909	67.949
			0.3	∞	1.8245	32.414	61.320	74.412
				1	5.723	32.437	61.366	74.493
				10	5.723[a]	32.229[a]	61.122[a]	74.121[a]
				∞	–	30.943	43.240	64.508

[a] Finite element method results.

Table 12
First four natural frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\frac{\rho h}{D_1}}$ of a clamped rectangular orthotropic plate with a rectangular hole and a spring-mass system; $\lambda = 3/2$ (Fig. 2b)

Hole		Spring-mass system			Ω_1	Ω_2	Ω_3	Ω_4
$x_1 = y_1$	$x_2 = y_2$	$x_0 = y_0$	m_0/m_p	K_0				
0.1	0.4	0.5	0.1	1	4.7392	49.328	90.011	115.99
				10	14.870	49.364	90.087	116.11
					14.868[a]	49.032[a]	89.705[a]	115.54[a]
				∞	–	48.843	84.060	104.22
			0.3	1	2.7361	49.328	90.011	115.99
				10	8.5863	49.361	90.085	116.11
				∞	–	47.046	64.994	96.258

[a] Finite element method results.

The given data for the patch are assumed take:

(A) $h_0 = h$; $\rho_0 = 1.5\rho$; $v_2 = 0.30$; $D_1 = D$; $D_2 = 2.8D$; $D_k = 0.30D$.

Table 13
First four natural frequency coefficients $\Omega_i = \omega_i a^2\sqrt{\frac{\rho h}{D}}$ of a simply supported isotropic plate with an orthotropic "patch" carrying an elastically mounted mass $m_0 = \frac{3}{10}m_p$ at its center; $\lambda = 3/2$

K_0	Ω_1	Ω_2	Ω_3	Ω_4	
1	2.7259	32.191	61.708	98.701	(a)
	2.7307	33.681	61.473	104.630	(b)
	2.7283	30.267	62.892	94.109	(c)
10	8.2645	33.253	61.919	98.838	(a)
	8.4043	34.395	61.612	104.680	(b)
	8.3210	31.192	63.028	94.203	(c)
20	11.1520	34.469	62.166	98.979	(a)
	11.5210	35.237	61.775	104.730	(b)
	11.2900	32.296	63.188	94.309	(c)
50	15.4300	38.071	62.983	99.398	(a)
	16.5530	37.947	62.320	104.870	(b)
	15.7500	35.836	63.720	94.641	(c)

$x_1 = y_1 = 0.2$; $x_2 = y_2 = 0.6$; $x_0 = y_0 = 0.4$

(a) Solid isotropic plate; (b) plate with an orthotropic patch made of material (A); (c) plate with an orthotropic patch made of material (B).

Table 14
First four natural frequency coefficients $\Omega_i = \omega_i a^2\sqrt{\frac{\rho h}{D}}$ of a simply supported rectangular isotropic plate with an orthotropic "patch" carrying an elastically mounted mass $m_0 = \frac{1}{2}m_p$ at its center; $\lambda = 3/2$

K_0	Ω_1	Ω_2	Ω_3	Ω_4	
1	2.1116	32.191	61.708	98.710	(a)
	2.1151	33.680	61.473	104.630	(b)
	2.1132	30.267	62.891	94.109	(c)
10	6.4080	33.222	61.917	98.837	(a)
	6.5139	34.375	61.610	104.68	(b)
	6.4524	31.160	63.027	94.203	(c)
20	8.6701	34.351	62.158	98.978	(a)
	8.9460	35.157	61.769	104.73	(b)
	8.7796	32.172	63.183	94.308	(c)
50	12.143	37.513	62.930	99.387	(a)
	12.983	37.502	62.284	104.86	(b)
	12.431	35.192	63.686	94.632	(c)

$x_1 = y_1 = 0.2$; $x_2 = y_2 = 0.6$; $x_0 = y_0 = 0.4$

(a) Solid isotropic plate; (b) plate with an orthotropic patch made of material (A); (c) plate with an orthotropic patch made of material (B).

(B) $h_0 = h$; $\rho_0 = 1.5\rho$; $\nu_2 = 0.84$; $D_1 = 2.8D$; $D_2 = D$; $D_k = 0.80D$.

Tables 13–16 correspond to the simply supported plate, see Fig. 3a and Tables 17–20 to clamped plates, Fig. 3b.

Table 15
First four natural frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\frac{\rho h}{D}}$ of a simply supported rectangular isotropic plate with an orthotropic "patch" carrying an elastically mounted mass $m_0 = \frac{4}{5} m_p$ at its center; $\lambda = 3/2$

K_0	Ω_1	Ω_2	Ω_3	Ω_4	
1	1.6694	32.190	61.707	98.710	(a)
	1.6724	33.680	61.473	104.63	(b)
	1.6706	30.266	62.891	94.109	(c)
10	5.0687	33.204	61.916	98.837	(a)
	5.1515	34.364	61.610	104.68	(b)
	5.1040	31.143	63.026	94.202	(c)
20	6.8673	34.288	62.154	98.977	(a)
	7.0813	35.115	61.767	104.73	(b)
	6.9552	32.107	63.180	94.307	(c)
50	9.6789	37.229	62.902	99.382	(a)
	10.329	37.281	62.264	104.86	(b)
	9.9220	34.872	63.668	94.627	(c)

$x_1 = y_1 = 0.2$; $x_2 = y_2 = 0.6$; $x_0 = y_0 = 0.4$

(a) Solid isotropic plate; (b) plate with an orthotropic patch made of material (A); (c) plate with an orthotropic patch made of material (B).

Table 16
First four natural frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\frac{\rho h}{D}}$ of a simply supported rectangular isotropic plate with an orthotropic "patch" carrying an elastically mounted mass $m_0 = m_p$, at its center; $\lambda = 3/2$

K_0	Ω_1	Ω_2	Ω_3	Ω_4	
1	1.4933	32.190	61.707	98.710	(a)
	1.4956	33.680	61.473	104.63	(b)
	1.4943	30.266	62.891	94.109	(c)
10	4.5344	33.199	61.916	98.837	(a)
	4.6081	34.360	61.610	104.68	(b)
	4.5660	31.137	63.026	94.202	(c)
20	6.1460	34.268	62.152	98.976	(a)
	6.3362	35.101	61.766	104.73	(b)
	6.2251	32.087	63.179	94.307	(c)
50	8.6797	37.138	62.893	99.380	(a)
	9.2572	37.212	62.258	104.86	(b)
	8.9012	34.772	63.662	94.625	(c)

$x_1 = y_1 = 0.2$; $x_2 = y_2 = 0.6$; $x_0 = y_0 = 0.4$

(a) Solid isotropic plate; (b) plate with an orthotropic patch made of material (A); (c) plate with an orthotropic patch made of material (B).

The first frequency coefficients correspond to the spring-mass system modified by the presence of the plate. The values of the subsequent frequencies, show the influence of the orthotropic patch and the spring-mass system upon the original structural element.

All the calculations have been performed with $M = N = 20$. A brief study of convergence is added in Table 21.

Table 17
First four natural frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\frac{\rho h}{D}}$ of a clamped rectangular isotropic plate with an orthotropic "patch" carrying an elastically mounted mass $m_0 = \frac{3}{10} m_p$ at its center; $\lambda = 3/2$

K_0	Ω_1	Ω_2	Ω_3	Ω_4	
1	2.7331	60.841	93.859	148.79	(a)
	2.7354	61.524	93.190	144.92	(b)
	2.7340	55.830	95.136	139.55	(c)
10	8.4838	61.560	94.090	148.92	(a)
	8.5456	62.017	93.330	144.96	(b)
	8.5152	56.468	95.271	139.65	(c)
20	11.756	62.346	94.354	149.04	(a)
	11.925	62.566	93.491	145.01	(b)
	11.840	57.183	95.424	139.77	(c)
50	17.527	64.603	95.184	149.23	(a)
	18.117	64.216	94.002	145.15	(b)
	17.798	59.350	95.909	140.12	(c)

$x_1 = y_1 = 0.2$; $x_2 = y_2 = 0.6$; $x_0 = y_0 = 0.4$

(a) Solid isotropic plate; (b) plate with an orthotropic patch made of material (A); (c) plate with an orthotropic patch made of material (B).

Table 18
First four natural frequency coefficients $\Omega_i = \omega_i a^2 \sqrt{\frac{\rho h}{D}}$ of a clamped rectangular isotropic plate with an orthotropic "patch" carrying an elastically mounted mass $m_0 = \frac{1}{2} m_p$ at its center; $\lambda = 3/2$

K_0	Ω_1	Ω_2	Ω_3	Ω_4	
1	2.1172	60.841	93.859	148.79	(a)
	2.1189	61.524	93.190	144.92	(b)
	2.1177	55.830	95.136	139.55	(c)
10	6.5723	61.554	94.090	148.92	(a)
	6.6200	62.013	93.330	144.96	(b)
	6.5967	56.461	95.270	139.65	(c)
20	9.1107	62.322	94.350	149.04	(a)
	9.2400	62.550	93.489	145.01	(b)
	9.1757	57.157	95.422	139.77	(c)
50	13.608	64.477	95.161	149.23	(a)
	14.057	64.120	93.987	145.15	(b)
	13.824	59.201	95.895	140.11	(c)

$x_1 = y_1 = 0.2$; $x_2 = y_2 = 0.6$; $x_0 = y_0 = 0.4$

(a) Solid isotropic plate; (b) plate with an orthotropic patch made of material (A); (c) plate with an orthotropic patch made of material (B).

4. Conclusion

As a general conclusion one may say that the Rayleigh–Ritz method, using beam functions provides an accurate and convenient procedure to attack problems of thin rectangular plates with a great variety of

Table 19
First four natural frequency coefficients $\Omega_i = \omega_i a^2\sqrt{\frac{\rho h}{D}}$ of a clamped rectangular isotropic plate with an orthotropic "patch" carrying an elastically mounted mass $m_0 = \frac{4}{5}m_p$ at its center; $\lambda = 3/2$

K_0	Ω_1	Ω_2	Ω_3	Ω_4	
1	1.6740	60.841	93.859	148.79	(a)
	1.6755	61.524	93.190	144.92	(b)
	1.6748	55.830	95.136	139.55	(c)
10	5.1961	61.550	94.089	148.92	(a)
	5.2337	62.010	93.330	144.96	(b)
	5.2156	56.457	95.270	139.65	(c)
20	7.2045	62.309	94.348	149.04	(a)
	7.3060	62.541	93.488	145.01	(b)
	7.2560	57.143	95.421	139.77	(c)
50	10.771	64.409	95.148	149.23	(a)
	11.123	64.068	94.979	145.15	(b)
	10.944	59.122	95.888	140.11	(c)

$x_1 = y_1 = 0.2$; $x_2 = y_2 = 0.6$; $x_0 = y_0 = 0.4$

(a) Solid isotropic plate; (b) plate with an orthotropic patch made of material (A); (c) plate with an orthotropic patch made of material (B).

Table 20
First four natural frequency coefficients $\Omega_i = \omega_i a^2\sqrt{\frac{\rho h}{D}}$ of a clamped rectangular isotropic plate with an orthotropic "patch" carrying an elastically mounted mass $m_0 = m_p$, at its center; $\lambda = 3/2$

K_0	Ω_1	Ω_2	Ω_3	Ω_4	
1	1.4975	60.841	93.859	148.79	(a)
	1.4983	61.524	93.190	144.92	(b)
	1.4975	55.830	95.136	139.55	(c)
10	4.6478	61.549	94.089	148.92	(a)
	4.6813	62.010	93.330	144.96	(b)
	4.6650	56.456	95.270	139.65	(c)
20	6.4443	62.305	94.348	149.04	(a)
	6.5351	62.538	93.487	145.01	(b)
	6.4905	57.138	95.420	139.77	(c)
50	9.6380	64.386	95.144	149.23	(a)
	9.9523	64.051	93.976	145.15	(b)
	9.7936	59.097	95.885	140.11	(c)

$x_1 = y_1 = 0.2$; $x_2 = y_2 = 0.6$; $x_0 = y_0 = 0.4$

(a) Solid isotropic plate; (b) plate with an orthotropic patch made of material (A); (c) plate with an orthotropic patch made of material (B).

structural and mechanical complexities like for the present situation where doubly connected domain and non-uniform material characteristics are present.

Table 21
Study of convergence for the frequency coefficients of a simply supported square orthotropic plate with a square hole ($x_1 = y_1 = 0.1$, $x_2 = y_2 = 0.4$) and carrying an elastically mounted mass $m_0 = 0.3$, at $x_0 = y_0 = 0.75$; $D_2 = D_k = D_1/2$

K_0	$M = N$	Ω_1	Ω_2	Ω_3	Ω_4
1	3	1.8207	19.9551	44.3012	53.0510
	5	1.8207	19.7622	44.1316	52.7812
	10	1.8204	19.4841	43.5439	52.0999
	15	1.8202	19.4234	43.3895	51.9542
	20	1.8202	19.3940	43.3298	51.8971
100	3	13.5807	23.2751	45.2689	56.9480
	5	13.5146	23.0341	45.1320	56.5909
	10	13.4125	22.6170	44.6187	55.7144
	15	13.3944	22.5370	44.4641	55.5511
	20	13.3874	22.5013	44.4064	55.4881
∞	3	16.6444	32.8855	46.8197	75.5613
	5	16.5118	32.4073	46.6810	74.6866
	10	16.3297	31.4098	46.2019	72.4264
	15	16.2923	31.2473	46.0474	72.0217
	20	16.2756	31.1858	45.9874	71.8855

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