Schumann's resonances: A particular example of a spherical resonant cavity

M. F. Ciappina[a] and M. Febbo
Departamento de Física, Universidad Nacional del Sur, Av. Alem 1253, 8000, Bahía Blanca, Argentina

(Received 19 June 2003; accepted 9 January 2004)

The resonances between the Earth as one boundary and the ionosphere as the other, known as Schumann's resonances, represent an interesting example of a spherical cavity. We consider a simple model in which the boundaries behave as perfect conductors and then take into account the finite conductivity of the boundaries. Numerical results are obtained for both models and compared with available data. Good agreement is shown to exist between the analytical results and the experimental values when finite conductivity of the walls is considered.

[DOI: 10.1119/1.1652037]

I. INTRODUCTION

If one thinks of a resonant cavity, the first thought that might come in mind is a microwave oven or a similar device. It might be difficult to believe that the Earth behaves like an enormous resonant cavity. This behavior was first realized by Nikola Tesla in the early 1900's. He thought that the Earth would be a conductor and be responsive to certain electromagnetic frequencies. The inner boundary of the cavity is provided by the Earth's surface while the ionosphere is the outer surface. The resonant frequencies of this cavity are known as *Schumann resonances* in honor of W. O. Schumann who predicted them in 1952[1] and detected them in 1954.[2]

To observe these electromagnetic waves, one has to excite the Earth resonant cavity near its resonance, which is what the electrical activity in the atmosphere does. The resonant waves manifest themselves as peaks in the electromagnetic noise spectrum and have average values of 8, 14, 20, 26, 32, 37, and 43 Hz, with a daily variation of about ± 0.5 Hz depending on the Earth's electromagnetic activity.[3] Their characteristic wavelengths are of the order of magnitude of the Earth's radius (6400 km).

In this paper, we determine the Schumann resonances by considering two models. The first, a rather crude one, considers the cavity's walls as perfect conductors of infinite conductivity. The more realistic model takes into account the finite conductivity of the walls.

The paper is organized as follows. In Sec. II, the theoretical approach for perfectly conducting walls is developed. In Sec. III, we present two methods for obtaining the corrections for real conductive walls. One method includes the calculation of the quality factor Q. The other approach is based on the method of perturbation of boundary conditions.[4] In Sec. IV, these methods are used to determine the improved eigenfrequencies of the system, and we compare these frequencies with the results for perfectly conducting walls and the available data. Concluding remarks are presented in Sec. V.

II. THEORY AND INFINITE CONDUCTIVITY APPROACH

Electric or magnetic fields in an electromagnetic cavity resonator can support different types of standing waves depending on the direction of their components. The directions of the fields determine the characteristics of the electromagnetic modes inside a hollow spherical cavity. There are two types: transverse magnetic (TM) modes and transverse electric (TE) modes, depending on the existence of transverse magnetic or electric fields, respectively. In our problem, TM (TE) designates the non-existence of radial magnetic (electric) field component, that is, $B_r = 0$ $(E_r = 0)$. We shall discuss only the TM modes because they present the lowest frequencies in comparison with TE modes for the Earth cavity.

We assume that the fields for a TM mode in our spherical cavity are independent of the azimuthal angle ϕ. From Maxwell's equation for zero divergence and the requirement that the fields must be finite at $\theta = 0$, we conclude that only the ϕ component of $\mathbf{B}$ is nonzero. Faraday's law requires that the ϕ component of the electric field $\mathbf{E}$ also must vanish. Hence, the nonvanishing field's components for the TM mode are B_ϕ, E_r, and E_θ.

We first consider two perfectly conducting concentric spheres with inner radius R_i (the Earth's radius) and outer radius R_o, where $R_o = R_i + h$ and h is the height of the ionosphere. Although the ionosphere is the region of the atmosphere between 90 and 3000 km above the Earth's surface, the major electron density $10^8 - 10^{11}\,\mathrm{m}^{-3}$ is at about 50 km during the day and 300 km at night. It can be shown that for these values, the resonant frequencies vary only about ± 0.5 Hz for each mode. Therefore, we will assume $h = 100\,\mathrm{km}$ to compare with literature.[5]

We start with the vector Helmholtz equation[4]

$$\nabla^2 \mathbf{B} + \frac{\omega^2}{c^2} \mathbf{B} = 0, \tag{1}$$

which can be written in spherical coordinates as

$$\frac{\omega^2}{c^2}(rB_\phi) + \frac{\partial}{\partial r^2}(rB_\phi) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial(rB_\phi)}{\partial\theta}\right) - \frac{(rB_\phi)}{\sin^2\theta}\right] = 0. \tag{2}$$

To solve Eq. (2), we apply the method of separation of variables. We have

$$B_\phi(r,\theta) = \frac{f_l(r)}{r} g(\theta). \tag{3}$$

The angular behavior can be determined using Legendre polynomials, $g(\theta)=P_l^1(\cos\theta)$. If we substitute this form for g in Eq. (2), we obtain

$$\frac{df_l(r)}{dr^2}+\left[\frac{\omega^2}{c^2}-\frac{l(l+1)}{r^2}\right]f_l(r)=0, \tag{4}$$

where $l=1,2,...$ determines the angular dependence of the modes. The solution for $f_l(r)$ is

$$f_l(r)=\sum_{l=1}^{\infty} A_l r j_l(kr)+B_l r \eta_l(kr), \tag{5}$$

where j_l and η_l are the spherical Bessel functions,[6] $k=\omega/c$ is the magnitude of the wave vector, and the constants A_l and B_l are determined by the boundary conditions.

The corresponding components of $\mathbf{E}$ are

$$E_r=\frac{ic^2}{\omega r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\phi)$$

$$=-\frac{ic^2}{\omega r}l(l+1)\frac{f_l(r)}{r}P_l(\cos\theta), \tag{6a}$$

$$E_\theta=-\frac{ic^2}{\omega r}\frac{\partial}{\partial r}(rB_\phi)=-\frac{ic^2}{\omega r}\frac{df_l(r)}{dr}P_l^1(\cos\theta). \tag{6b}$$

The boundary condition for perfect conductors implies the vanishing of E_θ at $r=R_i$ and $r=R_o$. Hence,

$$\left.\frac{df_l(r)}{dr}\right|_{r=R_i,R_o}=0. \tag{7}$$

If we substitute Eq. (5) for $f_l(r)$, we find

$$\frac{d}{dr}(A_l r j_l(kr)+B_l r\eta_l(kr))|_{r=R_i,R_o}=0, \tag{8}$$

or

$$A_l[kR_i j_{l-1}(kR_i)-l j_l(kR_i)]+B_l[kR_i\eta_{l-1}(kR_i)$$
$$-l\eta_l(kR_i)]=0, \tag{9a}$$

$$A_l[kR_o j_{l-1}(kR_o)-l j_l(kR_o)]+B_l[kR_o\eta_{l-1}(kR_o)$$
$$-l\eta_l(kR_o)]=0. \tag{9b}$$

To obtain a nontrivial solution of Eq. (9), the determinant of the coefficient matrix is set equal to zero, the solution of which yields the eigenvalues

$$k_l=\frac{\omega_l}{c}. \tag{10}$$

III. FINITE CONDUCTIVITY APPROACH

The use of perfectly conducting walls is an approximation that is far from reality, because the earth's ionosphere behaves like a real conductor with finite conductivity. A model that includes the important properties which determine the dynamic behavior of the ionosphere must consider ionization, the recombination of species, the variation of solar radiation, and the configuration of the Earth's magnetic field. We will not take into account all of these features, but will assume as a first approximation that the ionosphere behaves as a wall whose conductivity is determined from a Drude model for the electron gas.[7] In this approximation, the ionosphere is treated as an anisotropic cold plasma mainly due to the ionized gas of particles that exists in this region of the atmosphere and the presence of the Earth's magnetic field $\mathbf{B}_0$.

The current density in the presence of an arbitrarily oriented magnetic field is[8]

$$\mathbf{j}=\sigma_\parallel\mathbf{E}_\parallel+\sigma_P\mathbf{E}_\perp-\sigma_H(\mathbf{E}_\perp\times\mathbf{B}_\perp)/B, \tag{11}$$

where the symbols $\parallel$ and $\perp$ refer to the direction parallel and perpendicular to the magnetic field and σ_P, σ_H, and $\sigma_\parallel$ are the Pedersen, Hall, and parallel conductivities, respectively, and are given by

$$\sigma_P=\frac{\nu_c^2}{\nu_c^2+\nu_B^2}\sigma_0,\quad \sigma_H=-\frac{\nu_B\nu_c}{\nu_c^2+\nu_B^2}\sigma_0,\quad \sigma_\parallel=\sigma_0=\frac{n_e e^2}{m_e\nu_c}, \tag{12}$$

where m_e and e are the electron mass and charge, respectively, ν_c is the electron collision frequency, $\nu_B=eB_0/2\pi m$ is the precession frequency of a charged particle in a magnetic field (gyration frequency), and n_e is the electron density. For a typical value of $B_0=30\,\mu T$ as the Earth's magnetic field, $\nu_B\sim 10^6\,\mathrm{s}^{-1}$.

We can make approximations that allow us to simplify the calculations and still obtain useful results. One class of approximations consists of simplifications to the anisotropic cold plasma medium.

For typical day-time values of ν_c,[9] we have $\nu_c\gg\nu_B$ at the height of interest ($h\sim 100$ km). In this limit, there is electric current only in the direction of the electric field ($\sigma_H\to 0$ and $\sigma_P\to\sigma_0$), and the medium becomes an isotropic conductor ($\mathbf{j}\propto\mathbf{E}$). The reason is that the electrons collide with the ionic nuclei before they are influenced by the Earth's magnetic field leading to a net electronic movement parallel to the direction of the electric field. Thus, the ionosphere can be well described as an isotropic conductor whose current density is given by the Ohm's law:[7]

$$\mathbf{j}=\sigma_0\mathbf{E}. \tag{13}$$

Typical values for the ionospheric conductivity, σ_i, at of $h=100$ km vary from $\sigma_0=\sigma_i\sim 10^{-6}\,\Omega^{-1}\,\mathrm{m}^{-1}$ at night to $\sigma_i\sim 10^{-3}\,\Omega^{-1}\,\mathrm{m}^{-1}$ during the day. The Earth's surface conductivity, σ_s, can be taken as $\sigma_0=\sigma_s\sim 1\,\Omega^{-1}\,\mathrm{m}^{-1}$, which corresponds to seawater conductivity.[10]

The main consequence of considering real conductors in a cavity is the dissipation of energy in their walls and a shifting of the resonant frequencies. In the following, we will take into account the finite conductivity of the Earth cavity's walls by using two approaches: The perturbation of boundary conditions and the conservation of energy principle through the calculation of the quality factor Q of the cavity.

A. Perturbation of boundary conditions

The finite conductivity of the cavity walls can be taken into account using the perturbation of boundary conditions method in which the deviation from an exactly solvable problem occurs at the boundaries due to the alteration of the boundary conditions. Although the method allows for corrections to any degree of accuracy in powers of a perturbation parameter, we shall use only the lowest order.

We focus our analysis on a single, nondegenerate TM mode and consider the B_ϕ component. We denote by a zero superscript the unperturbed solution for infinite conductivity. Thus, the time independent wave equation for $\varphi^0 = rB_\phi$ is

$$(\nabla^2 + k^{(0)2})\varphi^0 = 0, \tag{14}$$

with homogenous Neumann boundary conditions

$$\left.\frac{\partial \varphi^0}{\partial n}\right|_S = 0, \tag{15}$$

where $k^0 = \omega^0/c$. For finite conductivity, the normal derivative of φ^0 is not zero on the walls, and thus

$$\left.\frac{\partial \varphi}{\partial n}\right|_S = C(\omega)\varphi, \tag{16}$$

where $C(\omega)$ is given by

$$C(\omega) = \frac{\delta}{2}\frac{\omega^2}{c^2}(1+i), \tag{17}$$

where δ is the skin depth defined as $\delta = \sqrt{2/(\mu_c \sigma \omega)}$, μ_c is the permeability of the conducting wall, and σ is its conductivity. The value of $C(\omega)$ was obtained by considering the boundary conditions on the tangential value of $\mathbf{E}$ just outside the surface of a real conductor[5]

$$E_t \simeq \sqrt{\frac{\mu_c \omega}{2\sigma}}(1-i)(\mathbf{n} \times H_t). \tag{18}$$

Because we are looking for the lowest order approximation to the perturbed problem,

$$(\nabla^2 + k^2)\varphi = 0, \tag{19}$$

with its boundary condition given by Eq. (16), the right-hand side of Eq. (16) can be replaced by the unperturbed field, that is

$$\left.\frac{\partial \varphi}{\partial n}\right|_S \simeq C(\omega^0)\varphi^0|_S. \tag{20}$$

Green's theorem[11] can be employed to find the eigenvalue k^2

$$\int_V [\phi \nabla^2 \psi - \psi \nabla^2 \phi] d^3x = \oint_S \left[\psi \frac{\delta \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n}\right] da, \tag{21}$$

where the right-hand side of Eq. (21) has an inwardly oriented normal (outside the conductor). If we let $\psi = \varphi$ and $\phi = \varphi^{0*}$ and use Eqs. (14) and (19), and the boundary conditions, (15) and (20), Eq. (21) becomes

$$(k^{(0)2} - k^2)\int_V \varphi^{0*}\varphi d^3x = C(\omega^0) \oint_S |\varphi^0|^2 da. \tag{22}$$

Because $C(\omega^0)$ is a small parameter, φ can be replaced by its unperturbed value φ^0 inside the volume integral. Finally, we have

$$(k^{(0)2} - k^2) \simeq C(\omega^0)\frac{\oint_S |\varphi^0|^2 da}{\int_V |\varphi^0|^2 d^3x}. \tag{23}$$

We write $k = \omega/c$ and rewrite Eq. (23) as

$$\omega^2 \simeq \omega^{(0)2}(1 - (1+i)I), \tag{24}$$

where we have defined

$$I = \frac{1}{2}\frac{\delta_e \oint_{S_e}|\varphi^0|^2 da + \delta_i \oint_{S_i}|\varphi^0|^2 da}{\int_V |\varphi^0|^2 d^3x}, \tag{25}$$

and δ_e, δ_i, S_e, and S_i are the skin depths and the spherical surfaces of the Earth and the ionosphere, respectively. Equation (25) includes all of the contributions to the surface integral determined by Eq. (23). If we evaluate Eq. (25) and replace it in Eq. (24), we obtain improved values for the eigenfrequencies.

B. The Q approach

Dissipation of energy causes a spread in the resonance frequencies in real conductors. A measure of the tuning of a cavity is the quality factor Q, defined by

$$Q = \omega^0 \frac{U}{P}, \tag{26}$$

where U is the energy stored in the cavity, P is the power loss per cycle, and ω^0 is the resonance frequency without considering ohmic losses. The energy stored by electromagnetic fields inside the cavity is

$$U = \frac{1}{4}\int \left(\epsilon |\mathbf{E}|^2 + \frac{1}{\mu}|\mathbf{B}|^2\right) d^3x. \tag{27}$$

We use the definition for $\mathbf{E}$ and $\mathbf{B}$ [Eqs. (3) and (6)] and separate the volume integral into angular and radial components. For the angular part of E_θ and E_r, we use

$$\int (P_l^1(\cos\theta))^2 d\Omega = \frac{4\pi l(l+1)}{(2l+1)}, \tag{28a}$$

$$\int (P_l(\cos\theta))^2 d\Omega = \frac{4\pi}{(2l+1)}, \tag{28b}$$

respectively. Their radial integrals are

$$\int_{R_i}^{R_o} |E_\theta|^2 r^2 dr = \frac{c^4}{\omega^{(0)2}} A_l^2 \int_{R_i}^{R_o} ([krj_{l-1}(kr) - lj_l(kr)] + C_l[kr\eta_{l-1}(kr) - l\eta_l(kr)])^2 dr \tag{29}$$

$$= \frac{c^4}{\omega^{(0)2}} A_l^2 I_l^{(\theta)}, \tag{30}$$

and

$$\int_{R_i}^{R_o} |E_r|^2 r^2 dr = \frac{c^4}{\omega^{(0)2}}(l(l+1))^2 A_l^2 \int_{R_i}^{R_o} (j_l(k_l r) + C_l \eta_l(k_l r))^2 dr$$

$$= \frac{c^4}{\omega^{(0)2}}(l(l+1))^2 A_l^2 I_l^{(r)}, \tag{31}$$

where we have defined $C_l = A_l/B_l$. Alternatively, for the B_ϕ component, we have the same angular part as E_θ [Eq. (6b)], while the radial integral is

$$\int_{R_i}^{R_o} |B_\phi|^2 r^2 dr = A_l^2 \int_{R_i}^{R_o} (j_l(k_l r) + C_l \eta_l(k_l r))^2 r^2 dr$$

$$= A_l^2 I_l^{(\phi)}. \tag{32}$$

By using the angular integrals in Eq. (28) and the radial integrals in Eqs. (30)–(32), the energy stored inside the cavity can be written as

$$U=\frac{A_l^2\pi}{(2l+1)}\left(\frac{\epsilon c^4}{\omega^{(0)2}}(l(l+1)I_l^{(r)}+I_l^{(\theta)})+\frac{1}{\mu}I_l^{(\phi)}\right). \quad (33)$$

Power loss arises from the cavity's surfaces and is defined by

$$P=\frac{1}{2\sigma\delta}\int|\mathbf{n}\times\mathbf{H}|^2da. \quad (34)$$

In our case, the power loss is

$$P=\frac{1}{2\sigma\delta\mu_c^2}\int|B_\phi|^2da, \quad (35)$$

where $\mathbf{H}=\mathbf{B}/\mu_c$. The surface integral must be expressed into two contributions, one for the inner spherical shell and the other for the outer one. Therefore, the power loss can be expressed as

$$P=\frac{2\pi l(l+1)}{(2l+1)\mu_c^2}A_l^2\left[\frac{R_i^2}{\sigma_s\delta_s}\left|f_l(R_i)\right|^2+\frac{R_0^2}{\sigma_i\delta_i}\left|f_l(R_0)\right|^2\right]. \quad (36)$$

Because

$$\delta=\left(\frac{2}{\mu_c\sigma\omega}\right)^{1/2}\rightarrow\frac{1}{\delta\sigma}=\frac{\delta\mu_c\omega}{2}, \quad (37)$$

Eq. (36) for P becomes

$$P=\frac{\pi l(l+1)\omega^0}{(2l+1)\mu_c}A_l^2[\delta_sR_i^2|f_l(R_i)|^2+\delta_iR_0^2|f_l(R_0)|^2]. \quad (38)$$

If we substitute Eqs. (33) and (38) in Eq. (26), we obtain

$$Q=\frac{\frac{\epsilon c^4}{\omega^{(0)2}}(l(l+1)I_l^{(r)}+I_l^{(\theta)})+\frac{1}{\mu}I_l^{(\phi)}}{\frac{1}{\mu_c}[\delta_sR_i^2|f_l(R_i)|^2+\delta_iR_0^2|u_l(R_0)|^2]}. \quad (39)$$

If we assume that both walls are nonmagnetic $(\mu_c\sim\mu)$ and use the equality $\mu\epsilon=1/c^2$, Eq. (39) reduces to

$$Q=\frac{\frac{c^2}{\omega^{(0)2}}(l(l+1)I_l^{(r)}+I_l^{(\theta)})+I_l^{(\phi)}}{[\delta_sR_i^2|f_l(R_i)|^2+\delta_iR_0^2|f_l(R_0)|^2]}. \quad (40)$$

Equation (40) can be evaluated numerically for the different resonant modes.

C. Connection between *Q* and perturbation of boundary conditions approaches

The correction for the resonant frequencies using the method of perturbation of boundary conditions is given by Eq. (24). We perform a Taylor expansion as $I\rightarrow 0$, and obtain for the imaginary part of ω:

$$\text{Im}\,\omega\simeq-\frac{I}{2\omega^0}+O(I^2). \quad (41)$$

Because $P=-dU/dt$, Eq. (26) implies

$$\frac{dU}{dt}=-\omega^0\frac{U}{Q}, \quad (42)$$

with the solution

$$U(t)=U_0e^{-\omega^0t/Q}. \quad (43)$$

The field dependence on U is obtained from Eq. (27), resulting in

$$\psi(t)=\psi_0(t)e^{-\omega^0t/2Q}, \quad (44)$$

where $\psi(t)$ represents the magnitude of **E** or **B** field and $\psi_0(t)$ contains the oscillatory part of the fields.

Because the imaginary part of ω is related to the evanescent behavior of the electromagnetic fields, we obtain from Eqs. (41) and (44), neglecting terms of order I^2:

$$I=\frac{\omega^{(0)2}}{Q}. \quad (45)$$

Equation (45) establishes the connection between the two methods as $I\rightarrow 0$. Conversely, it is useful to express ω in terms of Q. From Eq. (24) we have

$$\omega^2\simeq\omega^{(0)2}\left[1-\frac{(1+i)}{Q}\right]. \quad (46)$$

From Eq. (46), it is seen that Q modifies both the real and imaginary components of ω. The modification of the real part leads to a downward shift of the resonant frequencies, while the contribution for the imaginary component changes the rate of decay of the fields.

IV. NUMERICAL RESULTS AND COMPARISONS

Tables I to IV show the numerical values obtained solving the transcendental Eqs. (9a) and (9b), those calculated from the Q approach, and those determined using the perturbation of boundary conditions method. The data are obtained from averaged values between night and day.[3]

Table I illustrates the comparison between the measured data ($\nu_m=\omega_m/2\pi$) and the eigenfrequencies that are generated from the infinite conductivity approach (ν_a). The relative errors (without point) and the quality factor Q between the values for the different modes ($l=1,...,7$) are shown. Note that the maximum error value occurs for the first resonant mode.

Table I. Comparison between the measured frequency values ν_m and the calculated values ν_a (infinite conductivity approach) for the first seven modes; e_r is the relative percentage error between ν_a and ν_m. It can be seen that the first mode shows the maximum error value whereas the superior modes reflect errors of about 30%.

Mode (l)	$\nu_m\pm0.5$ (Hz)	ν_a (Hz)	e_r (%)
1	8	10.47	30.88
2	14	18.13	29.52
3	20	25.64	28.22
4	26	33.11	27.33
5	32	40.55	26.71
6	37	47.98	29.68
7	43	55.39	28.81

The computed values for the finite conductivity approaches are shown in Tables II, III, and IV using ionospheric conductivities ranging from an intermediate value of $\sigma_i\sim10^{-5}\,\Omega^{-1}\,\text{m}^{-1}$, a typical night conductivity value of or-

Table II. Frequency values for the finite conductivity approaches using $\sigma_i \sim 10^{-5}\,\Omega^{-1}\,\mathrm{m}^{-1}$: ν_Q (Q approach), ν_I (perturbation of boundary conditions approach). e_Q and e_I correspond to the relative percentage errors between the measured values and ν_Q and ν_I, respectively. δ_e and δ_i are the skin depths calculated for the Earth and the ionosphere. The minimum error is observed for the first resonant mode while it increases for the superior ones. The frequency values obtained here represent a considerable improvement compared to the infinite conductivity approach.

Mode (l)	δ_e (km)	δ_i (km)	Q	ν_Q (Hz)	e_Q (%)	ν_I (Hz)	e_I (%)
1	0.156	49.19	4.06	9.21	15.13	9.19	14.89
2	0.118	37.38	5.34	16.45	17.50	16.43	17.36
3	0.099	31.43	6.35	23.64	18.20	23.61	18.05
4	0.087	27.66	7.21	30.82	18.54	30.79	18.42
5	0.079	25.00	7.98	38.02	18.81	37.98	18.69
6	0.072	22.98	8.68	45.22	22.22	45.18	22.11
7	0.067	21.38	9.32	52.44	21.95	52.39	21.84

der $\sigma_i \sim 10^{-6}\,\Omega^{-1}\,\mathrm{m}^{-1}$, and a day value of $\sigma_i \sim 10^{-3}\,\Omega^{-1}\,\mathrm{m}^{-1}$, where ν_Q and ν_I represent the resonant frequencies calculated by the Q approach and the perturbation of boundary conditions method, respectively. The relative errors. and the quality factor Q computed for the different modes are also presented. A detailed discussion of the results shown in Tables I to IV will be presented in Sec. V.

V. SUMMARY

We have shown how a complex problem in electromagnetism can be treated from different viewpoints. First, we employed the usual infinite conductivity approach which consists of the solution of a partial differential equation with a homogenous Neumann boundary condition. This equation can be solved exactly and leads to a secular determinant for the eigenvalues. Numerical values obtained from its solution are not sufficiently accurate, which is not surprising because the assumption of perfectly conducting walls is far from reality for the Earth–ionosphere cavity.

We then incorporated the finite conductivity of the cavity's walls by two simple methods: The Q approach and the perturbation of boundary conditions. The ionosphere acts like an anisotropic conductor unlike the usual situations with resonant cavities for which the walls are isotropic. However, useful results may be obtained by treating the ionosphere as an isotropic conductor within a certain range of values of ν_c

Consider first the case when $\nu_c \gg \nu_B$ (day-time). For this limiting case, the ionosphere behaves as an isotropic conductor. For $\sigma_i \sim 10^{-3}\,\Omega^{-1}\,\mathrm{m}^{-1}$, a typical value, the resonant frequency values show an error that is comparable to the model of infinite conductivity walls (zero skin depth). This result can be easily understood because the ionospheric skin depth is negligible when compared to the height of the ionosphere (see Table IV). To test the sensitivity of our numerical approximations, we use a smaller conductivity value, say $\sigma_i \sim 10^{-5}\,\Omega^{-1}\,\mathrm{m}^{-1}$, which also is a reasonable value (in view of the fluctuation of the values of ν_c and n_e). We observe in Table II that the differences between the experimental and theoretical values are less than 25%. In particular, the difference for the fundamental mode is of $\approx$15%, which is a very significant improvement over the infinite conductivity model.

In the opposite limit, that is, $\nu_c \lesssim \nu_B$ (typical night values), the ionosphere does not behave like an isotropic conductor. However, it is illustrative to discuss the numerical results for $\sigma_i \sim 10^{-6}\,\Omega^{-1}\,\mathrm{m}^{-1}$. This value results from the values of ν_c and n_e at night. The calculations show the limitations of our numerical models and also are helpful for comparing with values in literature. Our results show a skin depth that is the order of and is even larger than the height of the ionosphere (see Table III). When the skin depth is large, the Q of the cavity is very small and this fact implies that the possibility of determining the frequency, using the previously determined value of Q, is rather poor. On the other hand, the application of the perturbation of boundary conditions method cannot be considered suitable because the perturbation of the boundary condition is large and satisfactory convergence is not achieved.

In summary, the two methods show accurate results as can be seen from Tables I and II without much algebraic and computational effort. We believe that the Q approach presents some advantages from a educational viewpoint because it has an intuitive interpretation in terms of energies and losses. On the other hand, the perturbation of boundary conditions method is a powerful mathematical tool not usually used in physics undergraduate or even graduate courses. The problem considered here provides an example of its power and versatility.

If we want to improve our results, further considerations

Table III. Frequency values for the finite conductivity approaches using $\sigma_i \sim 10^{-6}\,\Omega^{-1}\,\mathrm{m}^{-1}$. Although it can be seen a remarkable decrease on the relative error values in all cases, the calculated ionospheric skin depth, δ_i, can be, for some modes, even larger than the actual height of the ionosphere yielding a small value for the Q of the cavity and resulting in an inaccurate determination of the resonant frequency.

Mode (l)	δ_e (km)	δ_i (km)	Q	ν_Q (Hz)	e_Q (%)	ν_I (Hz)	e_I (%)
1	0.156	155.48	1.29	7.52	6.00	7.50	6.25
2	0.118	118.19	1.69	13.62	2.71	13.57	3.07
3	0.099	99.39	2.01	19.95	0.25	19.88	0.60
4	0.087	87.47	2.29	26.43	1.65	26.35	1.35
5	0.079	79.04	2.53	33.03	3.22	32.93	2.91
6	0.072	72.66	2.75	39.70	7.30	39.58	6.97
7	0.067	67.62	2.96	46.43	7.98	46.30	7.67

Table IV. Frequency values for the finite conductivity approaches using $\sigma_i \sim 10^{-3}\ \Omega^{-1}\ \mathrm{m}^{-1}$. In this case, the ionospheric skin depth δ_i is very small which implies that the frequency values (ν_I and ν_Q) tends to those calculated from the infinite conductivity approach, ν_a.

Mode (l)	δ_e (km)	δ_i (km)	Q	ν_Q (Hz)	e_Q (%)	ν_I (Hz)	e_I (%)
1	0.156	4.92	39.43	10.34	29.25	10.33	29.13
2	0.118	3.74	51.89	17.96	28.29	17.96	28.29
3	0.099	3.14	61.71	25.44	27.20	25.43	27.15
4	0.087	2.76	70.12	32.87	26.42	32.87	26.42
5	0.079	2.50	77.60	40.29	25.91	40.28	25.88
6	0.072	2.30	84.41	47.69	28.89	47.69	28.89
7	0.067	2.14	90.70	55.09	28.12	55.09	28.12

are needed. First, the relation between the fields outside the boundary surfaces should be accounted for in a more accurate way. Second, a more realistic model for both the ionosphere and the Earth's surface should be considered, including modeling the ionosphere as a full anisotropic plasma, the variation of the ionospheric conductivity during the day and at different seasons, and according to the sunspot cycle. Methods that implement these features, such as the finite difference time domain,[8] permit one to incorporate all of these characteristics of the ionosphere. However, they increase the algebraic and numeric complexity.

A very interesting and not so complicated project appropriate for undergraduate students is to measure these resonances. They can be detected by using a whip antenna that captures the radial electric field together with an array of electronic devices that amplifies and filters the collected signal. Next, this signal, digitally recorded, must be analyzed with a computer using fast Fourier transform algorithms to clean and fix the values of the resonance frequencies. At present, we are successfully carrying out this task with a sophomore student class.

ACKNOWLEDGMENTS

This work has been sponsored by CONICET Research and Development Program and Secretaría General de Ciencia y Tecnología of the Universidad Nacional del Sur through PGI 24/F032 and 24/F027. The authors wish to thank Professors W. R. Cravero, D. A. Vega, and P. A. A. Laura for their critical reading of a preliminary version of this paper and for their conceptual comments and corrections.

[a]Electronic mail: ciappina@criba.edu.ar

[1]W. O. Schumann, ''On the characteristic oscillations of a conducting sphere which is surrounded by an air layer and an ionospheric shell (in German),'' Z. Naturforsch. A **7**, 149–154 (1952).
[2]W. O. Schumann and H. L. König, ''Über die beobachtung von atmospherics bei geringsten frequenzen,'' Naturwissenschaften **41**, 183 (1954).
[3]T. R. Larsen and A. Egeland, ''Fine structure of Earth-ionosphere cavity resonances,'' J. Geophys. Res. **73**, 4986–4989 (1968).
[4]P. M. Morse and H. Feshbach, *Methods of Theoretical Physics, Part II* (McGraw–Hill, New York, 1953).
[5]J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1998).
[6]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, 9th ed. (Dover, New York, 1972).
[7]N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Saunders College, Philadelphia, 1976).
[8]S. A. Cummer, ''Modeling electromagnetic propagation in the Earth-ionosphere waveguide,'' IEEE Trans. Antennas Propag. **48**, 1420–1429 (2000).
[9]H. S. W. Massey, *Atomic and Molecular Collisions* (Taylor and Francis, London, 1979).
[10]G. J. Palacky, in *Electromagnetic Methods in Applied Geophysics-Theory*, edited by M. N. Nabighian (Society of Exploration Geophysics, Tulsa, OK, 1987), Vol. 1.
[11]P. M. Morse and H. Feshbach, *Methods of Theoretical Physics, Part I* (McGraw–Hill, New York, 1953).