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Functional bosonization of a Dirac field in $2+1$ dimensions, in the presence of a boundary

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ABSTRACT

We apply the functional bosonization procedure to a massive Dirac field defined on a $2+1$ dimensional spacetime which has a non-trivial boundary. We find the form of the bosonized current both for the bulk and boundary modes, showing that the gauge field in the bosonized theory contains a perfect-conductor boundary condition on the worldsheet spanned by the boundary. We find the bosonized action for the corresponding boundary modes.

1. Introduction

A seemingly obvious yet fruitful property of quantum field theory systems is that they must be susceptible of being described in terms of different sets of fields. This finds an extreme realization in the bosonization procedure, whereby a model can be defined in terms of either fermionic or bosonic quantum fields, the equivalence between those two formulations is made explicit by the existence of so called 'bosonization rules'. Besides mapping one set of fields into the other, they yield the dynamics the new variables are subjected to, in order to correspond to the same physical model.

In $1+1$ spacetime dimensions, bosonization is a very powerful tool which allows one to understand, and in some cases even to solve, some non-trivial Quantum Field Theory models (see [1] for a complete list of references). It is interesting to note that there is no fundamental theoretical stumbling block to the extension of this path-integral approach to higher dimensions. Indeed, there has been some progress in the application, although in an approximated form, of a path integral bosonization procedure to theories in more than two spacetime dimensions, dealing with both the Abelian and the non-Abelian cases.

We are concerned here with $2+1$ spacetime dimensions, where the path integral bosonization framework yields the exact form of the bosonized form of the current, while an inverse mass expansion can been used to determine the corresponding local terms in the dual bosonic action. Locality plus gauge invariance strongly constraint the form of the possible terms. Indeed, the leading term in the dual action becomes a Chern–Simons term, while the next-to-leading one corresponds, in the Abelian or non Abelian cases, to a (local) Maxwell [2,3] or Yang–Mills term [4], respectively. When the fermions are massless, the above procedure becomes more involved, since the even parity part of the dual action becomes non local, involving the squared root of the Laplacian [3]. Note, however, that the bosonization rule for the current is still the same as in the massive case, and that the dual action still contains a Chern–Simons term. The need for the latter has been shown explicitly, as a consequence of an *eta* function regularization required to have a consistent gauge invariant theory [5].

Let us finally point out that the path integral bosonization approach can be also employed in higher dimensions, and to situations where the fermionic theory has more than one conserved currents. For example, bosonization rules for fermionic currents in $3+1$ space–time dimensions have been found in terms of Kalb–Ramond fields [6].

In this paper, we are concerned with massive (mass $\equiv m$) fermions on a $2+1$ dimensional spacetime with a non-trivial boundary. Besides dealing with the necessary changes one has to implement to cope with it (a non-trivial boundary calls for a non-trivial boundary condition), we also include an auxiliary source for the fermionic current, localized on the boundary of the spacetime manifold. This last step will allow us, as we shall see, to express the current corresponding to the boundary modes in terms of the (bulk) bosonized current.

This paper is organized as follows: in Sect. 2 we present the derivation of the bosonized version of the model, within the context of the path integral formulation. Then, in Sect. 3, we study the

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properties of the resulting bosonic theory, and present our conclusions.

2. Generating functional

To begin with, let us introduce $\mathcal{S}_f(\bar{\psi}, \psi)$, the Euclidean action for a (free) massive Dirac field in $2+1$ dimensions:

$$\mathcal{S}_f(\bar{\psi}, \psi) = \int d^3x\, \bar{\psi}(\not\partial + m)\psi \,, \tag{1}$$

where, for Dirac's γ-matrices, we have adopted the conventions:

$$\gamma_\mu^\dagger = \gamma_\mu \,, \quad \{\gamma_\mu, \gamma_\nu\} = 2\,\delta_{\mu\nu} \,. \tag{2}$$

Letters from the middle of the Greek alphabet are assumed to run over the values 0, 1, 2. The Euclidean metric has been assumed to be the identity matrix $\delta_{\mu\nu}$. We shall sometimes raise of lower a spacetime index for notational convenience, although, for this metric tensor, there is no difference between them.

To proceed, we need to deal with the fermionic current, $J_\mu = \bar{\psi}\gamma_\mu\psi$. A first step will be to introduce an auxiliary source s_μ, which will allow us to generate correlation functions involving that operator, in the same way as when bosonization is constructed in the no-boundary case. This will amount to adding to the fermionic action an extra term $\mathcal{S}_J(s, J)$, where

$$\mathcal{S}_J(s, J) = i \int d^3x\, s_\mu(x) J_\mu(x) \,. \tag{3}$$

The current appears also as part of a constraint, namely, that its normal component, J_n, vanishes on $\mathcal{M} = \partial U$, the boundary of U, the spacetime region the field is confined to. The vanishing of the normal component of the current ensures that the fermions are indeed confined to U. Let us now introduce an explicit form for that constraint. To that end, we assume that a parametrization has been introduced for $\mathcal{M}$:

$$\sigma = (\sigma^0, \sigma^1) \to y_\mu(\sigma) \,, \ \mu = 0, 1, 2 \,, \tag{4}$$

with the two parameters σ^α, $\alpha = 0, 1$. In terms of the parametrization, the unit normal $\hat{n}_\mu$ may be written as follows:

$$\hat{n}_\mu(\sigma) = \frac{N_\mu(\sigma)}{\sqrt{N^2(\sigma)}} \,, \quad N_\mu(\sigma) = \frac{1}{2}\epsilon^{\alpha\beta}\,\epsilon_{\mu\nu\lambda} e^\nu_\alpha(\sigma) e^\lambda_\beta(\sigma) \,, \tag{5}$$

where we have introduced the tangent vectors $e^\mu_\alpha(\sigma) = \frac{\partial y^\mu}{\partial \sigma^\alpha}(\sigma)$.

Thus, the constraint can be conveniently introduced in terms of a functional Fourier representation, at the expense of using an auxiliary scalar field, $\xi(\sigma)$, living on $\mathcal{M}$:

$$\begin{aligned} \delta_{\mathcal{M}}(J_n) &= \int \mathcal{D}\xi\, e^{-S_{\mathcal{M}}(\xi, J)} \,, \\ S_{\mathcal{M}}(\xi, J) &= i \int d^2\sigma \sqrt{g(\sigma)}\, \xi(\sigma)\, \hat{n}_\mu(\sigma)\, J_\mu(y(\sigma)) \,, \end{aligned} \tag{6}$$

with $g(\sigma) \equiv \det[g_{\alpha\beta}(\sigma)]$, $g_{\alpha\beta}(\sigma) = e^\mu_\alpha(\sigma) e^\mu_\beta(\sigma)$ denoting the induced metric on $\mathcal{M}$.

Therefore, putting together the previous elements, we see that a generating functional of current correlation functions, $\mathcal{Z}(s)$, for a massive Dirac field in $2+1$ Euclidean dimensions, in the presence of a boundary $\mathcal{M}$, may be written as follows:

$$\mathcal{Z}(s) = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \delta_{\mathcal{M}}(J_n)\, e^{-\mathcal{S}(\bar{\psi}, \psi; s)} \,, \tag{7}$$

with

$$\mathcal{S}(\bar{\psi}, \psi; s) = \mathcal{S}_f(\bar{\psi}, \psi) + \mathcal{S}_J(s, J) \,. \tag{8}$$

Equivalently, recalling the representation (6),

$$\mathcal{Z}(s) = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \mathcal{D}\xi\, e^{-\mathcal{S}(\bar{\psi}, \psi; s) - S_{\mathcal{M}}(\xi, J)} \,. \tag{9}$$

The functional $\mathcal{S}_{\mathcal{M}}$, introduced in (6) is explicitly reparametrization invariant. Besides, since $\sqrt{N^2(\sigma)} = \sqrt{g(\sigma)}$, we see that it may be rendered also as follows:

$$S_{\mathcal{M}}(\xi, J) = i \int d^2\sigma\, \xi(\sigma)\, N_\mu(\sigma)\, J_\mu(y(\sigma)) \,, \tag{10}$$

or, more conveniently from the point of view of the next steps in our derivation, also as:

$$S_{\mathcal{M}}(\xi, J) = i \int d^3x\, c_\mu(x)\, J_\mu(x) \,, \tag{11}$$

with:

$$c_\mu(x) = \int d^2\sigma\, \xi(\sigma)\, N_\mu(\sigma)\, \delta[x - y(\sigma)] \,. \tag{12}$$

Then, the generating functional may be written as follows:

$$\mathcal{Z}(s) = \int \mathcal{D}\xi\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, e^{-\mathcal{S}_f(\bar{\psi}, \psi; s+c)} \,, \tag{13}$$

with

$$\mathcal{S}_f(\bar{\psi}, \psi; s) = \int d^3x\, \bar{\psi}(\not\partial + i\not s + m)\psi \,. \tag{14}$$

We then perform the change of variables:

$$\psi(x) \to e^{i\alpha(x)}\psi(x) \,, \quad \bar{\psi}(x) \to e^{-i\alpha(x)}\bar{\psi}(x) \,, \tag{15}$$

and integrate over α, to obtain (discarding immaterial factors)

$$\mathcal{Z}(s) = \int \mathcal{D}\alpha\, \mathcal{D}\xi\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, e^{-\mathcal{S}_f(\bar{\psi}, \psi; s+c+\partial\alpha)} \,. \tag{16}$$

Finally, we make the substitution $\partial_\mu\alpha \to b_\mu$,

$$\mathcal{Z}(s) = \int \mathcal{D}b\, \delta[\tilde{f}_\mu(b)]\, \mathcal{D}\xi\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, e^{-\mathcal{S}_f(\bar{\psi}, \psi; s+c+b)} \,, \tag{17}$$

where the condition $\tilde{f}_\mu(b) = \epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda = 0$, which implies that b_μ is a pure gradient [1] has been introduced in the measure.

Introducing yet another auxiliary field, A_μ, to implement that condition:

$$\delta[\tilde{f}_\mu(b)] = \int \mathcal{D}A\, e^{-i\int d^3x\, A_\mu \tilde{f}_\mu(b)} \,, \tag{18}$$

we get:

$$\mathcal{Z}(s) = \int \mathcal{D}A\, \mathcal{D}b\, \mathcal{D}\xi\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, e^{-\mathcal{S}_f(\bar{\psi}, \psi; s+c+b) - i\int d^3x A_\mu \tilde{f}_\mu(b)} \,. \tag{19}$$

Finally, we make the shift $b \to b - c - s$, to obtain:

$$\mathcal{Z}(s) = \int \mathcal{D}A\, \mathcal{D}b\, \mathcal{D}\xi\, e^{-W(b) - i\int d^3x A_\mu[\tilde{f}_\mu(b) - \tilde{f}_\mu(c) - \tilde{f}_\mu(s)]} \,, \tag{20}$$

where $W(b)$ is the effective action for the b_μ field due to the fermion loop, namely,

$$e^{-W(b)} = \det(\not\partial + i\not b + m) \,. \tag{21}$$

[1] We assume that U is a simply connected manifold.

The next step is to integrate out the auxiliary fields; to that end, we first rearrange the integrals as follows:

$$\mathcal{Z}(s) = \int \mathcal{D}A \, e^{i\int d^3x s_\mu \tilde{f}_\mu(A)} \times \int \mathcal{D}\xi \, e^{i\int d^3x c_\mu \tilde{f}_\mu(A)} \left(\int \mathcal{D}b \, e^{-W(b) - i\int d^3x b_\mu \tilde{f}_\mu(A)} \right). \tag{22}$$

The integral over the b_μ-field requires the knowledge of the fermionic determinant. Assuming that the large-mass expansion is applicable, we have, keeping the leading term [7]:

$$W(b) \simeq \pm \frac{i}{4\pi} \int d^3x \, \epsilon_{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda . \tag{23}$$

Thus, the integral over b_μ yields, in this approximation:

$$\int \mathcal{D}b \, e^{-W(b) - i\int d^3x b_\mu \tilde{f}_\mu(A)} = e^{\pm \frac{i}{2}\int d^3x \, 2\pi \, \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda} . \tag{24}$$

Performing the rescaling $A_\mu \to \frac{1}{\sqrt{2\pi}} A_\mu$, and defining

$$J_\mu \to \frac{i}{\sqrt{2\pi}} \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda \equiv \mathcal{J}_\mu , \tag{25}$$

which is the expression for the bosonized current, as seen by taking the functional derivative with respect to s_μ, we get:

$$\mathcal{Z}(s) = \int \mathcal{D}A \, e^{\int d^3x s_\mu \mathcal{J}_\mu} \times \int \mathcal{D}\xi \, e^{\int d^3x c_\mu \mathcal{J}_\mu \pm \frac{i}{2}\int d^3x \, \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda} . \tag{26}$$

Or,

$$\mathcal{Z}(s) = \int \mathcal{D}A \, \delta_{\mathcal{M}}(\mathcal{J}_n) \, e^{\int d^3x s_\mu \mathcal{J}_\mu \pm \frac{i}{2}\int d^3x \, \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda} . \tag{27}$$

Which is our final expression for the bosonized version of the system. Note that the original constraint has been converted into the vanishing of $\mathcal{J}_n \equiv \hat{n}_\mu \mathcal{J}_\mu$, the normal component of the *bosonized* current, $\mathcal{J}_\mu$, on $\mathcal{M}$.

Now, regarding A_μ as an Abelian gauge field, one can show, after some algebra, that the vanishing of the normal component of the bosonized current amounts to perfect conductor boundary conditions for that field. Indeed, the condition:

$$N^\mu(\sigma) \, \mathcal{J}_\mu(y(\sigma)) = 0 \tag{28}$$

becomes, in terms of $\mathcal{A}_\alpha(\sigma) \equiv A_\mu(y(\sigma)) e^\mu_\alpha(\sigma)$, the components of $A_\mu(x)$ projected to $\mathcal{M}$,

$$\partial_\alpha \mathcal{A}_\beta(\sigma) - \partial_\beta \mathcal{A}_\alpha(\sigma) = 0 , \tag{29}$$

which are perfect-conductor boundary conditions: since the boundary is two-dimensional, just the vanishing of the parallel component of the electric field.

A related observation is that one can verify that

$$\frac{i}{\sqrt{2\pi}} \int d^3x c_\mu \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda = \frac{i}{\sqrt{2\pi}} \int_{\mathcal{M}} d^2\sigma \, \xi(\sigma) \, \epsilon^{\alpha\beta} \, \partial_\alpha \mathcal{A}_\beta(\sigma) \tag{30}$$

where the rhs depends on the projected components of the gauge field. Now one can reinterpret the reasoning leading to the perfect-conductor boundary conditions as follows: the term (30), the only place where the auxiliary field ξ appears, is invariant under constant shifts of ξ: $\xi(\sigma) \to \xi(\sigma) + c_0$. This global continuous transformation implies, via Noether's theorem, the existence of a conserved current which is concentrated on the boundary:

$$\partial_\alpha j^\alpha(\sigma) = 0 , \quad j^\alpha(\sigma) = -\epsilon^{\alpha\beta} \, \mathcal{A}_\beta(\sigma) . \tag{31}$$

It is interesting to note here that there is some ambiguity regarding the form of the conserved current j^α, since it depends on the gauge field at the boundary, and we do allow for gauge transformations we are non-trivial there. More explicitly, since those gauge transformations have the form: $\delta \mathcal{A}_\alpha(\sigma) = \partial_\alpha \omega(\sigma)$, we see that the boundary current changes as follows:

$$\begin{aligned} j^\alpha(\sigma) &\to j^\alpha(\sigma) + \delta j^\alpha(\sigma) , \\ \delta j^\alpha(\sigma) &= -\epsilon^{\alpha\beta} \, \partial_\beta \omega(\sigma) . \end{aligned} \tag{32}$$

Now, the extra term has the form of a 'Noether superpotential term' [8], which, in d dimensions, amounts to the divergence of an antisymmetric tensor. Namely, for a Noether current j^μ, one can add $\delta j^\mu = \partial_\nu \theta^{\nu\mu}$, where $\theta^{\mu\nu}$ is antisymmetric. δj^μ is therefore topologically conserved. We also see that the resulting change in the conserved charge Q is the spatial integral of a divergence, and therefore dependent on the boundary conditions imposed on the antisymmetric field.

3. Discussion

Let us now consider the evaluation of the constrained path integral (27) deriving, as a consistency check for the present formulation, the Floreanini–Jackiw action [9] for the boundary modes. They are self-dual boson fields, which are solitons describing a charge-density wave of paired fermions. It is thus interesting, we believe, to see how those fields emerge in the present formalism.

The gauge field satisfies perfect-conductor boundary conditions on the boundary $\mathcal{M}$, and the exponent contains a Chern–Simons term, plus terms where the gauge field couples linearly to sources. We will proceed to split A_μ in the measure into a classical part A^{cl}_μ, satisfying the proper boundary conditions, plus a fluctuating field a^{cl}_μ, with trivial (Dirichlet) boundary conditions:

$$A_\mu(x) = A^{cl}_\mu(x) + a_\mu(x) , \tag{33}$$

such that $a_\mu(x)$ vanishes on $\mathcal{M}$.

Using the definition: $\mathcal{S}_{CS} \equiv \mp \frac{i}{2} \int d^3x \, \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$, the classical equation of motion have the form:

$$\delta \mathcal{S}_{CS}(A) = \delta\Big(\int d^3x \, s_\mu \mathcal{J}_\mu \Big) , \quad \partial_\alpha \delta \mathcal{A}_\beta(\sigma) - \partial_\beta \delta \mathcal{A}_\alpha(\sigma) = 0 , \tag{34}$$

where the last equation follows from the constraint.

The equations above will admit as a solution a sum:

$$A^{cl}_\mu(x) = A^{(0)}_\mu(x) + A^{(1)}_\mu(x) , \tag{35}$$

where $A^{(0)}_\mu$ is the general solution to the homogeneous system:

$$\delta \mathcal{S}_{CS}(A) = 0 , \quad \partial_\alpha \delta \mathcal{A}_\beta(\sigma) - \partial_\beta \delta \mathcal{A}_\alpha(\sigma) = 0 , \tag{36}$$

and $A^{(1)}_\mu(x)$ a particular solution to the inhomogeneous equation (i.e., including s_μ).

Let us then consider the equations for $A^{(0)}_\mu$. We see that, because of the non-trivial boundary, the homogeneous equations are:

$$\int d^3x\,\epsilon_{\mu\nu\lambda}\delta A_\mu \partial_\nu A_\lambda \;-\; \frac{1}{2}\int_{\mathcal{M}} d^2\sigma\,\epsilon^{\alpha\beta}\mathcal{A}_\alpha(\sigma)\delta\mathcal{A}_\beta(\sigma) \;=\; 0\,, \quad (37)$$

plus the second equation in (34).

The vanishing of the second term above leaves room for many different conditions which can be imposed on $\mathcal{A}_\alpha$ to make that happen. Recalling the conservation of the boundary current $j^\alpha(\sigma)$ on the boundary, if we want to keep the possibility of having non-vanishing values for that current, we cannot use trivial conditions for $\mathcal{A}_\alpha$, since those fields are proportional to components of the current.

In what follows, we assume the border to be static, namely, to have the form $\mathcal{M} = \mathcal{C} \times \mathbb{R}$, where $\mathcal{C}$ denotes a static closed curve: the spatial boundary. Then, assuming for the current j^α the form:

$$j^0(\sigma) = \rho(\sigma)\,, \quad j^1(\sigma) = \rho(\sigma)\,v \quad (38)$$

where v is a constant with dimensions of velocity, we see that the assumption above implies, from the continuity equation for the current:

$$\mathcal{A}_0 \;-\; v\mathcal{A}_1 \;=\; 0\,. \quad (39)$$

The second term in (37) then vanishes; indeed, one first deduces that $\delta\mathcal{A}_\alpha = \partial_\alpha\omega$, and then one uses the continuity equation for the surface current.

Then, the rest of the construction is rather standard [9] using general coordinates (rather than Cartesian ones) x_1 and x_2, such that the curve $\mathcal{C}$ may be regarded as the coordinate curve $x_2 = 0$, there are new coordinates $x'_0 = x_0$, $x'_1 = x_1 + vx_0$ and $x'_2 = x_2$, such that (39) becomes:

$$\mathcal{A}'_0 = 0\,, \quad (40)$$

where $\mathcal{A}'_0$ is the gauge field component in the new coordinates.

The other two components are pure gauges, and can be extended to pure gauges over U, because of the equations following from the bulk part of the variation: $A_i = \partial_i\phi$. Using the independence of the action on the metric, and extending (40) to all the spacetime region, as $A'_0(x') = 0$, we see that the action evaluated on this configuration yields:

$$\mathcal{A}_{CS}(A^{(0)}) \;=\; \pm\frac{i}{2}\int d^3x'\epsilon^{ij}A'_i\partial_0 A'_j \;=\; \pm\frac{i}{2}\int d^3x'\epsilon^{ij}\partial'_i\phi\,\partial_0\partial'_j\phi \quad (41)$$

or, recalling that the boundary is at $x'_2 = 0$, the action adopts the Floreanini–Jackiw [10] form:

$$\mathcal{A}_{CS}(A^{(0)}) \;=\; \pm\frac{i}{2}\int dx_0 dx'_1\left[\partial_0\varphi\partial_1\varphi - v(\partial_1\varphi)^2\right] \quad (42)$$

where $\varphi(x_0, x_1) \equiv \phi(x_0, x_1, 0)$.

Thus, the classical gauge field configurations contain the φ modes, concentrated on the boundary, which have to be integrated alongside the fluctuating part a_μ which has trivial boundary conditions.

The inhomogeneous equation can then be solved by imposing trivial boundary conditions on $A^{(1)}_\mu$. It is straightforward to see that the resulting equations and their solutions do not involve the boundary modes. Finally, the fluctuating part a_μ appears quadratically and does not involve the source s_μ, so we can discard it.

Let us end this work by noting that in the last three years there has been much interest in the application of dualities to analyze condensed matter systems like topological insulators, superconductors, and fractional quantum Hall effect systems [5], [11–13]. In these studies bosonization in $2+1$ dimensions play a relevant role [14], [15], [16], [17], and, in this context, the case of manifolds with boundary like those we discussed here would be of interest. We expect to discuss this issue in a future publication.

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