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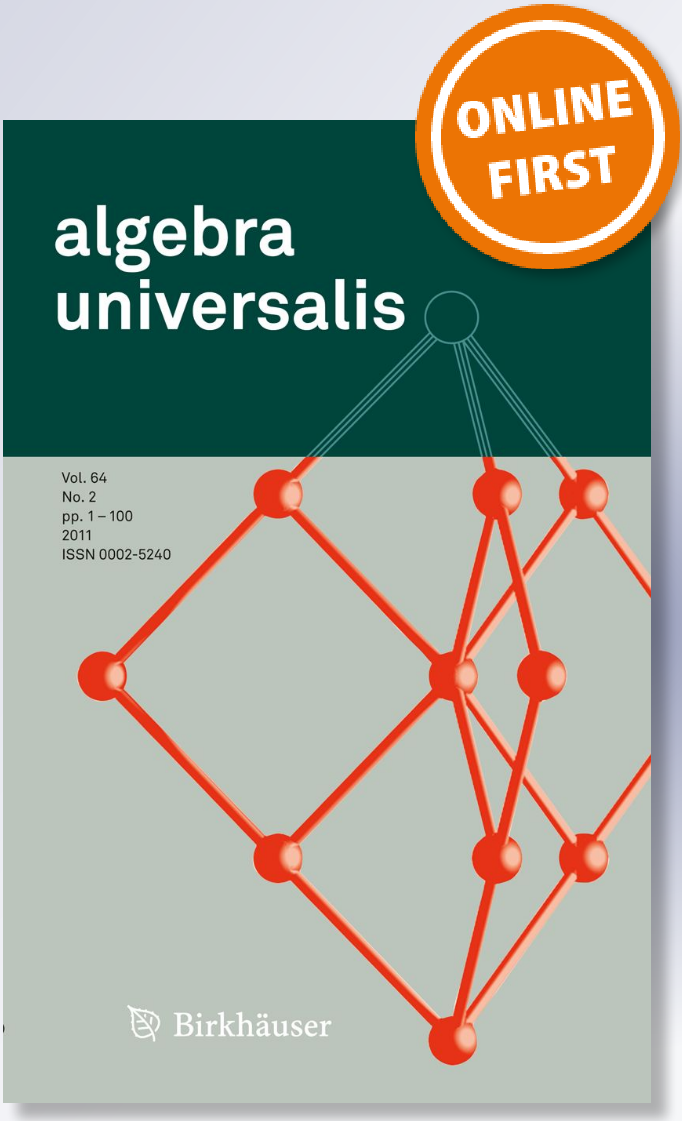
Algebra universalis

ISSN 0002-5240

Algebra Univers.
DOI 10.1007/s00012-017-0434-3



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Mariana Badano and Diego J. Vaggione

Abstract. We study four types of equational definability of factor congruences in varieties with $\vec{0}$ and $\vec{1}$. The paper completes the work of a previous paper on left equational definability of factor congruences.

1. Introduction

A *variety with* $\vec{0}$ *and* $\vec{1}$ is a variety $\mathcal{V}$ for which there are 0-ary terms $0_1, \ldots, 0_N$, $1_1, \ldots, 1_N$ such that

$$\mathcal{V} \models \vec{0} = \vec{1} \to x = y,$$

where $\vec{0} = (0_1, \ldots, 0_N)$ and $\vec{1} = (1_1, \ldots, 1_N)$. (If $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{b} = (b_1, \ldots, b_n)$ we write $\vec{a} = \vec{b}$ to express $\bigwedge_{i=1}^{n} a_i = b_i$.) This condition is equivalent to the fact that there is a nullary operation in the language of $\mathcal{V}$ and no non-trivial algebra in $\mathcal{V}$ has a trivial subalgebra. Classical examples of this type of varieties are the variety $\mathcal{S}_{01}^{\vee}$ of bounded join semilattices and the variety $\mathcal{R}$ of rings with identity (in both cases $N = 1$). If $\vec{a} \in A^N$ and $\vec{b} \in B^N$, then we use $[\vec{a}, \vec{b}]$ to denote the N-tuple $((a_1, b_1), \ldots, (a_N, b_N)) \in (A \times B)^N$. If $\mathbf{A} \in \mathcal{V}$, then we say that $\vec{e} \in A^N$ is a *central element* of $\mathbf{A}$ if there exists an isomorphism $\mathbf{A} \to \mathbf{A}_1 \times \mathbf{A}_2$ such that $\vec{e} \to [\vec{0}, \vec{1}]$. Also, we say that $\vec{e}$ and $\vec{f}$ are a *pair of complementary central elements* of $\mathbf{A}$ if there exists an isomorphism $\mathbf{A} \to \mathbf{A}_1 \times \mathbf{A}_2$ such that $\vec{e} \to [\vec{0}, \vec{1}]$ and $\vec{f} \to [\vec{1}, \vec{0}]$. As is well known, the direct product representations $\mathbf{A} \to \mathbf{A}_1 \times \mathbf{A}_2$ of an algebra $\mathbf{A}$ are closely related to the concept of factor congruence. A pair of congruences (θ, δ) of an algebra $\mathbf{A}$ is a *pair of complementary factor congruences* of $\mathbf{A}$ if $\theta \cap \delta = \Delta$ and $\theta \circ \delta = \nabla$ and in such a case θ and δ are called *factor congruences*.

Consider the following property.

(L) There is a first order formula $\lambda(\vec{z}, x, y)$ such that for every $\mathbf{A}, \mathbf{B} \in \mathcal{V}$,

$$\mathbf{A} \times \mathbf{B} \models \lambda([\vec{0}, \vec{1}], (a, b), (a', b')) \text{ iff } a = a'.$$

Presented by J. Raftery.
Received October 15, 2015; accepted in final form June 1, 2016.
2010 *Mathematics Subject Classification*: Primary: 03C05; Secondary: 08B05, 08B10.
Key words and phrases: central element, equationally definable factor congruences, Boolean factor congruences.

Birkhäuser

If $\mathcal{V} = \mathcal{S}_{01}^{\vee}$ observe that for every $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{S}_{01}^{\vee}$ and $a, a' \in A_1, b, b' \in A_2$ we have

$$(a, b) \vee (0, 1) = (a', b') \vee (0, 1) \text{ iff } (a, b \vee 1) = (a', b' \vee 1)$$
$$\text{iff } (a, 1) = (a', 1) \text{ iff } a = a'.$$

Then for the variety $\mathcal{S}_{01}^{\vee}$, we can take $\lambda := x \vee z_1 = y \vee z_1$ in order to satisfy (L). Similarly, we can see that if $\mathcal{V} = \mathcal{R}$, we can take $\lambda := x \cdot (1 - z_1) = y \cdot (1 - z_1)$. Assume that (L) holds and θ is any factor congruence of an algebra $\mathbf{A} \in \mathcal{V}$, and take $\vec{e}$ to be the unique $\vec{u} \in A^N$ such that $\vec{u} \equiv \vec{0}(\theta)$ and $\vec{u} \equiv \vec{1}(\delta)$, where δ is any factor complement of θ, and we write $\vec{a} \equiv \vec{b}(\theta)$ to express that $(a_i, b_i) \in \theta$, with $i = 1, \ldots, N$. Then we have that $\theta = \{(a, b) \in A^2 : \mathbf{A} \models \lambda(\vec{e}, a, b)\}$.

So, condition (L) says that *every* factor congruence θ can be defined by λ parameterized with an adequate central element (which we will see is uniquely determined by θ). Since $\mathbf{A} \times \mathbf{B}$ is isomorphic to $\mathbf{B} \times \mathbf{A}$ via $(a, b) \mapsto (b, a)$, it is trivial that a formula λ satisfying (L) also satisfies

$$\mathbf{A} \times \mathbf{B} \models \lambda([\vec{1}, \vec{0}], (a, b), (a', b')) \text{ iff } b = b',$$

for any $\mathbf{A}, \mathbf{B} \in \mathcal{V}$. Observe that this condition not only states the equality of the second coordinate but also $\vec{0}$ and $\vec{1}$ have been interchanged in the formula λ. Since in general $\vec{0}$ and $\vec{1}$ are not interchangeable, it is not obvious that (L) is equivalent to the following condition.

(R) There is a first order formula $\rho(\vec{z}, x, y)$ such that for every $\mathbf{A}, \mathbf{B} \in \mathcal{V}$,

$$\mathbf{A} \times \mathbf{B} \models \rho([\vec{0}, \vec{1}], (a, b), (a', b')) \text{ iff } b = b'.$$

If $\mathcal{V} = \mathcal{S}_{01}^{\vee}$, the reader can easily check that

$$\rho := \forall u \ (x \vee u \vee z_1 = y \vee u \vee z_1 \rightarrow x \vee u = y \vee u)$$

satisfies (R). Moreover, in [1], it is proved that for the variety $\mathcal{S}_{01}^{\vee}$, there is no positive nor existential formula satisfying (R), which says that the above ρ is as good as possible in the sense of its complexity. So, for $\mathcal{S}_{01}^{\vee}$, the best options are $\lambda := x \vee z_1 = y \vee z_1$ for property (L) and the above ρ for property (R).

A third definability condition is the following.

(W) There is a first order formula $\omega(\vec{z}, \vec{w}, x, y)$ such that for every $\mathbf{A}, \mathbf{B} \in \mathcal{V}$,

$$\mathbf{A} \times \mathbf{B} \models \omega([\vec{0}, \vec{1}], [\vec{1}, \vec{0}], (a, b), (a', b')) \text{ iff } a = a'.$$

Of course, (W) is implied by (L) and (R), taking $\omega(\vec{z}, \vec{w}, x, y) = \lambda(\vec{z}, x, y)$ and $\omega(\vec{z}, \vec{w}, x, y) = \rho(\vec{w}, x, y)$, respectively. Further, we note that as was shown above for (L), (W) guarantees that every factor congruence can be defined by ω parameterized with an adequate pair of complementary central elements.

In [6], the following theorem is proved.

Theorem 1.1. *For a variety $\mathcal{V}$ with $\vec{0}$ and $\vec{1}$, each of the properties* (L), (R), *and* (W) *are equivalent to that $\mathcal{V}$ has Boolean factor congruences, i.e., the set of factor congruences of any algebra in $\mathcal{V}$ is a distributive sublattice of its* *congruence lattice. Moreover the formulas in* (L)*,* (R)*, and* (W) *can be chosen to be preserved by direct products and direct factors.*

When a variety $\mathcal{V}$ with $\vec{0}$ and $\vec{1}$ satisfies the equivalent conditions (L), (R), and (W), we say that $\mathcal{V}$ has *definable factor congruences* (*DFC*, for short). As we exemplified with the semilattice case, the equivalence of (L), (R), and (W) does not preserve the complexity of the defining formula. So, several definitions arise, which we state now (we abbreviate with EDFC the phrase "equationally definable factor congruences"). We say that a variety $\mathcal{V}$ with $\vec{0}$ and $\vec{1}$ has *left* (*resp. right, weak*) *EDFC* if (L) (resp. (R), (W)) holds with λ (resp. ρ, ω) a conjunction of equations. We say that $\mathcal{V}$ has *twice EDFC* if $\mathcal{V}$ has left and right EDFC.

Examples of varieties with some type of EDFC abound. If $\mathcal{V}$ is a congruence modular variety with $\vec{0}$ and $\vec{1}$, then $\mathcal{V}$ has twice EDFC (see [7]). If $\mathcal{V}$ is a variety of bounded lattice expansions, then $\mathcal{V}$ has twice EDFC (folklore). If $\mathcal{V}$ is a variety of bounded join semilattice expansions, then $\mathcal{V}$ has left EDFC (see [1]), but does not necessarily have right EDFC. In [1], we extensively studied varieties with left EDFC, and the main theorem in [1] gives several equivalent properties for left EDFC. Also, first order axiomatizations of the properties "$\vec{e}$ is a central element" and "$\vec{e}$ and $\vec{f}$ are complementary central elements" are given, for the case of a variety with left EDFC. Furthermore, it is proved that such axiomatizations are optimal, in the sense of the complexity of the involved formulas.

In this paper, we make a similar study for the right, weak and twice cases, and new properties which are equivalent to left EDFC are added by means of two definability results proved in [3].

2. Notation and basic results

As usual, $\mathbb{I}(\mathcal{K})$, $\mathbb{S}(\mathcal{K})$, and $\mathbb{P}_u(\mathcal{K})$ denote the classes of isomorphic images, substructures, and ultraproducts of elements of $\mathcal{K}$. If $\mathcal{V}$ is a variety, we use $\mathcal{V}_{SI}$ (resp. $\mathcal{V}_{DI}$) to denote the class of subdirectly irreducible (resp. directly indecomposable) members of $\mathcal{V}$. If $\mathbf{A}, \mathbf{B}$ are algebras, we write $\mathbf{A} \leq \mathbf{B}$ to express that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$. By $\mathrm{Con}(\mathbf{A})$, we denote the congruence lattice of $\mathbf{A}$. As usual, the join operation of $\mathrm{Con}(\mathbf{A})$ is denoted by $\vee$. We use $\nabla^{\mathbf{A}}$ to denote the universal congruence on $\mathbf{A}$ and $\Delta^{\mathbf{A}}$ to denote the identity congruence on $\mathbf{A}$, or simply ∇ and Δ when the context is clear. If $\vec{a}, \vec{b} \in A^n$, then $\theta^{\mathbf{A}}(\vec{a}, \vec{b})$ denotes the congruence generated by $\{(a_k, b_k) : 1 \leq k \leq n\}$. If $\vec{a}, \vec{b} \in A^n$ and $\theta \in \mathrm{Con}(\mathbf{A})$, we write $\vec{a} \equiv \vec{b}(\theta)$ to express that $(a_i, b_i) \in \theta$, $i = 1, \ldots, n$. We use $FC(\mathbf{A})$ to denote the set of factor congruences of $\mathbf{A}$. A variety $\mathcal{V}$ has *Boolean factor congruences* if for every $\mathbf{A} \in \mathcal{V}$, the set $FC(\mathbf{A})$ is a distributive sublattice of $\mathrm{Con}(\mathbf{A})$. If $\theta \in FC(\mathbf{A})$, we use θ^* to denote the factor complement of θ. Observe that in a variety with Boolean factor congruences, $(FC(\mathbf{A}), \vee, \cap, ^*, \Delta^{\mathbf{A}}, \nabla^{\mathbf{A}})$ is a Boolean algebra. If $\mathbf{S} \leq \mathbf{A}$ and $\theta \in \mathrm{Con}(\mathbf{A})$, we use $\theta|_S$ to denote $\theta \cap (S \times S)$.

A *decomposition operation on* $\mathbf{A}$ is a homomorphism $d\colon A \times A \to A$ with

$$d(x,x) = x \quad \text{and} \quad d(d(x,y),z) = d(x,z) = d(x,d(y,z)).$$

With each pair (θ, δ) of complementary factor congruences, we associate a decomposition operation defined by

$$d(a,b) = \text{the unique } c \in A \text{ such that } (c,a) \in \theta \text{and } (c,b) \in \delta.$$

Reciprocally, given a decomposition operation d, the relations

$$\theta = \{(x,y) : d(x,y) = y\} \text{ and } \delta = \{(x,y) : d(x,y) = x\}$$

are a pair of complementary factor congruences. The above maps $(\theta, \delta) \mapsto d$ and $d \mapsto (\theta, \delta)$ are mutually inverse ([4, Theorem 4.33]).

Given a variety $\mathcal{V}$ and a set of variables X, we use $\mathbf{F}_{\mathcal{V}}(X)$ to denote the free algebra of $\mathcal{V}$ freely generated by X. If $X = \{x_1, \ldots, x_n\}$, then we use $\mathbf{F}_{\mathcal{V}}(x_1, \ldots, x_n)$ instead of $\mathbf{F}_{\mathcal{V}}(\{x_1, \ldots, x_n\})$.

Lemma 2.1. *Let $\mathcal{V}$ be a variety and let X be a set of variables. Let $r_1, \ldots, r_m$, $s_1, \ldots, s_m, r, s$ be terms with variables in X. The following are equivalent:*

(1) $(r,s) \in \theta^{\mathbf{F}_{\mathcal{V}}(X)}(\vec{r}, \vec{s})$;
(2) $\mathcal{V} \models \vec{r} = \vec{s} \to r = s$.

Lemma 2.2. *Let $\mathbf{A}$ and $\mathbf{B}$ be any algebras and let $\sigma\colon \mathbf{A} \to \mathbf{B}$ be a homomorphism. Then $(c,d) \in \theta^{\mathbf{A}}(\vec{a}, \vec{b})$ implies $(\sigma(c), \sigma(d)) \in \theta^{\mathbf{B}}(\sigma(\vec{a}), \sigma(\vec{b}))$.*

If $\mathbf{S} \leq \mathbf{A}_1 \times \mathbf{A}_2$ and $\mathbf{T} \leq \mathbf{B}_1 \times \mathbf{B}_2$, we say that a homomorphism $\sigma\colon \mathbf{S} \to \mathbf{T}$ is *left factorable* if there are homomorphisms $\sigma_1\colon \pi_1(\mathbf{S}) \to \mathbf{B}_1$ and $\sigma_2\colon \mathbf{S} \to \mathbf{B}_2$ such that for every $(a_1, a_2) \in S$, we have $\sigma(a_1, a_2) = (\sigma_1(a_1), \sigma_2(a_1, a_2))$. Similarly, we say that $\sigma\colon \mathbf{S} \to \mathbf{T}$ is *right factorable* if there are homomorphisms $\sigma_1\colon \mathbf{S} \to \mathbf{B}_1$ and $\sigma_2\colon \pi_2(\mathbf{S}) \to \mathbf{B}_2$ such that for every $(a_1, a_2) \in S$, we have $\sigma(a_1, a_2) = (\sigma_1(a_1, a_2), \sigma_2(a_2))$. We say that $\sigma\colon \mathbf{S} \to \mathbf{T}$ is *twice factorable* if there are homomorphisms $\sigma_1\colon \pi_1(\mathbf{S}) \to \mathbf{B}_1$ and $\sigma_2\colon \pi_2(\mathbf{S}) \to \mathbf{B}_2$ such that for every $(a_1, a_2) \in S$, we have $\sigma(a_1, a_2) = (\sigma_1(a_1), \sigma_2(a_2))$. Observe that σ is left factorable iff

$$(s_1, s_2) \in \ker \pi_1|_S \text{ implies } (\sigma(s_1), \sigma(s_2)) \in \ker \pi_1|_T.$$

Similarly, σ is twice factorable iff for $i = 1, 2$, we have that

$$(s_1, s_2) \in \ker \pi_i|_S \text{ implies } (\sigma(s_1), \sigma(s_2)) \in \ker \pi_i|_T.$$

Basic facts on varieties with DFC. Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$ and suppose $\mathcal{V}$ has DFC, i.e., $\mathcal{V}$ satisfies the equivalent conditions (L), (R), and (W) from the introduction. We use $Z(\mathbf{A})$ to denote the set of central elements of an algebra $\mathbf{A} \in \mathcal{V}$ and $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ to denote that $\vec{e}$ and $\vec{f}$ are a pair of complementary central elements of $\mathbf{A}$. It is obvious from the definition that $\vec{e} \in Z(\mathbf{A})$ iff there is a pair of complementary factor congruences (θ, δ) satisfying

$$\vec{e} \equiv \vec{0}(\theta) \text{ and } \vec{e} \equiv \vec{1}(\delta). \qquad (*)$$

Note that Theorem 1.1 implies that the central element $\vec{e}$ determines a unique pair of complementary factor congruences satisfying $(*)$ since (L) implies that $\lambda(\vec{e}, -, -)$ defines θ and (R) implies that $\rho(\vec{e}, -, -)$ defines δ. We denote this pair by $(\theta^{\mathbf{A}}_{\vec{0}\vec{e}}, \theta^{\mathbf{A}}_{\vec{1}\vec{e}})$. Thus, $Z(\mathbf{A})$ is naturally identified with the set of pairs of complementary factor congruences of $\mathbf{A}$. Since $\mathcal{V}$ has Boolean factor congruences (Theorem 1.1), factor complements are unique, and hence we obtain the following fundamental result.

Theorem 2.3. *Let $\mathcal{V}$ be a variety with DFC and $\mathbf{A} \in \mathcal{V}$. The maps*

$$Z(\mathbf{A}) \to FC(\mathbf{A}) \quad \text{given by } \vec{e} \mapsto \theta^{\mathbf{A}}_{\vec{0}\vec{e}}, \quad \text{and}$$

$$FC(\mathbf{A}) \to Z(\mathbf{A}) \quad \text{given by } \theta \mapsto \text{unique } \vec{e} \text{ satisfying } \vec{e} \equiv \vec{0}(\theta) \text{ and } \vec{e} \equiv \vec{1}(\theta^*)$$

are mutually inverse bijections.

Thus, we can define

$$\vec{e} \vee^{\mathbf{Z}(\mathbf{A})} \vec{f} = \text{the only } \vec{g} \in Z(\mathbf{A}) \text{ satisfying } \theta^{\mathbf{A}}_{\vec{0}\vec{g}} = \theta^{\mathbf{A}}_{\vec{0}\vec{e}} \vee \theta^{\mathbf{A}}_{\vec{0}\vec{f}},$$
$$\vec{e} \wedge^{\mathbf{Z}(\mathbf{A})} \vec{f} = \text{the only } \vec{g} \in Z(\mathbf{A}) \text{ satisfying } \theta^{\mathbf{A}}_{\vec{0}\vec{g}} = \theta^{\mathbf{A}}_{\vec{0}\vec{e}} \cap \theta^{\mathbf{A}}_{\vec{0}\vec{f}},$$
$$c^{\mathbf{Z}(\mathbf{A})}(\vec{e}) = \text{the only } \vec{g} \in Z(\mathbf{A}) \text{ satisfying } \theta^{\mathbf{A}}_{\vec{0}\vec{g}} = (\theta^{\mathbf{A}}_{\vec{0}\vec{e}})^*,$$

to obtain a Boolean algebra $\mathbf{Z}(\mathbf{A}) = (Z(\mathbf{A}), \vee^{\mathbf{Z}(\mathbf{A})}, \wedge^{\mathbf{Z}(\mathbf{A})}, c^{\mathbf{Z}(\mathbf{A})}, \vec{0}, \vec{1})$, which is naturally isomorphic to $(FC(\mathbf{A}), \vee, \cap, ^*, \Delta^{\mathbf{A}}, \nabla^{\mathbf{A}})$. When no confusion is possible, we will write $\vec{e} \vee \vec{f}$ in place of $\vec{e} \vee^{\mathbf{Z}(\mathbf{A})} \vec{f}$, $c(\vec{e})$ in place of $c^{\mathbf{Z}(\mathbf{A})}(\vec{e})$, etc. The following proposition states some basic properties involving central elements.

Proposition 2.4. *Let $\mathcal{V}$ be a variety with DFC and $\mathbf{A} \in \mathcal{V}$. The following properties hold.*

(a) $\theta^{\mathbf{A}}_{\vec{0}\vec{e}} = \{(a, b) : \mathbf{A} \models \lambda(\vec{e}, a, b)\}$ *for any λ satisfying* (L).
(b) $\theta^{\mathbf{A}}_{\vec{1}\vec{e}} = \{(a, b) : \mathbf{A} \models \rho(\vec{e}, a, b)\}$ *for any ρ satisfying* (R).
(c) $\theta^{\mathbf{A}}_{\vec{0}\vec{e}} = \{(a, b) : \mathbf{A} \models \omega(\vec{e}, c(\vec{e}), a, b)\}$ *for any ω satisfying* (W).
(d) $\theta^{\mathbf{A}}_{\vec{1}\vec{e}} = \{(a, b) : \mathbf{A} \models \omega(c(\vec{e}), \vec{e}, a, b)\}$ *for any ω satisfying* (W).
(e) $\vec{e} \leq^{\mathbf{Z}(\mathbf{A})} \vec{f}$ *iff* $\theta^{\mathbf{A}}_{\vec{0}\vec{e}} \subseteq \theta^{\mathbf{A}}_{\vec{0}\vec{f}}$.
(f) $\theta^{\mathbf{A}}_{\vec{0}\vec{e}} = \theta^{\mathbf{A}}_{\vec{1}c(\vec{e})}$.
(g) $\theta^{\mathbf{A}}_{\vec{0}\vec{0}} = \Delta^{\mathbf{A}}$ *and* $\theta^{\mathbf{A}}_{\vec{0}\vec{1}} = \nabla^{\mathbf{A}}$.
(h) *The map* $Z(\mathbf{A}_1) \times Z(\mathbf{A}_2) \to Z(\mathbf{A}_1 \times \mathbf{A}_2)$ *given by* $(\vec{e}_1, \vec{e}_2) \mapsto [\vec{e}_1, \vec{e}_2]$ *is a Boolean algebra isomorphism.*
(i) $\theta^{\mathbf{A}_1 \times \mathbf{A}_2}_{[\vec{0},\vec{0}][\vec{e}_1,\vec{e}_2]} = \theta^{\mathbf{A}_1}_{\vec{0}\vec{e}_1} \times \theta^{\mathbf{A}_2}_{\vec{0}\vec{e}_2}$.
(j) $\theta^{\mathbf{A}}(\vec{0}, \vec{e}) \subseteq \theta^{\mathbf{A}}_{\vec{0}\vec{e}}$ *and* $\theta^{\mathbf{A}}(\vec{1}, \vec{e}) \subseteq \theta^{\mathbf{A}}_{\vec{1}\vec{e}}$.

In [6], we give an example to show that the converse inclusion in (j) of the previous proposition is not true in general since the congruences $\theta^{\mathbf{A}}_{\vec{0}\vec{e}}$ and $\theta^{\mathbf{A}}_{\vec{1}\vec{e}}$ fail to be finitely generated.

If $\mathcal{V}$ is a variety with DFC, we say that a formula $\varphi(\vec{z}, x, y)$ *defines* $\theta^{\mathbf{A}}_{\vec{0}\vec{e}}$ *in* $\mathcal{V}$ (resp. *defines* $\theta^{\mathbf{A}}_{\vec{1}\vec{e}}$ *in* $\mathcal{V}$) if for every $\mathbf{A} \in \mathcal{V}$ and $\vec{e} \in Z(\mathbf{A})$, we have that $\theta^{\mathbf{A}}_{\vec{0}\vec{e}} = \{(a, b) \in A^2 : \mathbf{A} \models \varphi(\vec{e}, a, b)\}$ (resp. $\theta^{\mathbf{A}}_{\vec{1}\vec{e}} = \{(a, b) \in A^2 : \mathbf{A} \models \varphi(\vec{e}, a, b)\}$), i.e., φ satisfies (L) (resp. (R)) of the introduction. Analogously, we say that a formula $\varphi(\vec{z}, \vec{w}, x, y)$ *defines* $\theta^{\mathbf{A}}_{\vec{0}\vec{e}}$ *in* $\mathcal{V}$ if for every $\mathbf{A} \in \mathcal{V}$ and $\vec{e} \in Z(\mathbf{A})$, we have that $\theta^{\mathbf{A}}_{\vec{0}\vec{e}} = \{(a, b) \in A^2 : \mathbf{A} \models \varphi(\vec{e}, c(\vec{e}), a, b)\}$, i.e., φ satisfies (W) of the introduction.

We conclude the section with some examples showing that the notions of weak, left, right, and twice EDFC are in fact different one from the other. First, we note that the variety $\mathcal{S}^{\vee}_{01}$ of bounded join semilattices has left EDFC and $\mathcal{S}^{\vee}_{01}$ does not have right EDFC since by [1, Lemma 4.3], there is no positive formula satisfying (R) in $\mathcal{S}^{\vee}_{01}$. Similarly, the variety $\mathcal{S}^{\wedge}_{01}$ of bounded meet semilattices has right EDFC and $\mathcal{S}^{\wedge}_{01}$ does not have left EDFC.

We will give an example of a variety $\mathcal{W}$ with weak EDFC which does not have either left or right EDFC. Let $\mathcal{W}$ be the variety given by the identities

$$p(0, 1, x, y) \approx x \quad \text{and} \quad p(1, 0, x, y) \approx p(1, 0, y, x).$$

It is easy to check that the formula

$$\omega(z_1, w_1, x, y) := p(z_1, w_1, x, y) = p(z_1, w_1, y, x)$$

satisfies (W) and so $\mathcal{W}$ has weak EDFC. We note that there is a subvariety of $\mathcal{W}$ which is equivalent to $\mathcal{S}^{\vee}_{01}$ (axiomatize this subvariety by taking identities assuring that $p(x, y, z, w) := x \vee w$, for some join operation $\vee$ for which 0 is a bottom and 1 is a top). Similarly, there is a subvariety of $\mathcal{W}$ which is equivalent to $\mathcal{S}^{\wedge}_{01}$, and hence $\mathcal{W}$ does not have either left or right EDFC.

3. Characterizations of left, right, weak and twice EDFC

In this section, we prove theorems characterizing the different types of EDFC. First we complete the left case with some new equivalences which follow from two definability results proved in [3]. Then we approach the weak and twice cases. Since the concept of right EDFC is dual to that of left EDFC, we state without proof the corresponding theorem characterizing varieties with right EDFC.

Lemma 3.1 ([3]). *Let $\mathcal{L}$ be a first order language and let R be a n-ary relation symbol not belonging to $\mathcal{L}$. Let $\mathcal{K}$ be a class of $(\mathcal{L} \cup \{R\})$-structures which is closed under the formation of ultraproducts.*

(1) *The following are equivalent.*
  (a) *There is an open $\mathcal{L}$-formula φ such that $\mathcal{K} \models R(\vec{x}) \leftrightarrow \varphi(\vec{x})$.*

(b) *If* $\langle \mathbf{A}, R^{\mathbf{A}}\rangle, \langle \mathbf{B}, R^{\mathbf{B}}\rangle \in \mathcal{K}$, $\mathbf{A}_0 \leq \mathbf{A}$, $\mathbf{B}_0 \leq \mathbf{B}$, *and* $\sigma\colon \mathbf{A}_0 \to \mathbf{B}_0$ *is an isomorphism, then for every* $a_1, \dots, a_n \in A_0$, *we have that* $(a_1, \dots, a_n) \in R^{\mathbf{A}}$ *implies* $(\sigma(a_1), \dots, \sigma(a_n)) \in R^{\mathbf{B}}$.

(2) *The following are equivalent.*

(a) *There is an existential* $\mathcal{L}$*-formula* φ *such that* $\mathcal{K} \models R(\vec{x}) \leftrightarrow \varphi(\vec{x})$.

(b) *If* $\langle \mathbf{A}, R^{\mathbf{A}}\rangle, \langle \mathbf{B}, R^{\mathbf{B}}\rangle \in \mathcal{K}$ *and* $\sigma\colon \mathbf{A} \to \mathbf{B}$ *is an embedding, then for every* $a_1, \dots, a_n \in A$, *we have that* $(a_1, \dots, a_n) \in R^{\mathbf{A}}$ *implies* $(\sigma(a_1), \dots, \sigma(a_n)) \in R^{\mathbf{B}}$.

Lemma 3.2. *Let* $\mathcal{V}$ *be a variety with* $\vec{0}$ *and* $\vec{1}$ *for which there is a positive formula satisfying (L) of the introduction. Then, for every* $\mathbf{A} \in \mathcal{V}$ *and every* $\vec{e} \in Z(\mathbf{A})$, *we have that* $\theta^{\mathbf{A}}_{\vec{0}\vec{e}} = \theta^{\mathbf{A}}(\vec{0}, \vec{e})$.

Proof. This is proved in Claim 2 of [6, Proposition 18]. □

Those items of the following theorem which have a word in parentheses are double items, in the sense that both ways of reading them are equivalent to all the other items of the theorem. The same will happen with Theorems 3.5, 3.7, and 3.9.

Since in [1] the characterization of left EDFC is made under the assumption that $\mathcal{V}$ has DFC and in this paper we can drop this hypothesis, we include a detailed proof of the left case.

Theorem 3.3 (Left EDFC)**.** *Let* $\mathcal{V}$ *be a variety with* $\vec{0}$ *and* $\vec{1}$. *The following are equivalent.*

(L1) $\mathcal{V}$ *has left EDFC.*

(L2) *There is an open formula* $\lambda(\vec{z}, x, y)$ *which satisfies* (L) *of the introduction.*

(L3) *There are terms* p_i, q_i, $i = 1, \dots, n$ *such that*

$$\mathcal{V} \models (\bigwedge p_i(\vec{0}, x, y) = q_i(\vec{0}, x, y)) \leftrightarrow x = y,$$
$$\mathcal{V} \models \bigwedge p_i(\vec{1}, x, y) = q_i(\vec{1}, x, y).$$

(L4) *There is a* $(\bigwedge p = q)$*-formula* $\varphi(\vec{z}, x, y)$ *such that if* $\mathbf{A} \in \mathcal{V}$, $\vec{e} \in A^N$, *and* $\theta^{\mathbf{A}}(\vec{0}, \vec{e}) \cap \theta^{\mathbf{A}}(\vec{1}, \vec{e}) = \Delta^{\mathbf{A}}$, *then* $\theta^{\mathbf{A}}(\vec{0}, \vec{e}) = \{(a, b) : \mathbf{A} \models \varphi(\vec{e}, a, b)\}$.

(L5) *There are terms* v_i, *for* $i = 0, \dots, k$ *with* k *even, such that the following identities hold in* $\mathcal{V}$:

$$v_0(\vec{z}, x, y) = x,$$
$$v_k(\vec{z}, x, y) = y,$$
$$v_i(\vec{0}, x, y) = v_{i+1}(\vec{0}, x, y), \ \ i \text{ even},$$
$$v_i(\vec{1}, x, y) = v_{i+1}(\vec{1}, x, y), \ \ i \text{ odd},$$
$$v_i(\vec{0}, x, x) = x, \ \ i = 0, \dots, k.$$

(L6) $(x, y) \in \theta^{\mathbf{F}}(\vec{0}, \vec{z}) \vee ((\theta^{\mathbf{F}}(\vec{0}, \vec{z}) \vee \theta^{\mathbf{F}}(x, y)) \cap \theta^{\mathbf{F}}(\vec{1}, \vec{z}))$, *where we abbreviate* $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(\vec{z}, x, y)$.

(L7) *The following conditions hold in* $\mathcal{V}$.
 (i) *If* $\sigma\colon \mathbf{A}_1 \times \mathbf{A}_2 \to \mathbf{B}_1 \times \mathbf{B}_2$ *is a homomorphism (embedding) such that* $\sigma([\vec{0},\vec{1}]) = [\vec{0},\vec{1}]$, *then* σ *is left factorable.*
 (ii) *If* $\mathbf{S} \leq \mathbf{A}_1 \times \mathbf{A}_2$ *and* $[\vec{0},\vec{1}] \in S^N$, *then*

$$\theta^{\mathbf{S}}([\vec{0},\vec{0}],[\vec{0},\vec{1}]) = \theta^{\mathbf{A}_1\times\mathbf{A}_2}([\vec{0},\vec{0}],[\vec{0},\vec{1}])\Big|_S .$$

(L8) *If* $\mathbf{S} \leq \mathbf{A}_1 \times \mathbf{A}_2$ *and* $[\vec{0},\vec{1}] \in S^N$, *then* $\theta^{\mathbf{S}}([\vec{0},\vec{0}],[\vec{0},\vec{1}]) = \ker \pi_1|_S$.

(L9) *If* $\mathbf{S} \leq \mathbf{A}_1 \times \mathbf{A}_2$, $\mathbf{T} \leq \mathbf{B}_1 \times \mathbf{B}_2$, $[\vec{0},\vec{1}] \in S^N$, *and* $\sigma\colon \mathbf{S} \to \mathbf{T}$ *is a homomorphism (isomorphism) such that* $\sigma([\vec{0},\vec{1}]) = [\vec{0},\vec{1}]$, *then* σ *is left factorable.*

Moreover, when the above equivalent conditions hold, $\theta^{\mathbf{A}}_{0\vec{e}} = \theta^{\mathbf{A}}(\vec{0},\vec{e})$ *whenever* $\vec{e} \in Z(\mathbf{A})$.

Proof. (L1)⇒(L2): This is trivial.

(L2)⇒(L3): This is proved in the proof of (1)⇒(2) of [1, Theorem 3.2] under the hypothesis of $\mathcal{V}$ having DFC. Nevertheless, the exact same proof works for the more general case of a variety $\mathcal{V}$ with $\vec{0}$ and $\vec{1}$.

(L3)⇒(L1): Take $\lambda(\vec{z},x,y) := \bigwedge p_i(\vec{z},x,y) = q_i(\vec{z},x,y)$. It is easy to check that λ satisfies (L).

(L3)⇒(L4): This is proved in the proof of (2)⇒(3) of [1, Theorem 3.2] under the hypothesis of $\mathcal{V}$ having DFC. Nevertheless, the exact same proof works for the more general case of a variety $\mathcal{V}$ with $\vec{0}$ and $\vec{1}$.

(L4)⇒(L3): Let $\varphi(\vec{z},x,y) := \bigwedge p_i(\vec{z},x,y) = q_i(\vec{z},x,y)$ be such that if we have that $\mathbf{A} \in \mathcal{V}$, $\vec{e} \in A^N$, and $\theta^{\mathbf{A}}(\vec{0},\vec{e}) \cap \theta^{\mathbf{A}}(\vec{1},\vec{e}) = \Delta^{\mathbf{A}}$, then we have that $\theta^{\mathbf{A}}(\vec{0},\vec{e}) = \{(a,b) : \mathbf{A} \models \varphi(\vec{e},a,b)\}$. Taking $\vec{e} = \vec{0}$, we obtain that

$$\mathcal{V} \models (\bigwedge p_i(\vec{0},x,y) = q_i(\vec{0},x,y)) \leftrightarrow x = y.$$

Taking $\vec{e} = \vec{1}$, we obtain $\mathcal{V} \models \bigwedge p_i(\vec{1},x,y) = q_i(\vec{1},x,y)$ (since $\theta^{\mathbf{A}}(\vec{0},\vec{1}) = \nabla$).

(L1)⇒(L5): Since (L1) holds, we have that $\mathcal{V}$ has DFC, and hence we can apply [1, Theorem 3.2].

(L5)⇒(L1): Take $\lambda(\vec{z},x,y) := \bigwedge_{i \text{ odd}} v_i(\vec{z},x,y) = v_{i+1}(\vec{z},x,y)$.

(L5)⇔(L6): Of course, $(x,y) \in \theta^{\mathbf{F}}(\vec{0},\vec{z}) \vee ((\theta^{\mathbf{F}}(\vec{0},\vec{z}) \vee \theta^{\mathbf{F}}(x,y)) \cap \theta^{\mathbf{F}}(\vec{1},\vec{z}))$ iff there are terms $v_0(\vec{z},x,y),\ldots,v_k(\vec{z},x,y)$, with k even such that

$$v_0(\vec{z},x,y) = x, \quad \text{and} \quad v_k(\vec{z},x,y) = y,$$
$$(v_i(\vec{z},x,y), v_{i+1}(\vec{z},x,y)) \in \theta^{\mathbf{F}}(\vec{0},\vec{z}),\ i \text{ even},$$
$$(v_i(\vec{z},x,y), v_{i+1}(\vec{z},x,y)) \in (\theta^{\mathbf{F}}(\vec{0},\vec{z}) \vee \theta^{\mathbf{F}}(x,y)) \cap \theta^{\mathbf{F}}(\vec{1},\vec{z}),\ i \text{ odd}.$$

Thus, Lemma 2.1 naturally produces the equivalence (L5)⇔(L6).

(L1)⇒(L7). Let $\lambda(\vec{z},x,y)$ be a $(\bigwedge p = q)$-formula satisfying (L). First we will prove (i). Given that $\sigma([\vec{0},\vec{1}]) = [\vec{0},\vec{1}]$, we have that

$$\mathbf{A}_1 \times \mathbf{A}_2 \models \lambda([\vec{0},\vec{1}],x,y) \text{ implies } \mathbf{B}_1 \times \mathbf{B}_2 \models \lambda([\vec{0},\vec{1}],\sigma(x),\sigma(y)),$$

which means that $(x,y) \in \ker \pi_1$ implies $(\sigma(x),\sigma(y)) \in \ker \pi_1$. Therefore, σ is left factorable. Next we will prove (ii). Since (L1) implies (L5), we may assume that there is a $(\bigwedge p = q)$-formula $\varphi(\vec{z}, x, y)$ which defines $\theta^{\mathbf{A}}(\vec{0}, \vec{e})$, for every $\mathbf{A} \in \mathcal{V}$ and $\vec{e} \in A^N$ whenever $\theta^{\mathbf{A}}(\vec{0}, \vec{e}) \cap \theta^{\mathbf{A}}(\vec{1}, \vec{e}) = \Delta$. Given that $\theta^{\mathbf{A}_1 \times \mathbf{A}_2}([\vec{0}, \vec{0}], [\vec{0}, \vec{1}]) \subseteq \ker \pi_1$ and $\theta^{\mathbf{A}_1 \times \mathbf{A}_2}([\vec{1}, \vec{1}], [\vec{0}, \vec{1}]) \subseteq \ker \pi_2$, we have that

$$\theta^{\mathbf{A}_1 \times \mathbf{A}_2}([\vec{0}, \vec{0}], [\vec{0}, \vec{1}]) \cap \theta^{\mathbf{A}_1 \times \mathbf{A}_2}([\vec{1}, \vec{1}], [\vec{0}, \vec{1}]) = \Delta$$
$$\text{and} \quad \theta^{\mathbf{S}}([\vec{0}, \vec{0}], [\vec{0}, \vec{1}]) \cap \theta^{\mathbf{S}}([\vec{1}, \vec{1}], [\vec{0}, \vec{1}]) = \Delta,$$

hence φ defines both $\theta^{\mathbf{S}}([\vec{0}, \vec{0}], [\vec{0}, \vec{1}])$ and $\theta^{\mathbf{A}_1 \times \mathbf{A}_2}([\vec{0}, \vec{0}], [\vec{0}, \vec{1}])$. Now (ii) easily follows.

(L7)⇒(L8): Assume that the following condition, i.e., the embedding case of (L7)(i), holds.

$(i)_e$ For every $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{V}$, if $\sigma\colon \mathbf{A}_1 \times \mathbf{A}_2 \to \mathbf{B}_1 \times \mathbf{B}_2$ is an embedding such that $\sigma([\vec{0}, \vec{1}]) = [\vec{0}, \vec{1}]$, then σ is left factorable.

First we will prove that there is an existential formula $\lambda(\vec{z}, x, y)$ which satisfies (L) of the introduction. Let $\mathcal{L}$ be the expansion of the language of $\mathcal{V}$ with new constant symbols $c_1, \ldots, c_N$. Let R be a new binary relation symbol. Let $\mathcal{K}$ be the following class of $(\mathcal{L} \cup \{R\})$-structures:

$$\mathcal{K} = \{\langle \mathbf{A} \times \mathbf{B}, (0_1, 1_1), \ldots, (0_N, 1_N), \ker \pi_1 \rangle : \mathbf{A}, \mathbf{B} \in \mathcal{V}\}.$$

We will prove that $\mathbb{I}(\mathcal{K})$ is closed under ultraproducts. Let $\{\mathbf{A}_x \times \mathbf{B}_x : x \in X\}$ be an indexed family such that $\mathbf{A}_x, \mathbf{B}_x \in \mathcal{V}$ and let $\pi_{1x}\colon \mathbf{A}_x \times \mathbf{B}_x \to \mathbf{A}_x$ be the canonical projection. It is easy to check that for any ultrafilter u on X,

$$\prod\nolimits_{x \in X} \langle \mathbf{A}_x \times \mathbf{B}_x, (0_1, 1_1), \ldots, (0_N, 1_N), \ker \pi_{1x} \rangle \Big/ u$$

is naturally isomorphic to $\langle \mathbf{U} \times \mathbf{W}, (0_1, 1_1), \ldots, (0_N, 1_N), \ker \pi_1 \rangle$, such that $\mathbf{U} = \prod_{x \in X} \mathbf{A}_x / u$ and $\mathbf{W} = \prod_{x \in X} \mathbf{B}_x / u$. Thus, $\mathbb{I}(\mathcal{K})$ is closed under ultraproducts. Note that $(i)_e$ says that $\mathcal{K}$ satisfies (2)(b) of Lemma 3.1 and clearly $\mathbb{I}(\mathcal{K})$ satisfies it too. Then there is an existential $\mathcal{L}$-formula $\varphi(x, y)$ such that $\mathbb{I}(\mathcal{K}) \models R(x, y) \leftrightarrow \varphi(x, y)$. Let $\lambda(\vec{z}, x, y)$ be a formula of the language of $\mathcal{V}$ such that $\lambda(\vec{c}, x, y) = \varphi(x, y)$. Note that $\lambda(\vec{z}, x, y)$ is existential and since

$$\mathcal{K} \models R(x, y) \leftrightarrow \lambda(\vec{c}, x, y),$$

we have that $\lambda(\vec{z}, x, y)$ satisfies (L) of the introduction.

Now by [5], we have that there is a positive formula satisfying (L) of the introduction. By Lemma 3.2, we have that $\theta^{\mathbf{A}}_{\vec{0}\vec{e}} = \theta^{\mathbf{A}}(\vec{0}, \vec{e})$, for every $\mathbf{A} \in \mathcal{V}$, $\vec{e} \in Z(\mathbf{A})$. In particular, we have $\ker \pi_1 = \theta^{\mathbf{A}_1 \times \mathbf{A}_2}([\vec{0}, \vec{0}], [\vec{0}, \vec{1}])$. Now (L8) easily follows from (L7)(ii).

(L8)⇒(L9): By Lemma 2.2 and the fact that $\sigma([\vec{0}, \vec{1}]) = [\vec{0}, \vec{1}]$, we have that

$$(x, y) \in \theta^{\mathbf{S}}([\vec{0}, \vec{0}], [\vec{0}, \vec{1}]) \text{ implies } (\sigma(x), \sigma(y)) \in \theta^{\mathbf{T}}([\vec{0}, \vec{0}], [\vec{0}, \vec{1}]).$$

Therefore, by (L8), we have that

$$(x, y) \in \ker \pi_1|_S \text{ implies } (\sigma(x), \sigma(y)) \in \ker \pi_1|_T,$$

and then σ is left factorable.

(L9)⇒(L2): Assume that the following condition, i.e., the isomorphism case of (L9), holds.

(L9)$_i$ If $\mathbf{S} \leq \mathbf{A}_1 \times \mathbf{A}_2$, $\mathbf{T} \leq \mathbf{B}_1 \times \mathbf{B}_2$, $[\vec{0}, \vec{1}] \in S^N$, and $\sigma\colon \mathbf{S} \to \mathbf{T}$ is an isomorphism such that $\sigma([\vec{0}, \vec{1}]) = [\vec{0}, \vec{1}]$, then σ is left factorable.

Let $\mathcal{L}$ be the expansion of the language of $\mathcal{V}$ with new constant symbols $c_1, \ldots, c_N$. Let R be a new binary relation symbol. Let $\mathcal{K}$ be the following class of $(\mathcal{L} \cup \{R\})$-structures:

$$\mathcal{K} = \{\langle \mathbf{A} \times \mathbf{B}, (0_1, 1_1), \ldots, (0_N, 1_N), \ker \pi_1 \rangle : \mathbf{A}, \mathbf{B} \in \mathcal{V}\}.$$

As in the proof of (L7)⇒(L8), we can prove that $\mathbb{I}(\mathcal{K})$ is closed under ultraproducts. By (L9)$_i$, we have that $\mathcal{K}$ satisfies (1)(b) of Lemma 3.1, and clearly $\mathbb{I}(\mathcal{K})$ satisfies it too. Then there is an open $\mathcal{L}$-formula $\varphi(x, y)$ such that $\mathbb{I}(\mathcal{K}) \models R(x, y) \leftrightarrow \varphi(x, y)$. Let $\lambda(\vec{z}, x, y)$ be an open formula of the language of $\mathcal{V}$ such that $\lambda(\vec{c}, x, y) = \varphi(x, y)$. Note that $\lambda(\vec{z}, x, y)$ satisfies (L) of the introduction.

The last observation that $\theta^{\mathbf{A}}_{\vec{0}\vec{e}} = \theta^{\mathbf{A}}(\vec{0}, \vec{e})$, whenever $\vec{e} \in Z(\mathbf{A})$, follows from Lemma 3.2. □

Remark 3.4. If a variety $\mathcal{V}$ has terms $0_1, \ldots, 0_N, 1_1, \ldots, 1_N$, and $v_0, \ldots, v_k$ for k even, satisfying the identities in (L5), then $\mathcal{V} \models \vec{0} = \vec{1} \to x = y$ and the formula

$$\lambda(\vec{z}, x, y) := \bigwedge_{i \text{ odd}} v_i(\vec{z}, x, y) = v_{i+1}(\vec{z}, x, y)$$

satisfies (L) from the introduction. Thus, the existence of terms $0_1, \ldots, 0_N$, $1_1, \ldots, 1_N$, and $v_0, \ldots, v_k$ for k even, satisfying the identities in (L5), is a Maltsev condition for the property of being a variety with $\vec{0}$ and $\vec{1}$ which has left EDFC.

We state the analogue of the above theorem for the right EDFC case. The proof of this theorem is analogous to the proof of Theorem 3.3.

Theorem 3.5 (Right EDFC)**.** *Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$. The following are equivalent.*

(R1) *$\mathcal{V}$ has right EDFC.*

(R2) *There is an open formula $\rho(\vec{z}, x, y)$ which satisfies* (R) *of the introduction.*

(R3) *There are terms p_i, q_i, $i = 1, \ldots, n$ such that*

$$\mathcal{V} \models (\bigwedge p_i(\vec{1}, x, y) = q_i(\vec{1}, x, y)) \leftrightarrow x = y,$$

$$\mathcal{V} \models \bigwedge p_i(\vec{0}, x, y) = q_i(\vec{0}, x, y).$$

(R4) *There is a $(\bigwedge p = q)$-formula $\varphi(\vec{z}, x, y)$ such that if $\mathbf{A} \in \mathcal{V}$, $\vec{e} \in A^N$ and $\theta^{\mathbf{A}}(\vec{0}, \vec{e}) \cap \theta^{\mathbf{A}}(\vec{1}, \vec{e}) = \Delta^{\mathbf{A}}$, then $\theta^{\mathbf{A}}(\vec{1}, \vec{e}) = \{(a, b) : \mathbf{A} \models \varphi(\vec{e}, a, b)\}$.*

(R5) *There are terms, v_i for $i = 0, \dots, k$ with k even, such that the following identities hold in $\mathcal{V}$:*

$$\begin{aligned} v_0(\vec{z}, x, y) &= x, \quad \textit{and} \quad v_k(\vec{z}, x, y) = y, \\ v_i(\vec{1}, x, y) &= v_{i+1}(\vec{1}, x, y) \quad \textit{for } i \textit{ even}, \\ v_i(\vec{0}, x, y) &= v_{i+1}(\vec{0}, x, y) \quad \textit{for } i \textit{ odd}, \\ v_i(\vec{1}, x, x) &= x \quad \textit{for } i = 0, \dots, k. \end{aligned}$$

(R6) $(x, y) \in \theta^{\mathbf{F}}(\vec{1}, \vec{z}) \vee ((\theta^{\mathbf{F}}(\vec{1}, \vec{z}) \vee \theta^{\mathbf{F}}(x, y)) \cap \theta^{\mathbf{F}}(\vec{0}, \vec{z}))$*, where we abbreviate* $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(\vec{z}, x, y)$*.*

(R7) *The following conditions hold in $\mathcal{V}$.*
   (*i*) *If $\sigma \colon \mathbf{A}_1 \times \mathbf{A}_2 \to \mathbf{B}_1 \times \mathbf{B}_2$ is a homomorphism (embedding) such that $\sigma([\vec{0}, \vec{1}]) = [\vec{0}, \vec{1}]$, then σ is right factorable.*
   (*ii*) *If $\mathbf{S} \leq \mathbf{A}_1 \times \mathbf{A}_2$ and $[\vec{0}, \vec{1}] \in S^N$, then*

$$\theta^{\mathbf{S}}([\vec{1}, \vec{1}], [\vec{0}, \vec{1}]) = \theta^{\mathbf{A}_1 \times \mathbf{A}_2}([\vec{1}, \vec{1}], [\vec{0}, \vec{1}])\Big|_S.$$

(R8) *If $\mathbf{S} \leq \mathbf{A}_1 \times \mathbf{A}_2$ and $[\vec{0}, \vec{1}] \in S^N$, then $\theta^{\mathbf{S}}([\vec{1}, \vec{1}], [\vec{0}, \vec{1}]) = \ker \pi_2|_S$.*

(R9) *If $\mathbf{S} \leq \mathbf{A}_1 \times \mathbf{A}_2$, $\mathbf{T} \leq \mathbf{B}_1 \times \mathbf{B}_2$, $[\vec{0}, \vec{1}] \in S^N$, and $\sigma \colon \mathbf{S} \to \mathbf{T}$ is a homomorphism (isomorphism) such that $\sigma([\vec{0}, \vec{1}]) = [\vec{0}, \vec{1}]$, then σ is right factorable.*

Moreover, when the above equivalent conditions hold, we have $\theta^{\mathbf{A}}_{\vec{1}\vec{e}} = \theta^{\mathbf{A}}(\vec{1}, \vec{e})$ whenever $\vec{e} \in Z(\mathbf{A})$.

Remark 3.6. If a variety $\mathcal{V}$ has terms $0_1, \dots, 0_N, 1_1, \dots, 1_N$, and $v_0, \dots, v_k$ for k even, satisfying the identities in (R5), then $\mathcal{V} \models \vec{0} = \vec{1} \to x = y$ and the formula

$$\rho(\vec{z}, x, y) := \bigwedge_{i \text{ odd}} v_i(\vec{z}, x, y) = v_{i+1}(\vec{z}, x, y)$$

satisfies (R) from the introduction. Thus, the existence of terms $0_1, \dots, 0_N$, $1_1, \dots, 1_N$, and $v_0, \dots, v_k$ for k even, satisfying the identities in (R5), is a Maltsev condition for the property of being a variety with $\vec{0}$ and $\vec{1}$ which has right EDFC.

Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$ and suppose that $\mathcal{V}$ has DFC. Given an algebra $\mathbf{A} \in \mathcal{V}$ and $\vec{e} \in Z(\mathbf{A})$, we define $d^{\mathbf{A}}_{\vec{e}} \colon A \times A \to A$ as follows:

$$d^{\mathbf{A}}_{\vec{e}}(x, y) = \text{the only } z \in A \text{ satisfying } (x, z) \in \theta^{\mathbf{A}}_{\vec{0}\vec{e}} \text{ and } (z, y) \in \theta^{\mathbf{A}}_{\vec{0}c(\vec{e})}.$$

If $\mathbf{A}$ and $\mathbf{B}$ are any algebras, define $d^{\mathbf{A} \times \mathbf{B}} \colon (A \times B) \times (A \times B) \to A \times B$ as $d^{\mathbf{A} \times \mathbf{B}}((a_1, b_1), (a_2, b_2)) = (a_1, b_2)$. Of course, $d^{\mathbf{A} \times \mathbf{B}}$ is a decomposition operation on $\mathbf{A} \times \mathbf{B}$ and every decomposition operation on an algebra $\mathbf{A}$ is of this form via the isomorphism $\mathbf{A} \to \mathbf{A}/\theta_d \times \mathbf{A}/\delta_d$, where

$$\theta_d = \{(x, y) \in A^2 : d(x, y) = y\} \text{ and } \delta_d = \{(x, y) \in A^2 : d(x, y) = x\}.$$

We say that $\mathcal{V}$ has *equationally definable decomposition operations* if there is a $(\bigwedge p = q)$-formula $\delta(\vec{z}, x, y, z)$ such that the following condition holds.

(D) For every $\mathbf{A}, \mathbf{B} \in \mathcal{V}$,

$$\mathbf{A} \times \mathbf{B} \models \delta([\vec{0}, \vec{1}], x, y, z) \text{ iff } d^{\mathbf{A}\times\mathbf{B}}(x, y) = z.$$

We say that $\mathcal{V}$ has *weak equationally definable decomposition operations* if there is a $(\bigwedge p = q)$-formula $\delta(\vec{z}, \vec{w}, x, y, z)$ such that the following condition holds.

($\mathrm{D_w}$) For every $\mathbf{A}, \mathbf{B} \in \mathcal{V}$,

$$\mathbf{A} \times \mathbf{B} \models \delta([\vec{0}, \vec{1}], [\vec{1}, \vec{0}], x, y, z) \text{ iff } d^{\mathbf{A}\times\mathbf{B}}(x, y) = z.$$

Of course, the best case of definability of decomposition operations is where there is a term $u(\vec{z}, x, y)$ such that for $\mathbf{A}, \mathbf{B} \in \mathcal{V}$, $d^{\mathbf{A}\times\mathbf{B}}(x, y) = u([\vec{0}, \vec{1}], x, y)$. For basic facts on this type of variety, we refer the reader to [2].

Theorem 3.7. (Weak EDFC) *Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$. The following are equivalent.*

(W1) *$\mathcal{V}$ has weak EDFC.*

(W2) *There is an open formula $\omega(\vec{z}, \vec{w}, x, y)$ satisfying* (W) *of the introduction.*

(W3) *There are terms p_i, q_i, for $i = 1, \ldots, n$, such that*

$$\mathcal{V} \models (\bigwedge p_i(\vec{0}, \vec{1}, x, y) = q_i(\vec{0}, \vec{1}, x, y)) \leftrightarrow x = y,$$
$$\mathcal{V} \models \bigwedge p_i(\vec{1}, \vec{0}, x, y) = q_i(\vec{1}, \vec{0}, x, y).$$

(W4) *There is a $(\bigwedge p = q)$-formula $\varphi(\vec{z}, \vec{w}, x, y)$ such that if $\vec{e}, \vec{f} \in A^N$ and $(\theta^{\mathbf{A}}(\vec{0}, \vec{e}) \vee \theta^{\mathbf{A}}(\vec{1}, \vec{f})) \cap (\theta^{\mathbf{A}}(\vec{1}, \vec{e}) \vee \theta^{\mathbf{A}}(\vec{0}, \vec{f})) = \Delta^{\mathbf{A}}$, then*

$$\theta^{\mathbf{A}}(\vec{0}, \vec{e}) \vee \theta^{\mathbf{A}}(\vec{1}, \vec{f}) = \{(a, b) : \mathbf{A} \models \varphi(\vec{e}, \vec{f}, a, b)\}.$$

(W5) *There are terms, v_i for $i = 0, \ldots, k$ with k even, such that the following identities hold in $\mathcal{V}$:*

$$v_0(\vec{z}, \vec{w}, x, y) = x, \quad \textit{and} \quad v_k(\vec{z}, \vec{w}, x, y) = y,$$
$$v_i(\vec{0}, \vec{1}, x, y) = v_{i+1}(\vec{0}, \vec{1}, x, y), \quad \textit{for } i \textit{ even},$$
$$v_i(\vec{1}, \vec{0}, x, y) = v_{i+1}(\vec{1}, \vec{0}, x, y), \quad \textit{for } i \textit{ odd},$$
$$v_i(\vec{0}, \vec{1}, x, x) = x, \quad \textit{for } i = 0, \ldots, k.$$

(W6) *If $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(\vec{z}, \vec{w}, x, y)$ and $\theta = \theta^{\mathbf{F}}(\vec{0}, \vec{z}) \vee \theta^{\mathbf{F}}(\vec{1}, \vec{w})$, then*

$$(x, y) \in \theta \vee ((\theta \vee \theta^{\mathbf{F}}(x, y)) \cap (\theta^{\mathbf{F}}(\vec{1}, \vec{z}) \vee \theta^{\mathbf{F}}(\vec{0}, \vec{w}))).$$

(W7) *The following conditions hold in $\mathcal{V}$.*

(*i*) *If $\sigma\colon \mathbf{A}_1 \times \mathbf{A}_2 \to \mathbf{B}_1 \times \mathbf{B}_2$ is a homomorphism (embedding) such that $\sigma([\vec{0}, \vec{1}]) = [\vec{0}, \vec{1}]$ and $\sigma([\vec{1}, \vec{0}]) = [\vec{1}, \vec{0}]$, then σ is left factorable.*

(*ii*) *If $\mathbf{S} \leq \mathbf{A}_1 \times \mathbf{A}_2$ and $\vec{e} = [\vec{0}, \vec{1}]$, $\vec{f} = [\vec{1}, \vec{0}]$ are in S^N, then*

$$\theta^{\mathbf{S}}(\vec{0}, \vec{e}) \vee \theta^{\mathbf{S}}(\vec{1}, \vec{f}) = \left.(\theta^{\mathbf{A}_1\times\mathbf{A}_2}(\vec{0}, \vec{e}) \vee \theta^{\mathbf{A}_1\times\mathbf{A}_2}(\vec{1}, \vec{f}))\right|_S.$$

(W8) *If $\mathbf{S} \leq \mathbf{A}_1 \times \mathbf{A}_2$ and $\vec{e} = [\vec{0}, \vec{1}]$, $\vec{f} = [\vec{1}, \vec{0}]$ are in S^N, then we have $\theta^{\mathbf{S}}(\vec{0}, \vec{e}) \vee \theta^{\mathbf{S}}(\vec{1}, \vec{f}) = \ker \pi_1|_S$.*

(W9) *If* $\mathbf{S} \leq \mathbf{A}_1 \times \mathbf{A}_2$, *if* $\mathbf{T} \leq \mathbf{B}_1 \times \mathbf{B}_2$, $[\vec{0}, \vec{1}], [\vec{1}, \vec{0}] \in S^N$, *and if* $\sigma \colon \mathbf{S} \to \mathbf{T}$ *is a homomorphism (isomorphism) such that* $\sigma([\vec{0}, \vec{1}]) = [\vec{0}, \vec{1}]$ *and* $\sigma([\vec{1}, \vec{0}]) = [\vec{1}, \vec{0}]$, *then* σ *is left factorable.*

(W10) $\mathcal{V}$ *has weak equationally definable decomposition operations.*

(W11) *There is an open formula satisfying* $(\mathrm{D_w})$.

(W12) *There are terms,* p_i, q_i *for* $i = 1, \ldots, n$, *such that*

$$\mathcal{V} \models (\bigwedge p_i(\vec{0}, \vec{1}, x, y, z) = q_i(\vec{0}, \vec{1}, x, y, z)) \leftrightarrow z = x,$$

$$\mathcal{V} \models (\bigwedge p_i(\vec{1}, \vec{0}, x, y, z) = q_i(\vec{1}, \vec{0}, x, y, z)) \leftrightarrow z = y.$$

(W13) *If* $\mathbf{S} \leq \mathbf{A}_1 \times \mathbf{A}_2$, $\mathbf{T} \leq \mathbf{B}_1 \times \mathbf{B}_2$, $[\vec{0}, \vec{1}], [\vec{1}, \vec{0}] \in S^N$ *and* $\sigma \colon \mathbf{S} \to \mathbf{T}$ *is a homomorphism (isomorphism) such that* $\sigma([\vec{0}, \vec{1}]) = [\vec{0}, \vec{1}]$ *and* $\sigma([\vec{1}, \vec{0}]) = [\vec{1}, \vec{0}]$, *then*

$$x, y, d^{\mathbf{A}_1 \times \mathbf{A}_2}(x, y) \in S \text{ implies } \sigma(d^{\mathbf{A}_1 \times \mathbf{A}_2}(x, y)) = d^{\mathbf{B}_1 \times \mathbf{B}_2}(\sigma(x), \sigma(y)).$$

Moreover, when the above equivalent conditions hold, we have that $\theta^{\mathbf{A}}_{\vec{0}\vec{e}} = \theta^{\mathbf{A}}(\vec{0}, \vec{e}) \vee \theta^{\mathbf{A}}(\vec{1}, \vec{f})$, *whenever* $\vec{e} \diamond_{\mathbf{A}} \vec{f}$.

Proof. The proof of the equivalence of (W1)-(W9): This is completely analogous to the proof of Theorem 3.3. The results quoted from [1] in the proof of Theorem 3.3 also are completely analogous to the ones required for the proof of the equivalence of (W1)-(W9).

(W1)⇒(W10): Let $\omega(\vec{z}, \vec{w}, x, y)$ be a $(\bigwedge p = q)$-formula satisfying (W) of the introduction. Since $\omega([\vec{0}, \vec{1}], [\vec{1}, \vec{0}], -, -)$ defines π_1 for every product $\mathbf{A} \times \mathbf{B}$, we have that $\omega([\vec{1}, \vec{0}], [\vec{0}, \vec{1}], -, -)$ defines π_2 for every product $\mathbf{A} \times \mathbf{B}$. Thus, $\delta(\vec{z}, \vec{w}, x, y, z) := \omega(\vec{z}, \vec{w}, x, z) \wedge \omega(\vec{w}, \vec{z}, z, y)$ is a $(\bigwedge p = q)$-formula satisfying $(\mathrm{D_w})$.

(W10)⇒(W11): This is rivial.

(W11)⇒(W12): Suppose $\delta(\vec{z}, \vec{w}, x, y, z)$ is an open formula satisfying $(\mathrm{D_w})$. Note that $\omega(\vec{z}, \vec{w}, x, y) := \delta(\vec{z}, \vec{w}, x, y, y)$ is an open formula which satisfies (W) of the introduction. Thus, by (W2)⇒(W1), there is a $(\bigwedge p = q)$-formula $\tilde{\omega}(\vec{z}, \vec{w}, x, y)$ which satisfies (W) of the introduction. Hence, we have that $\tilde{\delta}(\vec{z}, \vec{w}, x, y, z) := \tilde{\omega}(\vec{z}, \vec{w}, x, z) \wedge \tilde{\omega}(\vec{w}, \vec{z}, z, y)$ satisfies $(\mathrm{D_w})$. Let p_i, q_i be such that $\tilde{\delta}(\vec{z}, \vec{w}, x, y, z) := \bigwedge p_i(\vec{z}, \vec{w}, x, y, z) = q_i(\vec{z}, \vec{w}, x, y, z)$. We will prove that

$$\mathcal{V} \models (\bigwedge p_i(\vec{0}, \vec{1}, x, y, z) = q_i(\vec{0}, \vec{1}, x, y, z)) \leftrightarrow z = x.$$

Let $\mathbf{A} \in \mathcal{V}$ and let $\mathbf{T}$ be a trivial algebra. Let $a \in T$ and $\vec{a} = (a, \ldots, a) \in T^N$. We have that

$$\begin{aligned} \mathbf{A} \models \tilde{\delta}(\vec{0}, \vec{1}, x, y, z) &\text{ iff } \mathbf{A} \times \mathbf{T} \models \tilde{\delta}([\vec{0}, \vec{a}], [\vec{1}, \vec{a}], (x, a), (y, a), (z, a)) \\ &\text{ iff } \mathbf{A} \times \mathbf{T} \models \tilde{\delta}([\vec{0}, \vec{1}], [\vec{1}, \vec{0}], (x, a), (y, a), (z, a)) \\ &\text{ iff } (z, a) = d^{\mathbf{A} \times \mathbf{T}}((x, a), (y, a)) \\ &\text{ iff } (z, a) = (x, a) \text{ iff } z = x. \end{aligned}$$

Similarly, we can prove that $\mathcal{V} \models (\bigwedge p_i(\vec{1}, \vec{0}, x, y, z) = q_i(\vec{1}, \vec{0}, x, y, z)) \leftrightarrow z = y$.

(W12)⇒(W13): It is easy to check that

$$\delta(\vec{z},\vec{w},x,y,z) := \bigwedge p_i(\vec{z},\vec{w},x,y,z) = q_i(\vec{z},\vec{w},x,y,z)$$

satisfies ($\mathrm{D_w}$). So, (W13) holds since homomorphisms preserve ($\bigwedge p = q$)-formulas.

(W13)⇒(W9): Note that the condition

$$x, y, d^{\mathbf{A}_1\times\mathbf{A}_2}(x,y) \in S \text{ implies } \sigma(d^{\mathbf{A}_1\times\mathbf{A}_2}(x,y)) = d^{\mathbf{B}_1\times\mathbf{B}_2}(\sigma(x),\sigma(y))$$

guarantees that σ is left factorable. □

Remark 3.8. We note that the existence of terms $0_1,\dots,0_N$, $1_1,\dots,1_N$, and $v_0,\dots,v_k$ for k even, satisfying the identities in (W5), is a Maltsev condition for the property of being a variety with $\vec{0}$ and $\vec{1}$ having weak EDFC.

If $\vec{a}\in A^N$, $\vec{b}\in B^N$, and $\vec{c}\in C^N$, then we use $[\vec{a},\vec{b},\vec{c}]$ to denote the N-tuple $((a_1,b_1,c_1),\dots,(a_N,b_N,c_N)) \in (A\times B\times C)^N$.

Theorem 3.9 (Twice EDFC)**.** *Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$. The following are equivalent.*

(T1) *$\mathcal{V}$ has twice EDFC.*

(T2) *There are terms, p_i, q_i for $i = 1,\dots,n$, such that*

$$\mathcal{V} \models (\bigwedge p_i(\vec{0},\vec{1},x,y) = q_i(\vec{0},\vec{1},x,y)) \leftrightarrow x = y,$$
$$\mathcal{V} \models \bigwedge p_i(\vec{1},\vec{0},x,y) = q_i(\vec{1},\vec{0},x,y),$$
$$\mathcal{V} \models \bigwedge p_i(\vec{0},\vec{0},x,y) = q_i(\vec{0},\vec{0},x,y),$$
$$\mathcal{V} \models \bigwedge p_i(\vec{1},\vec{1},x,y) = q_i(\vec{1},\vec{1},x,y).$$

(T3) *There is a ($\bigwedge p = q$)-formula (open formula) $\varphi(\vec{z},\vec{w},x,y)$ such that $\varphi(\vec{z},\vec{1},x,y)$ satisfies* (L), *$\varphi(\vec{0},\vec{z},x,y)$ satisfies* (R), *and $\varphi(\vec{z},\vec{w},x,y)$ satisfies* (W) *in $\mathcal{V}$.*

(T4) *There is a ($\bigwedge p = q$)-formula $\varphi(\vec{z},\vec{w},x,y)$ such that*

$$\theta^{\mathbf{A}}(\vec{0},\vec{e}) \vee \theta^{\mathbf{A}}(\vec{1},\vec{f}) = \{(a,b) : \mathbf{A} \models \varphi(\vec{e},\vec{f},a,b)\},$$

whenever $\mathbf{A}\in\mathcal{V}$, $\vec{e},\vec{f}\in A^N$ and

$$(\theta^{\mathbf{A}}(\vec{0},\vec{e}) \vee \theta^{\mathbf{A}}(\vec{1},\vec{f})) \cap (\theta^{\mathbf{A}}(\vec{0},\vec{e}) \vee \theta^{\mathbf{A}}(\vec{0},\vec{f}))$$
$$\cap\ (\theta^{\mathbf{A}}(\vec{1},\vec{e}) \vee \theta^{\mathbf{A}}(\vec{1},\vec{f})) \cap (\theta^{\mathbf{A}}(\vec{1},\vec{e}) \vee \theta^{\mathbf{A}}(\vec{0},\vec{f})) = \Delta.$$

(T5) *There is a ($\bigwedge p = q$)-formula (open formula) $\varphi(\vec{z},\vec{w},x,y)$ such that, for any $\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3\in\mathcal{V}$,*

$$\mathbf{A}_1\times\mathbf{A}_2\times\mathbf{A}_3 \models \varphi([\vec{0},\vec{0},\vec{1}],[\vec{0},\vec{1},\vec{1}],x,y) \text{ iff } (x,y)\in\ker\pi_2.$$

(T6) *There are terms* v_i, $i = 0, \dots, k$, k *even, such that the following identities hold in* $\mathcal{V}$*:*

$$v_0(\vec{z}, \vec{w}, x, y) = x, \quad \textit{and} \quad v_k(\vec{z}, \vec{w}, x, y) = y,$$
$$v_i(\vec{0}, \vec{1}, x, y) = v_{i+1}(\vec{0}, \vec{1}, x, y), \textit{ for } i \textit{ even},$$
$$v_i(\vec{1}, \vec{0}, x, y) = v_{i+1}(\vec{1}, \vec{0}, x, y), \textit{ for } i \textit{ odd},$$
$$v_i(\vec{0}, \vec{0}, x, y) = v_{i+1}(\vec{0}, \vec{0}, x, y), \textit{ for } i \textit{ odd},$$
$$v_i(\vec{1}, \vec{1}, x, y) = v_{i+1}(\vec{1}, \vec{1}, x, y), \textit{ for } i \textit{ odd},$$
$$v_i(\vec{0}, \vec{1}, x, x) = x, \textit{ for } i = 0, \dots, k.$$

(T7) *If* $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(\vec{z}, \vec{w}, x, y)$, $\theta_{01} = \theta^{\mathbf{F}}(\vec{0}, \vec{z}) \vee \theta^{\mathbf{F}}(\vec{1}, \vec{w})$, $\theta_{10} = \theta^{\mathbf{F}}(\vec{1}, \vec{z}) \vee \theta^{\mathbf{F}}(\vec{0}, \vec{w})$, $\theta_{00} = \theta^{\mathbf{F}}(\vec{0}, \vec{z}) \vee \theta^{\mathbf{F}}(\vec{0}, \vec{w})$, *and* $\theta_{11} = \theta^{\mathbf{F}}(\vec{1}, \vec{z}) \vee \theta^{\mathbf{F}}(\vec{1}, \vec{w})$, *then*

$$(x, y) \in \theta_{01} \vee ((\theta_{01} \vee \theta^{\mathbf{F}}(x, y)) \cap \theta_{10} \cap \theta_{00} \cap \theta_{11}).$$

(T8) *The following conditions hold in* $\mathcal{V}$*.*
- (*i*) *If* $\sigma\colon \mathbf{A}_1 \times \mathbf{A}_2 \to \mathbf{B}_1 \times \mathbf{B}_2$ *is a homomorphism (embedding) such that* $\sigma([\vec{0}, \vec{1}]) = [\vec{0}, \vec{1}]$, *then* σ *is twice factorable.*
- (*ii*) *If* $\mathbf{S} \leq \mathbf{A} = \mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{A}_3$ *and* $\vec{e} = [\vec{0}, \vec{0}, \vec{1}]$, $\vec{f} = [\vec{0}, \vec{1}, \vec{1}]$ *are in* S^N, *then*

$$\theta^{\mathbf{S}}(\vec{0}, \vec{e}) \vee \theta^{\mathbf{S}}(\vec{1}, \vec{f}) = (\theta^{\mathbf{A}}(\vec{0}, \vec{e}) \vee \theta^{\mathbf{A}}(\vec{1}, \vec{f}))\Big|_S .$$

(T9) *If* $\mathbf{S} \leq \mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{A}_3$ *and* $\vec{e} = [\vec{0}, \vec{0}, \vec{1}]$, $\vec{f} = [\vec{0}, \vec{1}, \vec{1}]$ *are in* S^N, *then* $\theta^{\mathbf{S}}(\vec{0}, \vec{e}) \vee \theta^{\mathbf{S}}(\vec{1}, \vec{f}) = \ker \pi_2|_S$.

(T10) *If* $\mathbf{S} \leq \mathbf{A}_1 \times \mathbf{A}_2$, $\mathbf{T} \leq \mathbf{B}_1 \times \mathbf{B}_2$, *and* $\sigma\colon \mathbf{S} \to \mathbf{T}$ *is a homomorphism (isomorphism) such that* $[\vec{0}, \vec{1}] \in S^N$ *and* $\sigma([\vec{0}, \vec{1}]) = [\vec{0}, \vec{1}]$, *then* σ *is twice factorable.*

(T11) $\mathcal{V}$ *has equationally definable decomposition operations.*

(T12) *There is an open formula satisfying* (D)*.*

(T13) *There are terms,* p_i, q_i *for* $i = 1, \dots, n$, *such that*

$$\mathcal{V} \models (\bigwedge p_i(\vec{0}, x, y, z) = q_i(\vec{0}, x, y, z)) \leftrightarrow z = x,$$
$$\mathcal{V} \models (\bigwedge p_i(\vec{1}, x, y, z) = q_i(\vec{1}, x, y, z)) \leftrightarrow z = y.$$

(T14) *If* $\mathbf{S} \leq \mathbf{A}_1 \times \mathbf{A}_2$, $\mathbf{T} \leq \mathbf{B}_1 \times \mathbf{B}_2$, $[\vec{0}, \vec{1}] \in S^N$, *and* $\sigma\colon \mathbf{S} \to \mathbf{T}$ *is a homomorphism (isomorphism) such that* $\sigma([\vec{0}, \vec{1}]) = [\vec{0}, \vec{1}]$, *then*

$$x, y, d^{\mathbf{A}_1 \times \mathbf{A}_2}(x, y) \in S \textit{ implies } \sigma(d^{\mathbf{A}_1 \times \mathbf{A}_2}(x, y)) = d^{\mathbf{B}_1 \times \mathbf{B}_2}(\sigma(x), \sigma(y)).$$

Moreover, when the above equivalent conditions hold, if $\vec{e}, \vec{f} \in Z(\mathbf{A})$, *then* $\theta^{\mathbf{A}}_{\vec{0}(\vec{e} \vee c(\vec{f}))} = \theta^{\mathbf{A}}(\vec{0}, \vec{e}) \vee \theta^{\mathbf{A}}(\vec{1}, \vec{f})$.

Proof. (T1)⇒(T2): By (L3) and (R3), we know there are terms u_l, v_l, r_j, s_j such that

$$\mathcal{V} \models (\bigwedge u_l(\vec{0}, x, y) = v_l(\vec{0}, x, y)) \leftrightarrow x = y, \quad \mathcal{V} \models \bigwedge u_l(\vec{1}, x, y) = v_l(\vec{1}, x, y),$$

$$\mathcal{V} \models (\bigwedge r_j(\vec{1}, x, y) = s_j(\vec{1}, x, y)) \leftrightarrow x = y, \quad \mathcal{V} \models \bigwedge r_j(\vec{0}, x, y) = s_j(\vec{0}, x, y).$$

Let

$$\psi(\vec{z}, \vec{w}, x, y) := \bigwedge_{j,l} u_l(\vec{z}, r_j(\vec{w}, x, y), s_j(\vec{w}, x, y)) = v_l(\vec{z}, r_j(\vec{w}, x, y), s_j(\vec{w}, x, y))$$

and observe that

$$\mathcal{V} \models \psi(\vec{0}, \vec{w}, x, y) \leftrightarrow \bigwedge r_j(\vec{w}, x, y) = s_j(\vec{w}, x, y), \qquad \mathcal{V} \models \psi(\vec{1}, \vec{w}, x, y).$$

So (T2) easily follows by taking p_i, q_i such that

$$\psi(\vec{z}, \vec{w}, x, y) := \bigwedge p_i(\vec{z}, \vec{w}, x, y) = q_i(\vec{z}, \vec{w}, x, y).$$

(T2)⇒(T3): Take $\varphi(\vec{z}, \vec{w}, x, y) := \bigwedge p_i(\vec{z}, \vec{w}, x, y) = q_i(\vec{z}, \vec{w}, x, y)$.

(T3)⇒(T1): Assume that the following condition, i.e., the open case of (T3), holds.

$(\mathrm{T3})_o$ There is an open formula $\psi(\vec{z}, \vec{w}, x, y)$ such that $\psi(\vec{z}, \vec{1}, x, y)$ satisfies (L), $\psi(\vec{0}, \vec{z}, x, y)$ satisfies (R), and $\psi(\vec{z}, \vec{w}, x, y)$ satisfies (W) in $\mathcal{V}$.

By (L2)⇒(L1) of Theorem 3.3 and (R2)⇒(R1) of Theorem 3.5, we have that (T1) holds.

(T2)⇒(T4): By (T2), there is a $(\bigwedge p = q)$-formula $\varphi(\vec{z}, \vec{w}, x, y)$ such that

$$\mathcal{V} \models \varphi(\vec{0}, \vec{1}, x, y) \leftrightarrow x = y, \tag{3.1}$$

$$\mathcal{V} \models \varphi(\vec{0}, \vec{0}, x, y) \wedge \varphi(\vec{1}, \vec{0}, x, y) \wedge \varphi(\vec{1}, \vec{1}, x, y). \tag{3.2}$$

Let $\vec{e}, \vec{f} \in A^N$ and define

$$\theta_{01} = \theta^{\mathbf{A}}(\vec{0}, \vec{e}) \vee \theta^{\mathbf{A}}(\vec{1}, \vec{f}), \qquad \theta_{00} = \theta^{\mathbf{A}}(\vec{0}, \vec{e}) \vee \theta^{\mathbf{A}}(\vec{0}, \vec{f}),$$

$$\theta_{11} = \theta^{\mathbf{A}}(\vec{1}, \vec{e}) \vee \theta^{\mathbf{A}}(\vec{1}, \vec{f}), \qquad \theta_{10} = \theta^{\mathbf{A}}(\vec{1}, \vec{e}) \vee \theta^{\mathbf{A}}(\vec{0}, \vec{f}).$$

Let $\theta_{01} \cap \theta_{00} \cap \theta_{11} \cap \theta_{10} = \Delta$. We claim $\theta_{01} = \{(a, b) : \mathbf{A} \models \varphi(\vec{e}, \vec{f}, a, b)\}$ holds. Let $\mathbf{A} \models \varphi(\vec{e}, \vec{f}, a, b)$. So we have $\mathbf{A}/\theta_{01} \models \varphi(\vec{e}/\theta_{01}, \vec{f}/\theta_{01}, a/\theta_{01}, b/\theta_{01})$, and therefore $\mathbf{A}/\theta_{01} \models \varphi(\vec{0}, \vec{1}, a/\theta_{01}, b/\theta_{01})$, which by (3.1) implies that $(a, b) \in \theta_{01}$. Now assume $(a, b) \in \theta_{01}$. By (3.1), we have that $\mathbf{A} \models \varphi(\vec{0}, \vec{1}, a, a)$, and hence $\mathbf{A}/\theta_{01} \models \varphi(\vec{e}/\theta_{01}, \vec{f}/\theta_{01}, a/\theta_{01}, b/\theta_{01})$. Similarly, we can use (3.2) to prove that $\mathbf{A}/\delta \models \varphi(\vec{e}/\delta, \vec{f}/\delta, a/\delta, b/\delta)$, for $\delta = \theta_{00}, \theta_{11}, \theta_{10}$. Since $\theta_{01} \cap \theta_{00} \cap \theta_{11} \cap \theta_{10} = \Delta$ and φ is a conjunction of equations, we have that $\mathbf{A} \models \varphi(\vec{e}, \vec{f}, a, b)$.

(T4)⇒(T1): Let $\varphi(\vec{z}, \vec{w}, x, y)$ be an open formula satisfying (T4). Note that $\varphi(\vec{z}, \vec{1}, x, y)$ satisfies (L4), and hence $\mathcal{V}$ has left EDFC. Similarly, we have that $\varphi(\vec{0}, \vec{z}, x, y)$ satisfies (R4), and hence $\mathcal{V}$ has right EDFC.

(T5)⇒(T1): Assume the open version of (T5) holds, i.e., there is an open formula $\varphi(\vec{z}, \vec{w}, x, y)$ such that for any $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 \in \mathcal{V}$,

$$\mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{A}_3 \models \varphi([\vec{0}, \vec{0}, \vec{1}], [\vec{0}, \vec{1}, \vec{1}], (a_1, a_2, a_3), (b_1, b_2, b_3)) \text{ iff } a_2 = b_2.$$

Taking $\mathbf{A}_1$ to be a trivial algebra, we obtain that for any $\mathbf{A}_2, \mathbf{A}_3 \in \mathcal{V}$,

$$\mathbf{A}_2 \times \mathbf{A}_3 \models \varphi([\vec{0}, \vec{1}], [\vec{1}, \vec{1}], (a_2, a_3), (b_2, b_3)) \text{ iff } a_2 = b_2,$$

which says that (L2) of Theorem 3.3 holds, and hence $\mathcal{V}$ has left EDFC. Taking $\mathbf{A}_3$ to be a trivial algebra, we obtain that for any $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{V}$,

$$\mathbf{A}_1 \times \mathbf{A}_2 \models \varphi([\vec{0}, \vec{0}], [\vec{0}, \vec{1}], (a_1, a_2), (b_1, b_2)) \text{ iff } a_2 = b_2,$$

which says that (R2) of Theorem 3.5 holds, and hence $\mathcal{V}$ has right EDFC.

(T2)⇒(T5): Take $\varphi(\vec{z}, \vec{w}, x, y) := \bigwedge p_i(\vec{z}, \vec{w}, x, y) = q_i(\vec{z}, \vec{w}, x, y)$.

(T6)⇒(T3): Take $\varphi(\vec{z}, \vec{w}, x, y) := \bigwedge_{i \text{ odd}} v_i(\vec{z}, \vec{w}, x, y) = v_{i+1}(\vec{z}, \vec{w}, x, y)$.

(T1)⇒(T6): By (L5) of Theorem 3.3 and (R5) of Theorem 3.5, there are terms p_i, q_j, $i = 0, \ldots, k$, $j = 0, \ldots, m$, with k and m even, such that the following identities hold in $\mathcal{V}$:

$$\begin{aligned}
&p_0(\vec{z}, x, y) = x, && q_0(\vec{z}, x, y) = x,\\
&p_k(\vec{z}, x, y) = y, && q_m(\vec{z}, x, y) = y,\\
&p_i(\vec{0}, x, y) = p_{i+1}(\vec{0}, x, y) \text{ for } i \text{ even}, && q_i(\vec{1}, x, y) = q_{i+1}(\vec{1}, x, y) \text{ for } i \text{ even},\\
&p_i(\vec{1}, x, y) = p_{i+1}(\vec{1}, x, y) \text{ for } i \text{ odd}, && q_i(\vec{0}, x, y) = q_{i+1}(\vec{0}, x, y)\ i \text{ odd},\\
&p_i(\vec{0}, x, x) = x \text{ for } i = 0, \ldots, k, && q_i(\vec{1}, x, x) = x \text{ for } i = 0, \ldots, m.
\end{aligned}$$

Let v_i be the terms defined as follows:

$$\begin{aligned}
v_0(\vec{z}, \vec{w}, x, y) &= x,\\
v_i(\vec{z}, \vec{w}, x, y) &= p_i(\vec{z}, q_1(\vec{w}, x, y), q_2(\vec{w}, x, y)) \text{ for } i = 1, \ldots, k,\\
v_{k+i}(\vec{z}, \vec{w}, x, y) &= p_i(\vec{z}, q_3(\vec{w}, x, y), q_4(\vec{w}, x, y)) \text{ for } i = 1, \ldots, k,\\
v_{2k+i}(\vec{z}, \vec{w}, x, y) &= p_i(\vec{z}, q_5(\vec{w}, x, y), q_6(\vec{w}, x, y)) \text{ for } i = 1, \ldots, k,\\
&\ \vdots\\
v_{(\frac{m}{2}-1)k+i}(\vec{z}, \vec{w}, x, y) &= p_i(\vec{z}, q_{m-1}(\vec{w}, x, y), q_m(\vec{w}, x, y)) \text{ for } i = 1, \ldots, k.
\end{aligned}$$

It is easy to check that the terms $v_0, \ldots, v_{\frac{m}{2}k}$ satisfy the equations in (T6).

(T6)⇔(T7): This is similar to (L5)⇔(L6) of Theorem 3.3.

(T8)⇒(T1): Note that taking in (T8)(*ii*) the factor $\mathbf{A}_1$ to be a trivial algebra, we obtain (L7), and hence by Theorem 3.3, (T8) implies (L1). Similarly, taking in (T8)(*ii*) the factor $\mathbf{A}_3$ to be a trivial algebra, we obtain (R7), and hence by Theorem 3.5, (T8) implies (R1).

(T4)⇒(T8): This is similar to the proof of (T4)⇒(T9) (see below).

(T9)⇒(T1): This is similar to the proof of (T8)⇒(T1).

(T4)⇒(T9): Let $\varphi(\vec{z}, \vec{w}, x, y)$ be a $(\bigwedge p = q)$-formula witnessing (T4). Since $\theta^{\mathbf{S}}(\vec{0}, \vec{e}) \vee \theta^{\mathbf{S}}(\vec{0}, \vec{f}) \subseteq \ker \pi_1$, $\theta^{\mathbf{S}}(\vec{0}, \vec{e}) \vee \theta^{\mathbf{S}}(\vec{1}, \vec{f}) \subseteq \ker \pi_2$, and $\theta^{\mathbf{S}}(\vec{1}, \vec{e}) \vee \theta^{\mathbf{S}}(\vec{1}, \vec{f}) \subseteq \ker \pi_3$, we have that

$$(\theta^{\mathbf{S}}(\vec{0}, \vec{e}) \vee \theta^{\mathbf{S}}(\vec{1}, \vec{f})) \cap (\theta^{\mathbf{S}}(\vec{0}, \vec{e}) \vee \theta^{\mathbf{S}}(\vec{0}, \vec{f})) \cap (\theta^{\mathbf{S}}(\vec{1}, \vec{e}) \vee \theta^{\mathbf{S}}(\vec{1}, \vec{f})) \cap (\theta^{\mathbf{S}}(\vec{1}, \vec{e}) \vee \theta^{\mathbf{S}}(\vec{0}, \vec{f})) = \Delta,$$

and hence

$$\theta^{\mathbf{S}}(\vec{0}, \vec{e}) \vee \theta^{\mathbf{S}}(\vec{1}, \vec{f}) = \{(a, b) : \mathbf{S} \models \varphi(\vec{e}, \vec{f}, a, b)\}.$$

Similarly, we can prove that

$$\theta^{\mathbf{A}}(\vec{0},\vec{e}) \vee \theta^{\mathbf{A}}(\vec{1},\vec{f}) = \{(a,b) : \mathbf{A} \models \varphi(\vec{e},\vec{f},a,b)\}.$$

Of course, this assures that $\theta^{\mathbf{S}}(\vec{0},\vec{e}) \vee \theta^{\mathbf{S}}(\vec{1},\vec{f}) = \theta^{\mathbf{A}}(\vec{0},\vec{e}) \vee \theta^{\mathbf{A}}(\vec{1},\vec{f})\Big|_S$. Thus, we only need to prove that $\theta^{\mathbf{A}}(\vec{0},\vec{e}) \vee \theta^{\mathbf{A}}(\vec{1},\vec{f}) = \ker \pi_2$. Since we have proved that (T4)⇒(T1), Theorems 3.3 and 3.5 say that $\theta^{\mathbf{A}}_{\vec{0}\vec{g}} = \theta^{\mathbf{A}}(\vec{0},\vec{g})$ and $\theta^{\mathbf{A}}_{\vec{1}\vec{g}} = \theta^{\mathbf{A}}(\vec{1},\vec{g})$, whenever $\vec{g} \in Z(\mathbf{A})$. Also, we note that $\vec{e},\vec{f} \in Z(\mathbf{A})$. Thus, we have

$$\begin{aligned}
\theta^{\mathbf{A}}(\vec{0},\vec{e}) \vee \theta^{\mathbf{A}}(\vec{1},\vec{f}) &= \theta^{\mathbf{A}}([\vec{0},\vec{0},\vec{0}],[\vec{0},\vec{0},\vec{1}]) \vee \theta^{\mathbf{A}}([\vec{1},\vec{1},\vec{1}],[\vec{0},\vec{1},\vec{1}]) \\
&= \theta^{\mathbf{A}}_{[\vec{0},\vec{0},\vec{0}][\vec{0},\vec{0},\vec{1}]} \vee \theta^{\mathbf{A}}_{[\vec{1},\vec{1},\vec{1}][\vec{0},\vec{1},\vec{1}]} \\
&= (\theta^{\mathbf{A}_1}_{\vec{0}\vec{0}} \times \theta^{\mathbf{A}_2}_{\vec{0}\vec{0}} \times \theta^{\mathbf{A}_3}_{\vec{0}\vec{1}}) \vee (\theta^{\mathbf{A}_1}_{\vec{1}\vec{0}} \times \theta^{\mathbf{A}_2}_{\vec{1}\vec{1}} \times \theta^{\mathbf{A}_3}_{\vec{1}\vec{1}}) \\
&= (\Delta \times \Delta \times \nabla) \vee (\nabla \times \Delta \times \Delta) = \ker \pi_2.
\end{aligned}$$

(T10)⇔(T1): Note that (T10) is the conjunction of (L9) and (R9). Thus, Theorems 3.3 and 3.5 say that (T10) and (T1) are equivalent conditions.

(T1)⇒(T11): Let $\lambda(\vec{z},x,y)$ be a $(\bigwedge p = q)$-formula satisfying (L) of the introduction and let $\rho(\vec{z},x,y)$ be a $(\bigwedge p = q)$-formula satisfying (R) of the introduction. Note that $\delta(\vec{z},x,y,z) := \lambda(\vec{z},x,z) \wedge \rho(\vec{z},z,y)$ is a $(\bigwedge p = q)$-formula satisfying (D).

(T11)⇒(T12): This is trivial.

(T12)⇒(T13): This is similar to the proof of (W11)⇒(W12) of Theorem 3.7.

(T13)⇒(T14): It is easy to check that

$$\delta(\vec{z},x,y,z) := \bigwedge p_i(\vec{z},x,y,z) = q_i(\vec{z},x,y,z)$$

satisfies (D). So, (T14) holds as homomorphisms preserve $(\bigwedge p = q)$-formulas.

(T14)⇒(T10): Note that the condition

$$x, y, d^{\mathbf{A}_1\times\mathbf{A}_2}(x,y) \in S \text{ implies } \sigma(d^{\mathbf{A}_1\times\mathbf{A}_2}(x,y)) = d^{\mathbf{B}_1\times\mathbf{B}_2}(\sigma(x),\sigma(y))$$

guarantees that σ is twice factorable.

The proof that (T1) implies $\theta^{\mathbf{A}}_{\vec{0}(\vec{e}\vee c(\vec{f}))} = \theta^{\mathbf{A}}(\vec{0},\vec{e}) \vee \theta^{\mathbf{A}}(\vec{1},\vec{f})$ whenever we have $\vec{e},\vec{f} \in Z(\mathbf{A})$ is left to the reader. □

Remark 3.10. It is easy to check that if a variety $\mathcal{V}$ has terms $0_1,\dots,0_N$, $1_1,\dots,1_N$, and $v_0,\dots,v_k$ for k even, satisfying the identities in (T6), then $\mathcal{V} \models \vec{0} = \vec{1} \to x = y$ and the formula

$$\alpha(\vec{z},\vec{w},x,y) := \bigwedge_{i \text{ odd}} v_i(\vec{z},\vec{w},x,y) = v_{i+1}(\vec{z},\vec{w},x,y)$$

is such that $\alpha(\vec{z},\vec{1},x,y)$ satisfies (L) and $\alpha(\vec{0},\vec{z},x,y)$ satisfies (R). Thus, the existence of terms $0_1,\dots,0_N$, $1_1,\dots,1_N$, and $v_0,\dots,v_k$ for k even, satisfying the identities in (T6), is a Maltsev condition for the property of being a variety with $\vec{0}$ and $\vec{1}$ having twice EDFC.

Under the assumption that the variety $\mathcal{V}$ has DFC, some of the conditions in Theorems 3.3, 3.5, 3.7, and 3.9 can be written in an intrinsic manner, which could be useful in some circumstances. For example, the intrinsic versions of (L1), (L7), and (L9) can be written as follows.

(L1)′ There is a $(\bigwedge p = q)$-formula which defines $\theta^{\mathbf{A}}_{\vec{0}\vec{e}}$ in $\mathcal{V}$.

(L7)′ The following conditions hold in $\mathcal{V}$.

(*i*) If $\sigma\colon \mathbf{A} \to \mathbf{B}$ is a homomorphism, $\vec{e} \in Z(\mathbf{A})$, and $\sigma(\vec{e}) \in Z(\mathbf{B})$, then $(x, y) \in \theta^{\mathbf{A}}_{\vec{0}\vec{e}}$ implies $(\sigma(x), \sigma(y)) \in \theta^{\mathbf{B}}_{\vec{0}\sigma(\vec{e})}$.

(*ii*) If $\mathbf{S} \leq \mathbf{A}$ and $\vec{e} \in Z(\mathbf{A})$ with $\vec{e} \in S^N$, then

$$\theta^{\mathbf{S}}(\vec{0}, \vec{e}) = \theta^{\mathbf{A}}(\vec{0}, \vec{e})\Big|_S .$$

(L9)′ If $\mathbf{S} \leq \mathbf{A}$, $\mathbf{T} \leq \mathbf{B}$, $\vec{e} \in Z(\mathbf{A})$, with $\vec{e} \in S^N$ and $\sigma\colon \mathbf{S} \to \mathbf{T}$ is an homomorphism such that $\sigma(\vec{e}) \in Z(\mathbf{B})$, then $(x, y) \in \theta^{\mathbf{A}}_{\vec{0}\vec{e}}\Big|_S$ implies $(\sigma(x), \sigma(y)) \in \theta^{\mathbf{B}}_{\vec{0}\sigma(\vec{e})}$.

4. Definability of "$\vec{e} \in Z(\mathbf{A})$" and "$\vec{e} \diamond_{\mathbf{A}} \vec{f}$"

Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$ and let $\mathcal{L}$ be the language of $\mathcal{V}$. We say that a set of first order $\mathcal{L}$-formulas $\{\varphi_r(\vec{z}) : r \in R\}$ *defines the property* "$\vec{e} \in Z(\mathbf{A})$" *in* $\mathcal{V}$ if for every $\mathbf{A} \in \mathcal{V}$ and $\vec{e} \in A^N$, we have that $\vec{e} \in Z(\mathbf{A})$ iff $\mathbf{A} \models \varphi_r(\vec{e})$, for every $r \in R$. We say that a set $\{\varphi_r(\vec{z}, \vec{w}) : r \in R\}$ *defines the property* "$\vec{e} \diamond_{\mathbf{A}} \vec{f}$" *in* $\mathcal{V}$ if for every $\mathbf{A} \in \mathcal{V}$ and $\vec{e}, \vec{f} \in A^N$, we have that $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ iff $\mathbf{A} \models \varphi_r(\vec{e}, \vec{f})$, for every $r \in R$. In [1, Proposition 3.4], we give, for the case of a variety with left EDFC, sets of formulas which define the properties "$\vec{e} \in Z(\mathbf{A})$" and "$\vec{e} \diamond_{\mathbf{A}} \vec{f}$". In such work, we also show that these axiomatizations are optimal in the sense of the complexity of the formulas. In this section, we will do the same thing for the weak and twice cases. Since left EDFC and right EDFC are notions which are each dual of the other, a simple translation produces optimal axiomatizations for the properties "$\vec{e} \in Z(\mathbf{A})$" and "$\vec{e} \diamond_{\mathbf{A}} \vec{f}$" in the right case.

Definability of "$\vec{e} \diamond_{\mathbf{A}} \vec{f}$" for the weak EDFC case. Let $\omega(\vec{z}, \vec{w}, x, y)$ be an $\mathcal{L}$-formula. Define

$$Ref^L(\vec{z}, \vec{w}) = \forall x\ \omega(\vec{z}, \vec{w}, x, x),$$

$$Sym^L(\vec{z}, \vec{w}) = \forall x, y\ (\omega(\vec{z}, \vec{w}, x, y) \to \omega(\vec{z}, \vec{w}, y, x)),$$

$$Tra^L(\vec{z}, \vec{w}) = \forall x, y, z\ (\omega(\vec{z}, \vec{w}, x, y) \wedge \omega(\vec{z}, \vec{w}, y, z) \to \omega(\vec{z}, \vec{w}, x, z)),$$

$$Exis(\vec{z}, \vec{w}) = \forall x, y \exists z\ \omega(\vec{z}, \vec{w}, x, z) \wedge \omega(\vec{w}, \vec{z}, z, y),$$

$$Int(\vec{z}, \vec{w}) = \forall x, y\ (\omega(\vec{z}, \vec{w}, x, y) \wedge \omega(\vec{w}, \vec{z}, x, y) \to x = y),$$

$$Bel(\vec{z}, \vec{w}) = \bigwedge_{i=1}^{N} \omega(\vec{z}, \vec{w}, 0_i, z_i) \wedge \omega(\vec{z}, \vec{w}, 1_i, w_i) \wedge \omega(\vec{w}, \vec{z}, 0_i, w_i) \wedge \omega(\vec{w}, \vec{z}, 1_i, z_i).$$

For each n-ary function symbol $F \in \mathcal{L}$, define

$$Pres_F^L(\vec{z}) = \forall x_1, y_1, \ldots, x_n, y_n \bigwedge_{j=1}^{n} \omega(\vec{z}, \vec{w}, x_j, y_j) \rightarrow \omega(\vec{z}, \vec{w}, F(\vec{x}), F(\vec{y})).$$

Finally, define Ref^R, Sym^R, Tra^R, and $Pres_F^R$ to be the result of interchanging $\omega(\vec{z}, \vec{w}, -, -)$ with $\omega(\vec{w}, \vec{z}, -, -)$ in Ref^L, Sym^L, Tra^L, and $Pres_F^L$, respectively. Let Σ_ω denote the set consisting of the following formulas

$$Bel, Exis, Int, Ref^L, Sym^L, Tra^L, Ref^R, Sym^R, Tra^R,$$
$$Pres_F^L, Pres_F^R, \text{ for } F \in \mathcal{L}.$$

Proposition 4.1. *Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$ and let $\mathcal{L}$ be the language of $\mathcal{V}$. Suppose that $\omega(\vec{z}, \vec{w}, x, y)$ satisfies* (W) *of the introduction. The set Σ_ω defines "$\vec{e} \diamond_{\mathbf{A}} \vec{f}$" in $\mathcal{V}$. So if $\mathcal{V}$ has weak EDFC, then the property "$\vec{e} \diamond_{\mathbf{A}} \vec{f}$" is definable in $\mathcal{V}$ by a set of $\forall\exists$-formulas.*

Proof. We note that $\omega(\vec{w}, \vec{z}, x, y)$ defines $(\theta^{\mathbf{A}}_{\vec{0}\vec{e}})^* = \theta^{\mathbf{A}}_{\vec{1}\vec{e}}$ in $\mathcal{V}$. Thus, $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ iff $L = \{(x, y) : \mathbf{A} \models \omega(\vec{e}, \vec{f}, x, y)\}$ and $R = \{(x, y) : \mathbf{A} \models \omega(\vec{f}, \vec{e}, x, y)\}$ are a pair of complementary factor congruences and $\vec{e} \equiv \vec{0}(L)$, $\vec{e} \equiv \vec{1}(R)$, $\vec{f} \equiv \vec{1}(L)$, and $\vec{f} \equiv \vec{0}(R)$. The axioms

$$Ref^L(\vec{e}, \vec{f}), Sym^L(\vec{e}, \vec{f}), Tra^L(\vec{e}, \vec{f}), Ref^R(\vec{e}, \vec{f}), Sym^R(\vec{e}, \vec{f}), Tra^R(\vec{e}, \vec{f})$$
$$Pres_F^L(\vec{e}, \vec{f}), Pres_F^R(\vec{e}, \vec{f}), \text{ for } F \in \mathcal{L}$$

say that L and R are congruences. Also, $Exis(\vec{e}, \vec{f})$ says that $L \circ R = \nabla^{\mathbf{A}}$, $Int(\vec{e}, \vec{f})$ says that $L \cap R = \Delta^{\mathbf{A}}$, and $Bel(\vec{e}, \vec{f})$ says that $\vec{e} \equiv \vec{0}(L)$, $\vec{e} \equiv \vec{1}(R)$, $\vec{f} \equiv \vec{1}(L)$, and $\vec{f} \equiv \vec{0}(R)$. □

As defined in Section 2, we use $\mathcal{V}_{DI}$ to denote the class of directly indecomposable algebras in $\mathcal{V}$.

Corollary 4.2. *If $\mathcal{V}$ has DFC* (*resp. weak EDFC*) *and the language of $\mathcal{V}$ is finite, then $\mathcal{V}_{DI}$ is a first order class* (*resp. $\forall\exists\forall$-first order class*).

Proof. Suppose that $\omega(\vec{z}, \vec{w}, x, y)$ satisfies (W) of the introduction. Since the language of $\mathcal{V}$ is finite, so is Σ_ω. By the above proposition, Σ_ω defines "$\vec{e} \diamond_{\mathbf{A}} \vec{f}$" in $\mathcal{V}$ and hence

$$\vec{0} \neq \vec{1} \wedge \forall \vec{z}, \vec{w} \ (\bigwedge \Sigma_\omega \rightarrow (\vec{z} = \vec{0} \vee \vec{z} = \vec{1}))$$

says "$\mathbf{A}$ is directly indecomposable". When ω is a conjunction of equations, the above formula is $\forall\exists\forall$. □

The axiomatization of Proposition 4.1 is in fact optimal, as it will be shown in Lemma 4.5.

Definability of "$\vec{e} \in Z(\mathbf{A})$" for the weak EDFC case. Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$ and let $\mathcal{L}$ be the language of $\mathcal{V}$. Suppose that $\omega(\vec{z}, \vec{w}, x, y)$ is a conjunction of equations satisfying (W) in the introduction. When $\mathcal{L}$ is finite, Proposition 4.1 gives a finite set of $\forall\exists$-formulas axiomatizing "$\vec{e} \diamond_{\mathbf{A}} \vec{f}$" and hence we can use the fact that $\vec{e} \in Z(\mathbf{A})$ iff $\exists \vec{f} \in A^N \; \vec{e} \diamond_{\mathbf{A}} \vec{f}$ to give a formula defining the property "$\vec{e} \in Z(\mathbf{A})$". The formula obtained is $\exists\forall\exists$ and it is not optimal in the sense of its quantificational complexity. As we will see, the property "$\vec{e} \in Z(\mathbf{A})$" can be defined using a set of formulas which are either $\forall\exists$ or $\exists\forall$, even for the case in which the language is infinite.

In order to give the optimal axiomatization of "$\vec{e} \in Z(\mathbf{A})$", we shall need formulas $\lambda(\vec{z}, x, y)$ and $\rho(\vec{z}, x, y)$ defining $\theta^{\mathbf{A}}_{\vec{0}\vec{e}}$ and $\theta^{\mathbf{A}}_{\vec{1}\vec{e}}$, respectively. Of course, if $\mathcal{L}$ is finite, we can take $\lambda(\vec{z}, x, y)$ to be a formula saying

$$\exists \vec{w} (\vec{z} \diamond \vec{w} \wedge \omega(\vec{z}, \vec{w}, x, y))$$

and $\rho(\vec{z}, x, y)$ to be a formula saying

$$\exists \vec{w} (\vec{z} \diamond \vec{w} \wedge \omega(\vec{w}, \vec{z}, x, y)).$$

However, λ and ρ are $\exists\forall\exists$-formulas and things can be done much better, as is shown in the following proposition which holds for an arbitrary language.

Proposition 4.3. *Let $\mathcal{V}$ be a variety with weak EDFC and suppose that*

$$\omega(\vec{z}, \vec{w}, x, y) := \bigwedge\nolimits_{i=1}^{n} p_i(\vec{z}, \vec{w}, x, y) = q_i(\vec{z}, \vec{w}, x, y)$$

defines $\theta^{\mathbf{A}}_{\vec{0}\vec{e}}$ in $\mathcal{V}$. Define

$$\beta(\vec{z}, \vec{w}, \vec{x}, \vec{y}, x, y) := \bigwedge\nolimits_{i=1}^{n} \omega(\vec{z}, \vec{w}, p_i(\vec{x}, \vec{y}, x, y), q_i(\vec{x}, \vec{y}, x, y)).$$

Then the formula

$$\lambda(\vec{z}, x, y) := \forall \vec{w} \; \beta(\vec{z}, \vec{w}, \vec{z}, \vec{1}, x, x) \wedge \beta(\vec{z}, \vec{w}, \vec{1}, \vec{z}, x, y) \to \beta(\vec{z}, \vec{w}, \vec{z}, \vec{1}, x, y)$$

defines $\theta^{\mathbf{A}}_{\vec{0}\vec{e}}$ in $\mathcal{V}$ and the formula

$$\rho(\vec{z}, x, y) := \forall \vec{w} \; \beta(\vec{w}, \vec{z}, \vec{0}, \vec{z}, x, x) \wedge \beta(\vec{w}, \vec{z}, \vec{z}, \vec{0}, x, y) \to \beta(\vec{w}, \vec{z}, \vec{0}, \vec{z}, x, y)$$

defines $\theta^{\mathbf{A}}_{\vec{1}\vec{e}}$ in $\mathcal{V}$.

Proof. We will prove that ρ defines $\theta^{\mathbf{A}}_{\vec{0}\vec{e}}$ in $\mathcal{V}$ i.e., that ρ satisfies (R) of the introduction. Let $\mathbf{A}, \mathbf{B} \in \mathcal{V}$. Since ρ is preserved by direct products, it is easy to check that $\mathbf{A} \times \mathbf{B} \models \rho([\vec{0}, \vec{1}], (a, b), (a', b))$, for every $a, a' \in A$ and $b \in B$. Using that $\mathcal{V} \models \omega(\vec{0}, \vec{1}, x, x) \wedge \omega(\vec{1}, \vec{0}, x, y)$, we can easily prove the following result.

(1) For every $a, a' \in A$ and $b, b' \in B$, we have that

$$\mathbf{A} \times \mathbf{B} \models \beta([\vec{1}, \vec{0}], [\vec{0}, \vec{1}], [\vec{0}, \vec{0}], [\vec{0}, \vec{1}], (a, b), (a, b)),$$
$$\mathbf{A} \times \mathbf{B} \models \beta([\vec{1}, \vec{0}], [\vec{0}, \vec{1}], [\vec{0}, \vec{1}], [\vec{0}, \vec{0}], (a, b), (a', b')).$$

Now suppose that $\mathbf{A}\times\mathbf{B} \models \rho([\vec{0},\vec{1}],(a,b),(a',b'))$. We will prove that $b=b'$. Note that (1) says that the antecedent in ρ, for $\vec{w}=[\vec{1},\vec{0}]$, holds, which implies that $\mathbf{A}\times\mathbf{B} \models \beta([\vec{1},\vec{0}],[\vec{0},\vec{1}],[\vec{0},\vec{0}],[\vec{0},\vec{1}],(a,b),(a',b'))$. In particular we obtain that $\mathbf{B} \models \beta(\vec{0},\vec{1},\vec{0},\vec{1},b,b')$. Now using that $\mathcal{V} \models \omega(\vec{0},\vec{1},x,y) \to x=y$, we can easily prove that $b=b'$.

By using an analogous argument, we can prove that λ defines $\theta^{\mathbf{A}}_{\vec{0}\vec{e}}$ in $\mathcal{V}$. □

Proposition 4.4. *Let $\mathcal{V}$ be a variety with weak EDFC.*

(a) *There is a set of $\forall\exists\forall$-formulas which defines "$\vec{e}\in Z(\mathbf{A})$" in $\mathcal{V}$.*
(b) *There is a set Σ_1 of $\forall\exists$-formulas and a set Σ_2 of $\exists\forall$-formulas such that $\Sigma_1\cup\Sigma_2$ defines "$\vec{e}\in Z(\mathbf{A})$" in $\mathcal{V}$.*

Proof. Let $\omega(\vec{z},\vec{w},x,y)$ be a $(\bigwedge p=q)$ -formula satisfying (W) of the introduction. Let λ and ρ be the formulas defined in the statement of Proposition 4.3. For $\vec{e},\vec{f}\in A^N$ let

$$L_{\vec{e}}=\{(x,y):\mathbf{A}\models\lambda(\vec{e},x,y)\},\qquad R_{\vec{e}}=\{(x,y):\mathbf{A}\models\rho(\vec{e},x,y)\},$$
$$L_{\vec{e}\vec{f}}=\{(x,y):\mathbf{A}\models\omega(\vec{e},\vec{f},x,y)\},\qquad R_{\vec{e}\vec{f}}=\{(x,y):\mathbf{A}\models\omega(\vec{f},\vec{e},x,y)\}.$$

(a) Note that $\vec{e}\in Z(\mathbf{A})$ iff $L_{\vec{e}}$ and $R_{\vec{e}}$ are a pair of complementary factor congruences such that $\vec{e}\equiv\vec{0}(L_{\vec{e}})\wedge\vec{e}\equiv\vec{1}(R_{\vec{e}})$.

(b) Note that $\vec{e}\in Z(\mathbf{A})$ iff the following conditions hold.

$$\forall\vec{f}\ ((\vec{f}\equiv\vec{1}(L_{\vec{e}})\wedge\vec{f}\equiv\vec{0}(R_{\vec{e}}))\to L_{\vec{e}\vec{f}}\text{ is a equivalence relation}),$$
$$\forall\vec{f}\ ((\vec{f}\equiv\vec{1}(L_{\vec{e}})\wedge\vec{f}\equiv\vec{0}(R_{\vec{e}}))\to F^{\mathbf{A}}\text{ preserves }L_{\vec{e}\vec{f}}),\text{ for each }F\in\mathcal{L},$$
$$\forall\vec{f}\ ((\vec{f}\equiv\vec{1}(L_{\vec{e}})\wedge\vec{f}\equiv\vec{0}(R_{\vec{e}}))\to R_{\vec{e}\vec{f}}\text{ is a equivalence relation}),$$
$$\forall\vec{f}\ ((\vec{f}\equiv\vec{1}(L_{\vec{e}})\wedge\vec{f}\equiv\vec{0}(R_{\vec{e}}))\to F^{\mathbf{A}}\text{ preserves }R_{\vec{e}\vec{f}}),\text{ for each }F\in\mathcal{L},$$
$$\forall\vec{f}\ ((\vec{f}\equiv\vec{1}(L_{\vec{e}})\wedge\vec{f}\equiv\vec{0}(R_{\vec{e}}))\to L_{\vec{e}\vec{f}}\cap R_{\vec{e}\vec{f}}=\Delta^{\mathbf{A}}),$$
$$\forall\vec{f}\ ((\vec{f}\equiv\vec{1}(L_{\vec{e}})\wedge\vec{f}\equiv\vec{0}(R_{\vec{e}}))\to L_{\vec{e}\vec{f}}\circ R_{\vec{e}\vec{f}}=\nabla^{\mathbf{A}}),$$
$$\forall\vec{f}\ ((\vec{f}\equiv\vec{1}(L_{\vec{e}})\wedge\vec{f}\equiv\vec{0}(R_{\vec{e}}))\to(\vec{e}\equiv\vec{0}(L_{\vec{e}\vec{f}})\wedge\vec{e}\equiv\vec{1}(R_{\vec{e}\vec{f}}))),$$
$$\exists\vec{f}\ (\vec{f}\equiv\vec{1}(L_{\vec{e}})\wedge\vec{f}\equiv\vec{0}(R_{\vec{e}})).$$

It is easy to check that all the above conditions are expressible by $\forall\exists$-formulas except the last one, which is expressible with a $\exists\forall$-formula. □

The following lemma guarantees that the axiomatizations of Propositions 4.1 and 4.4 for the weak EDFC case are as good as possible.

Lemma 4.5. ([1]) *Let $\mathcal{S}^{\vee}_{01}$ denote the variety of bounded join semilattices.*

(a) *"$e\in Z(\mathbf{A})$" is not definable in $\mathcal{S}^{\vee}_{01}$ by a set of positive formulas.*
(b) *"$e\in Z(\mathbf{A})$" is not definable in $\mathcal{S}^{\vee}_{01}$ by a set of $\forall\exists$-formulas.*
(c) "$e\in Z(\mathbf{A})$" is not definable in $\mathcal{S}^{\vee}_{01}$ by a set of $\exists\forall$-formulas.
(d) "$e\diamond_{\mathbf{A}} f$" is not definable in $\mathcal{S}^{\vee}_{01}$ by a set of positive formulas.
(e) "$e\diamond_{\mathbf{A}} f$" is not definable in $\mathcal{S}^{\vee}_{01}$ by a set of $\exists\forall$-formulas.

Definability of "$\vec{e} \in Z(\mathbf{A})$" and "$\vec{e} \diamond_{\mathbf{A}} \vec{f}$" for the twice EDFC case. We conclude the paper by giving optimal axiomatizations for "$\vec{e} \in Z(\mathbf{A})$" and "$\vec{e} \diamond_{\mathbf{A}} \vec{f}$" for the case of $\mathcal{V}$ having twice EDFC.

Proposition 4.6. *Let $\mathcal{V}$ be a variety with twice EDFC.*

(a) *There is a set of $\forall\exists$-formulas which defines "$\vec{e} \in Z(\mathbf{A})$" in $\mathcal{V}$.*
(b) *There is a set of $\forall\exists$-formulas which defines "$\vec{e} \diamond_{\mathbf{A}} \vec{f}$" in $\mathcal{V}$.*

Proof. This is a straightforward argument. □

In order to prove that the axiomatizations of the above proposition are optimal, we introduce an example. Let $\mathcal{L} = \{0, 1, ^c, \vee\}$ where 0 and 1 are constant symbols, ^{c} is a unary function symbol, and $\vee$ is a binary function symbol. Let $\mathcal{S}_{01}^{\vee c}$ be the variety of $\mathcal{L}$-algebras defined by the identities of bounded join semilattices together with the identities $0^c = 1$ and $1^c = 0$.

The following lemma guarantees that the axiomatizations of Proposition 4.6 are as good as possible.

Lemma 4.7. *The variety $\mathcal{S}_{01}^{\vee c}$ has twice EDFC and has the following properties.*

(a) *"$e \in Z(\mathbf{A})$" is not definable in $\mathcal{S}_{01}^{\vee c}$ by a set of positive formulas.*
(b) *"$e \in Z(\mathbf{A})$" is not definable in $\mathcal{S}_{01}^{\vee c}$ by a set of $\exists\forall$-formulas.*
(c) *"$e \diamond_{\mathbf{A}} f$" is not definable in $\mathcal{S}_{01}^{\vee c}$ by a set of positive formulas.*
(d) *"$e \diamond_{\mathbf{A}} f$" is not definable in $\mathcal{S}_{01}^{\vee c}$ by a set of $\exists\forall$-formulas.*

Proof. It is easy to check that the formulas $x \vee z_1 = y \vee z_1$ and $x \vee z_1^c = y \vee z_1^c$ satisfy (L) and (R) of the introduction, respectively. Thus, $\mathcal{S}_{01}^{\vee c}$ has twice EDFC.

(a): Since positive formulas are preserved by quotients, in order to prove (a), we will give a central element e and a congruence θ such that e/θ is not central. Let $\mathbf{T} = \langle\{0, 1/2, 1\}, 0, 1, ^c, \max\rangle$, where $0^c = 1$ and $(1/2)^c = 1^c = 0$. Let $\mathbf{S}$ be the subalgebra of $\mathbf{T}$ with universe equal to $\{0, 1\}$. Let $\mathbf{A} = \mathbf{T} \times \mathbf{S}$ and let θ be the binary relation on A given by the following partition: $\{\{(1/2, 1), (1, 1)\}, \{(0, 0)\}, \{(0, 1)\}, \{(1/2, 0)\}, \{(1, 0)\}\}$. It can be easily checked that θ is a congruence of $\mathbf{A}$. Note that $|Z(\mathbf{A}/\theta)| = 2$ since $|\mathbf{A}/\theta| = 5$ is a prime number. Thus, we have that $(0, 1) \in Z(\mathbf{A})$ and $(0, 1)/\theta \notin Z(\mathbf{A}/\theta)$.

(b): Let $\mathbf{C} = \langle[0, 1], 0, 1, ^c, \max\rangle$ where

$$a^c = \begin{cases} 1, & \text{if } a = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\mathbf{C}$ is in $\mathcal{S}_{01}^{\vee c}$. Suppose "$e \in Z(\mathbf{A})$" is definable in $\mathcal{S}_{01}^{\vee c}$ by a set of $\exists\forall$-formulas. By compactness, we can assume that this set is finite. Let $\mathbf{A} = \mathbf{C} \times \mathbf{C}$. Since $(0, 1) \in Z(\mathbf{A})$, there are $p_1, \ldots, p_m$ such that in every subalgebra of $\mathbf{A}$ containing $(0, 1), p_1, \ldots, p_m$, we have that $(0, 1)$ is central. Let $\mathbf{I} \leq \mathbf{C}$ be a finite subalgebra such that $\{(0, 1), p_1, \ldots, p_m\} \subseteq I \times I$. We observe that if $\max(I - \{1\}) < a_1 < a_2 < \cdots < a_k < 1$, then we have that $B = (I \times I) \cup \{(a_1, 1), (a_2, 1), \ldots, (a_k, 1)\}$ is the universe of a subalgebra $\mathbf{B}$ of $\mathbf{A}$ containing $(0, 1), p_1, \ldots, p_m$. Thus, $(0, 1) \in Z(\mathbf{B})$. But this is absurd since we can choose k in such a manner that the cardinality of B is a prime number.

(c): Suppose "$e \diamond_{\mathbf{A}} f$" is definable in $\mathcal{S}_{01}^{\vee c}$ by a set of positive formulas. Since the language of $\mathcal{S}_{01}^{\vee c}$ is finite, by compactness we have that there is a finite set of positive formulas which defines "$e \diamond_{\mathbf{A}} f$". So, the formula $\exists w \ (z \diamond w)$ is a positive formula which defines "$e \in Z(\mathbf{A})$" in $\mathcal{S}_{01}^{\vee c}$, which contradicts (a).

(d): This is similar to the proof of (c). □

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