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H_∞ control of a Wiener-type system

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A Wiener system is a system which can be modelled as a linear dynamic followed by a static gain. The goal of this paper is to develop a robust H_∞ compensator for controlling an SISO Wiener system. The controller also takes the form of a Wiener model. The design approach consists of the approximation of the non-linear gain using a piecewise linear (PWL) function and in using a linear controller for each sector obtained from this approximation. Therefore, the general controller structure can be stated as a linear dynamic compensator in series with a PWL static gain.

As an illustrative case, a neutralization pH reaction between a strong acid and a strong base in the presence of a buffer agent is dealt with. Computer simulations are developed for showing the performance of the proposed controller.

1. Introduction

In the past few decades, a considerable amount of research has been carried out on modelling, identification and control of non-linear systems. Most dynamical systems can be better represented by non-linear models than linear ones. This is because the former models are able to describe the behaviour of the system over a wider operating range, while the latter ones are only able to approximate the system around a given operating point. One of the most frequently studied classes of non-linear models are the so-called 'block-oriented non-linear models' (Pearson and Pottmann 2000). These models consist of the interconnection of linear time invariant (LTI) systems and static (memoryless) non-linearities. Within this class, two of the most common model structures are:

- The 'Hammerstein' model, which consists of the cascade connection of a static (memoryless) non-linearity followed by an LTI system (see, e.g. Eskinat *et al.* (1991) for a review on identification of Hammerstein models).
- The 'Wiener' model, in which the order of the linear and the non-linear blocks is reversed (see, e.g. Greblicki (1994) and Wigren (1993) for different methods for the identification of Wiener models).

These model structures have been successfully used to represent non-linear systems in a number of practical applications in the areas of chemical processes (Eskinat *et al.* 1991, Kalafatis *et al.* 1995, Pearson and Pottmann 2000), biological processes (Koremberg 1973), signal processing, communications and control (Zhu and Seborg 1994, Fruzzetti *et al.* 1997, Norquay *et al.* 1998, Gerkšič *et al.* 2000, Lussón Cervantes *et al.* 2003*a*). An analysis of the approximation properties of these models can be found in Boyd and Chua (1985).

In the context of process control, most of the applications of the structured models are in the area of model predictive control. Harris and Palazoglu (2001) present a controller based on robust control theory, which can deal with more general block-oriented models known as functional expansion models.

In this paper, we propose a particular formulation of the Wiener model where the non-linear static gain is represented by a piecewise linear (PWL) function. The PWL function is chosen as a convenient representation for approximating the non-linear function. It replaces the global non-linear function by a set of linear sub-functions which are defined in properly partitioned sub-regions of the original non-linear function domain. Then, it is possible to design a particular H_∞ controller for each domain partition. The totality of the resulting controllers can be written as a single expression in the form of a Wiener controller, with interesting stability and robustness properties.

The work is organized as follows. In § 2, a description of the Wiener model used in this paper is presented. The controller design is discussed in § 3, and its stability and robustness properties are analysed in § 4. In § 5, the evaluation of the controller performance is presented via simulation. Finally, in § 6, the conclusions are presented.

2. Wiener model

A Wiener model consists of a linear dynamics block (H_1) in cascade with a static non-linearity (H_2) at the output, as shown in figure 1, where $v \in \Re^1$ is an intermediate signal which does not necessarily have a physical meaning.

The following SISO state-space description is used to represent the linear block (H_1) as

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$$\dot{x} = Ax + B(u - u_{ss}) \tag{1}$$
$$v = Cx + D(u - u_{ss}) + v_{ss} \tag{2}$$

where the matrices and vectors stand for the typical variables of state-space models. The constants u_{ss} and v_{ss} are the steady state values of the variables u and v, respectively.

For the static non-linear element H_2, the use of continuous piecewise linear (PWL) functions is proposed (Chua and Ying 1983, Julián *et al.* 1999). PWL functions have been proved to be a very powerful tool for modelling and analysing non-linear systems. It can be demonstrated (Julián 1999) that any continuous non-linear function h, $h: \Re^m \longrightarrow \Re^m$ can uniquely be approximated by $E\Lambda_\beta(v) = h(v)$. In our application, h, $h: \Re^1 \longrightarrow \Re^1$, then we can use any description for the PWL. In particular, we choose a non-simplicial partition of the domain. In this way a more general description for the PWL functions is selected. Let us consider the domain ($\Re$) divided into $\sigma + 1$ segments by parameters β_j, with $\beta_1 \le \beta_2 \le \cdots \le \beta_\sigma$. The location of the segments is chosen in order to attain a good approximation of the non-linear function. With these assumptions in mind, the PWL function is defined as

$$h(v(t)) = E\Lambda_\beta(v) \tag{3}$$

where $E \in \Re^{\sigma+1}$ is the vector of parameters that describes the non-linear function. A standard identification algorithm and the Toolbox (Julián 2000) based on the least-squares method can be respectively used to obtain vector E (Lussón Cervantes *et al.* 2003 *a*). $\Lambda_\beta(v)$ depends

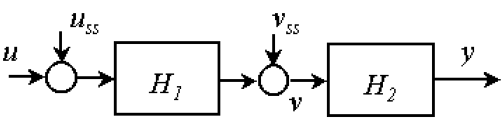

Figure 1. Scheme of a Wiener model structure.

only on the variable v and the domain segmentation described by the parameters β_j as

$$\Lambda_\beta(v) = \begin{bmatrix} 1 \\ \frac{1}{2}(v - \beta_1 + |v - \beta_1|) \\ \frac{1}{2}(v - \beta_2 + |v - \beta_2|) \\ \vdots \\ \frac{1}{2}(v - \beta_\sigma + |v - \beta_\sigma|) \end{bmatrix}. \tag{4}$$

Figure 2 represents a typical PWL approximation in $\Re^1$. Note that in each sector in which the domain is divided, the model is affine in the variable v. Let us define the sector $\aleph^{(i)} = \{v: \beta_i \le v \le \beta_{i+1}\}$, then the affine expression of h in this sector is defined as $h^{(i)} = J^{(i)}v + w^{(i)}$ for $v \in \aleph^{(i)}$, where

$$J^{(i)} = \sum_{j=1}^{i} E(j+1) \tag{5}$$

and

$$w^{(i)} = E(1) - \sum_{j=1}^{i} \beta_j E(j+1). \tag{6}$$

In the next section, this model will be used to design an H_∞ controller.

3. Controller design

In this section, an H_∞ compensator will be designed to control any system that matches the modelling description defined in the previous section. The necessary tools on H_∞ control can be read in Zhou *et al.* (1995) and Rossi and Figueroa (1997).

Consider the LTI system described by the block diagram of figure 3, where G is the generalized plant

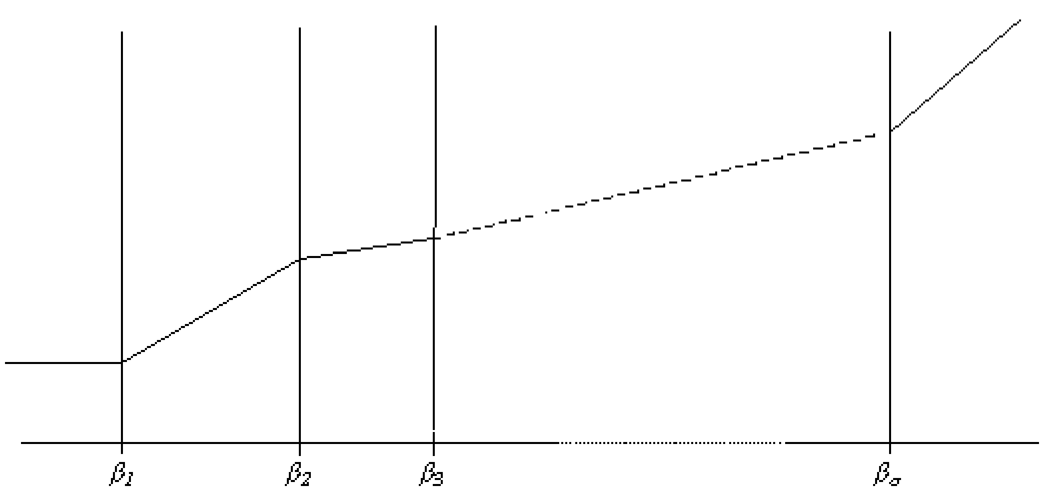

Figure 2. PWL model for static gain.

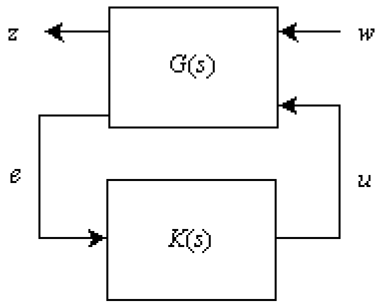

Figure 3. Linear fractional transformation.

and K is the controller. $G(s)$ is defined as

$$G(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}.$$

The generalized plant G contains what it is usually called the model in a control problem plus all weighting functions. The signal w contains all external inputs, including disturbances, sensor noise and commands, the output z is an error signal, e is the controller input, and u is the control input. The diagram in figure 3 is referred to as linear fractional transformation (LFT) on K, and G is called the coefficient matrix for the LFT.

The objective is to obtain a controller to track a set-point change over a wide operation region. Let us consider the closed-loop scheme depicted in figure 4, where the manipulated variable is u, the measured variable is y and the control objective is to minimize $z = [e_o \;\; u]^{\mathrm{T}}$ under any input reference change $w = y_{sp}$ (see figure 3).

Our intention is to design an H_∞ compensator for the Wiener system of figure 1. However, it is already known (Zhou *et al.* 1995) that the order of these controllers is at least of the same order as the plant model. To reduce the controller complexity we propose to design a Wiener-type controller. Then, the H_∞ compensator will be based on the linear model of the system H_1, and additionally its gain will be modified in concordance with the non-linear gain of the process H_2. Let us call

$$P(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

the process model, and

$$W_o(s) = \begin{bmatrix} A_o & B_o \\ C_o & D_o \end{bmatrix}$$

the weighing matrix used to include the control specification. Note that the information given by both $P(s)$ and $W_o(s)$ is used to transform the representation in figure 4 to the one in figure 3. Then, the linear fractional

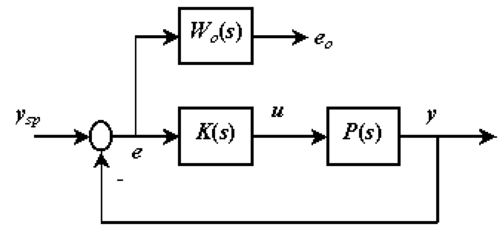

Figure 4. Closed-loop system for design.

transformation of the system would be represented as

$$G(s) = \begin{bmatrix} A_{ol} & B_{1ol} & B_{2ol} \\ C_{1ol} & D_{11ol} & D_{12ol} \\ C_{2ol} & D_{21ol} & D_{22ol} \end{bmatrix}$$

where

$$A_{ol} = \begin{bmatrix} A & 0 \\ -B_oC & A_o \end{bmatrix}, \quad B_{1ol} = \begin{bmatrix} 0 \\ B_o \end{bmatrix}, \quad B_{2ol} = \begin{bmatrix} B \\ -B_oD \end{bmatrix}$$

$$C_{1ol} = \begin{bmatrix} -D_oC & C_o \\ 0 & 0 \end{bmatrix}, \quad D_{11ol} = \begin{bmatrix} -D_o \\ 0 \end{bmatrix}$$

$$D_{12ol} = \begin{bmatrix} -D_oD \\ 1 \end{bmatrix}, \quad C_{2ol} = \begin{bmatrix} -C & C_o \end{bmatrix}, \quad D_{21ol} = \begin{bmatrix} 1 \end{bmatrix}$$

$$D_{22ol} = \begin{bmatrix} -D \end{bmatrix}.$$

Taking into account the previous expressions, it is now possible to compute an H_∞ controller. Let the controller be

$$K(s) = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}.$$

Now if we consider the complete non-linear Wiener model shown in figure 1, it is clear that in each sector $\aleph^{(i)}$ the gain $J^{(i)}$ of the plant model will change. Then, if we add in series with the controller $K(s)$ a gain $J_K^{(i)}$ which varies from sector to sector, it is possible to retain the linear closed-loop performance along the complete operative region. Therefore, the value of the controller gains for sector $\aleph^{(i)}$ should be $J_K^{(i)} = 1/J^{(i)}$.

In light of this, it is an appealing idea to conjugate all these gains in a unique PWL function. To perform it, we should consider that the plant domain (and for extension the partition), is not the same as the controller domain. To analyse the relation between them, let us consider the steady state version of figure 1. We know that the partition of the plant domain is completely defined by the parameters β_i. Then, the plant input must be

$$u = \gamma_i \triangleq \frac{(\beta_i - v_{ss})}{CA^{-1}B + D} - u_{ss}$$

to obtain the output β_i from the block H_1. Therefore, to compute the γ_i will require accurate system matrices

(A, B, C, D) and A must have an inverse. Now, if we consider a block-type structure for the controller as the one shown in figure 5, it is possible to define the relations (see figure 6)

$$\gamma_i = \gamma_{i-1} + J_K^{(i)}(\beta_i^K - \beta_{i-1}^K) \qquad i = 2, \ldots, \sigma. \tag{7}$$

In addition, from the steady state situation it becomes clear that for $e = 0$ the controller's output should be $u - u_{ss} = 0$. From these equalities, if we assume that $\gamma_j \le 0 \le \gamma_{j+1}$, it is clear that

$$\gamma_{j+1} = J_K^{(j)}\beta_j^K \tag{8}$$

and

$$\gamma_j = J_K^{(j)}\beta_j^K. \tag{9}$$

Then, using expressions (7)–(9) it is possible to compute the values for the parameters β_i^K that in conjunction with the gains in each sector define a PWL representation for the controller gain. This can be expressed as

$$h_K(v_e(t)) = E_K \Lambda_{\beta^K}(v_e) \tag{10}$$

where $v_e(t)$ is the output of the linear controller. It is important to remark that the controller partition is not simplicial, i.e. the segment length is not the same, even if the plant partition is simplicial. In the next section, some considerations related to stability and robustness of this approach will be discussed.

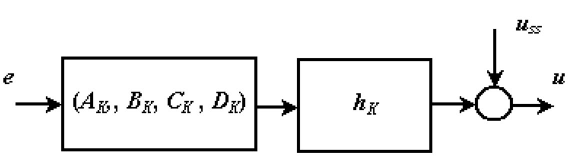

Figure 5. Wiener model for controller.

4. Stability and robustness considerations

In this section, some aspects regarding stability and robustness of the controller introduced in the previous section are treated.

4.1. *Nominal stability*

Let us consider the closed-loop system of figure 7. Note the explicit mention of the partition in the PWL blocks. Now, let us assume that the process behaviour is described exactly by the plant model. If we consider the linearization of the plant/controller system around all the possible operation points, then the closed-loop linear system can be posed as

$$\dot{x}_{cl} = A_{cl}^{(i,j)} x_{cl} + B_{cl}^{(i,j)} y_{sp}$$
$$y = C_{cl}^{(i,j)} x_{cl} + D_{cl}^{(i,j)} y_{sp}$$

where

$$x_{cl} = \begin{bmatrix} x \\ x_K \end{bmatrix}$$

$$A_{cl}^{(i,j)} = \begin{bmatrix} A - BJ_K^{(j)} D_K \Gamma C J^{(i)} & J_K^{(j)} B \Gamma C_K \\ -B_K \Gamma C J^{(i)} & A_K - B_K \Gamma J_K^{(j)} D J^{(i)} C_K \end{bmatrix}$$

$$B_{cl}^{(i,j)} = \begin{bmatrix} J_K^{(j)} B D_K \Gamma \\ B_K \Gamma \end{bmatrix}, \quad C_{cl}^{(i,j)} = \begin{bmatrix} \Gamma C J^{(i)} & \Gamma J_K^{(j)} D J^{(i)} C_K \end{bmatrix}$$

$$D_{cl}^{(i,j)} = \begin{bmatrix} \Gamma J_K^{(j)} D J^{(i)} D_K \end{bmatrix} \quad \text{and} \quad \Gamma = (I + J_K^{(j)} D J^{(i)} D_K)^{-1}.$$

Note that the system's equations explicitly depend on the controller sector (j) and the plant sector (i).

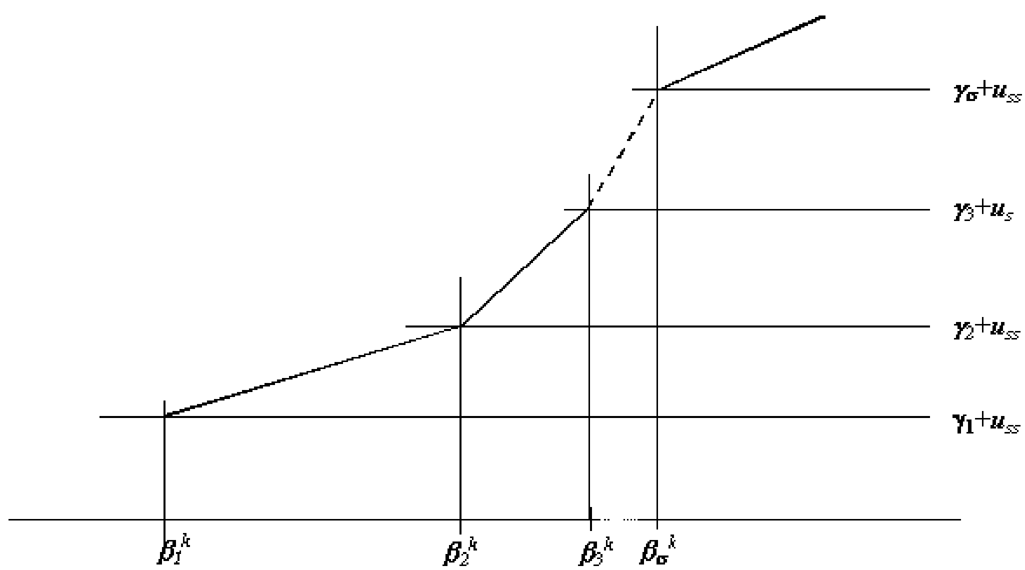

Figure 6. PWL model for controller gain.

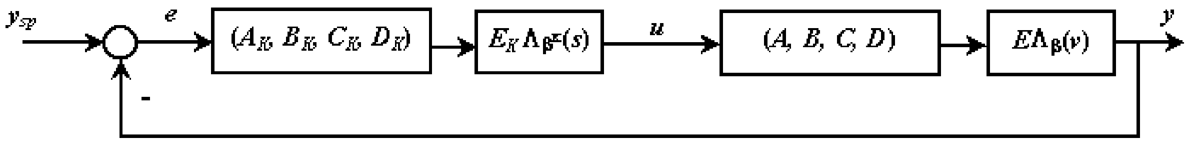

Figure 7. Closed-loop system for analysis.

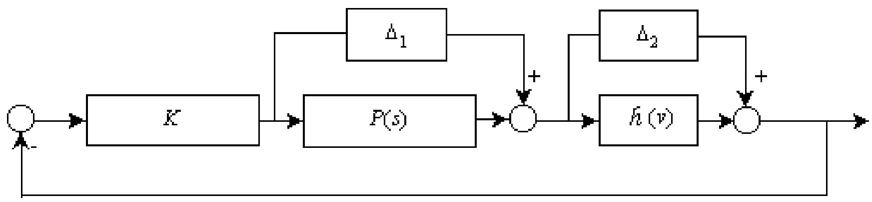

Figure 8. Block diagram for robustness analysis.

The following considerations can be mentioned:

4.1.1. *Nominal stability for exact switching.* If we consider that the switching between the partitions of the controller and the plant model is perfectly synchronized, then in the matrix $A_{cl}^{(i,j)}$ of the closed-loop system we should consider that $J_K^{(i)} = 1/J^{(i)}$, and when the controller is in the region i, the plant will also be in the sector i. Under these considerations, it is clear that

$$A_{cl}^{(i,i)} = \begin{bmatrix} A - BD_K\Gamma C & (B\Gamma C_K)/J^{(i)} \\ (-B_K\Gamma C)J^{(i)} & A_K - B_K\Gamma DC_K \end{bmatrix}$$

whose eigenvalues are independent of $J^{(i)}$. Then, as the controller stabilizes the plant for $J^{(i)} = 1$ (this happens by definition), the closed loop will be stable for any region. However, this assumption of exact switching implies that the transition is perfect from the dynamics point of view, but this is only true for steady state. The following statement represents a sufficient condition for nominal stability for smooth changes in the set-point (i.e. the set-point variation constitutes a continuous function).

4.1.2. *Nominal stability.* If we consider a smooth set-point change, then the closed-loop system will be stable if the eigenvalues of the matrices $A_{cl}^{(i,j)}$ have negative real parts for all $i = j + 1$ and $i = j - 1$. This implies that we require the controller to stabilize the plant in the regions next to the one used for the design. In the case that it is not possible to ensure a smooth switch between sectors, we can use the results on robustness presented in the next sub-section to ensure stability. In this case, the uncertainty model should include the gains of the adjacent regions.

4.2. *Robustness*

In order to analyse the robustness of the proposed control scheme, let us consider the block diagram shown in figure 8, where it was assumed that the system is linearized around some operation point. Note that we consider uncertainties in the block of linear dynamics and in the static gain, independently. This is a general description of the uncertainties for Wiener models. The $M - \Delta$ structure of this scheme is shown in figure 9, where

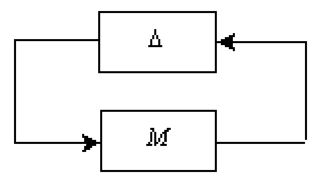

Figure 9. $M - \Delta$ structure.

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$$

and

$$M = \begin{bmatrix} -(I + KhP)^{-1}Kh & -(I + KhP)^{-1}K \\ (I - PKh)^{-1} & -(I + PKh)^{-1}PK \end{bmatrix}.$$

Note that M depends on the non-linear block. In this case we should compute a different M for each sector, considering the respective gain $J^{(i)}$. The same consideration holds for the controller block. In order to characterize the uncertainty associated to the non-linear block, we propose to bound it by using the upper and lower gain bounds. To accomplish this, it is possible to use the bounds for PWL developed by Lussón Cervantes *et al.* (2003*b*). Further details on this method can be found in the example section. Now, at this operation point, the structured singular value (Doyle 1982) can be used to analyse robustness.

Theorem 1: *The linear closed-loop system of figure 9 is robustly stable if and only if*

$$\mu_\Delta(M) < 1 \quad \forall \omega.$$

Now, if this condition is satisfied for all operation points, the non-linear closed-loop system will be robustly stable. Then, it is possible to draw the μ index as a function depending on the frequency and the operation point to determine the robustness properties of the closed-loop system. Note that whenever the whole uncertainty can be incorporated into the static gain (i.e. $\Delta_1 = 0$), it is possible to replace $\mu_\Delta(M)$ by $|M|$. It is also possible to include some of these robustness measures in the design step to ensure that a 'robust' controller is achieved (Zhou *et al.* 1995).

5. Example: pH neutralization

In order to illustrate the design procedure and to evaluate the controller performance, simulation results were obtained. A chemical process with marked non-linearity was selected. The example consists of the neutralization reaction between a strong acid (HA) and a strong base (BOH) in the presence of a buffer agent (BX) as described by Galán (2000). The neutralization takes place in a continuous stirred tank reactor (CSTR) with a constant volume V.

It is a well-known fact that pH processes control is particularly difficult to deal with. The main reason is that those processes are highly nonlinear. The slope of a chemical system's titration curve can vary several orders of magnitude over a modest range of pH values, causing the overall process gain to vary accordingly. The regions of high and low slope on the titration curve correspond to conditions of high and low gain for a pH control loop, respectively.

In figure 10, a scheme of the continuous pH neutralization process is presented. An acidic solution with a time-varying volumetric flow $q_A(t)$ of a composition $x_{1i}(t)$ is neutralized using an alkaline solution with volumetric flow $q_B(t)$ of known composition made up of base x_{2i} and buffer agent x_{3i}. Due to the high reaction rates of the acid-base neutralization, chemical equilibrium conditions are instantaneously achieved. Moreover, under the assumptions that the acid, the base and the buffer are strong enough, then the total dissociation of the three compounds takes place.

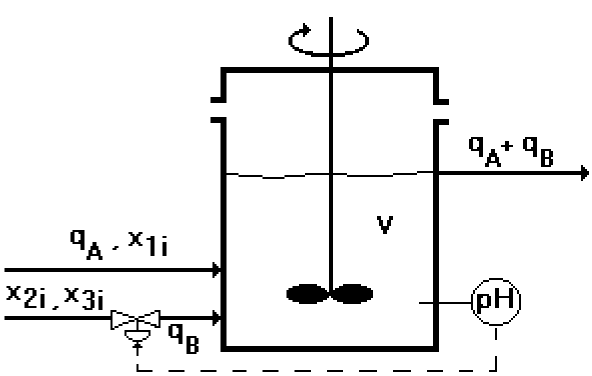

Figure 10. pH neutralization process scheme.

The process dynamics model can be obtained by considering the electroneutrality condition (which is always preserved) and through mass balances of equivalent chemical species (known as chemical invariants) that were introduced by Gustafsson and Waller (1983). For this specific case, under the previous assumptions, the dynamic behaviour of the process can be described considering the state variables

$$x_1 = [A^-] \tag{11}$$

$$x_2 = [B^+] \tag{12}$$

$$x_3 = [X^-]. \tag{13}$$

Therefore, the mathematical model of the process can be written in the following way (Galán 2000)

$$\dot{x}_1 = 1/\theta\,(x_{1i} - x_1) - 1/V\,x_1 q_B \tag{14}$$

$$\dot{x}_2 = -1/\theta\,x_2 + 1/V\,(x_{2i} - x_2) q_B \tag{15}$$

$$\dot{x}_3 = -1/\theta\,x_3 + 1/V\,(x_{3i} - x_3) q_B \tag{16}$$

$$F(x,\xi) \equiv \xi + x_2 + x_3 - x_1 - K_w/\xi - x_3/[1 + (K_x\,\xi/K_w)] = 0 \tag{17}$$

where $\xi = 10^{-pH}$ and $\theta = V/q_A$. K_w and K_x are the dissociation constants of the buffer and the water, respectively. The parameters of the system represented by (14)–(17) are addressed in table 1. Equation (17) was deduced by McAvoy *et al.* (1972), and it takes the standard form of the widely used implicit expression that connects pH with the states of the process. Note that (17) can be rewritten as a third-order polynomial

$$\begin{aligned} h(x,\xi) &\equiv \xi^3 + [K_w/K_x + x_3 + x_2 - x_1]\xi^2 \\ &\quad + \Big(x_2 - x_1 - K_x\Big) K_w/K_x\,\xi - (K_w^2)/(K_x) \\ &= 0. \end{aligned} \tag{18}$$

5.1. *Wiener model*

To obtain a linear description of the process, a linearization around the operation point $x = x_s$, $u = u_s$ (corresponding to pH = 7) was performed. The manipulated variable is $u = q_B$ and the output variable y is the

Parameter	Value
x_{2i}	0.0020 mol NaOH/l
x_{3i}	0.0025 mol $NaHCO_3$/l
K_x	10^{-7} mol/l
K_w	10^{-14} mol^2/l^2
V	2.5 l

Table 1. Non-linear model parameters.

pH measured. The resulting linear model is

$$A = \begin{bmatrix} -(1/\theta + u_s/V) & 0 & 0 \\ 0 & -(1/\theta + u_s/V) & 0 \\ 0 & 0 & -(1/\theta + u_s/V) \end{bmatrix}$$

$$B = \begin{bmatrix} -x_{1,s}/V \\ 1/V\,(x_{2,i} - x_{2,s}) \\ 1/V\,(x_{3,i} - x_{3,s}) \end{bmatrix}$$

$$C = [\eta_1 \quad \eta_2 \quad \eta_3]$$

where

$$\eta_k \triangleq \partial\eta/\partial x_k = \frac{\partial h/\partial x_k}{\xi \ln(10)(\partial h/\partial \xi)}$$

$$\frac{\partial h}{\partial \xi} = 3\xi^2 + 2[K_w/K_x + x_3 + x_2 - x_1]\xi + \left(x_2 - x_1 - K_x\right) K_w/K_x$$

$$\frac{\partial h}{\partial x_1} = -K_x\xi^2 - K_w\xi$$

$$\frac{\partial h}{\partial x_2} = K_x\xi^2 + K_w\xi$$

$$\frac{\partial h}{\partial x_3} = K_x\xi^2.$$

To determine the values for the static non-linear gain, a particular partition of the domain was performed. The necessary information to accomplish the partition was taken from the process titration curve. The bounds of each region are defined using the vector

$$\beta = \begin{bmatrix} 4.8000 \\ 6.3000 \\ 6.6000 \\ 8.2000 \\ 8.4000 \end{bmatrix}$$

and the set of parameters is

$$E = [2.6843 \quad 0.7845 \quad 8.1539 \quad -7.7951 \quad 6.2660 \quad -6.9771].$$

Figure 11 shows the plot of the real output function and the PWL approximation for this partition. In the next sub-section a PWL-H_∞ controller will be designed for the pH neutralization plant. This controller should be able to achieve the control specifications along the whole operative region.

5.2. *Controller design*

In this section, the controller formulation defined in §3 will be applied to the compensator design for the neutralization reactor. Based on the linear block part of the process model, a linear H_∞ controller is designed, using the weighing matrix W_o to reduce the steady state error for step set-point changes.

$$W_o = \frac{10^{-5}s + 1}{s + 10^{-3}}.$$

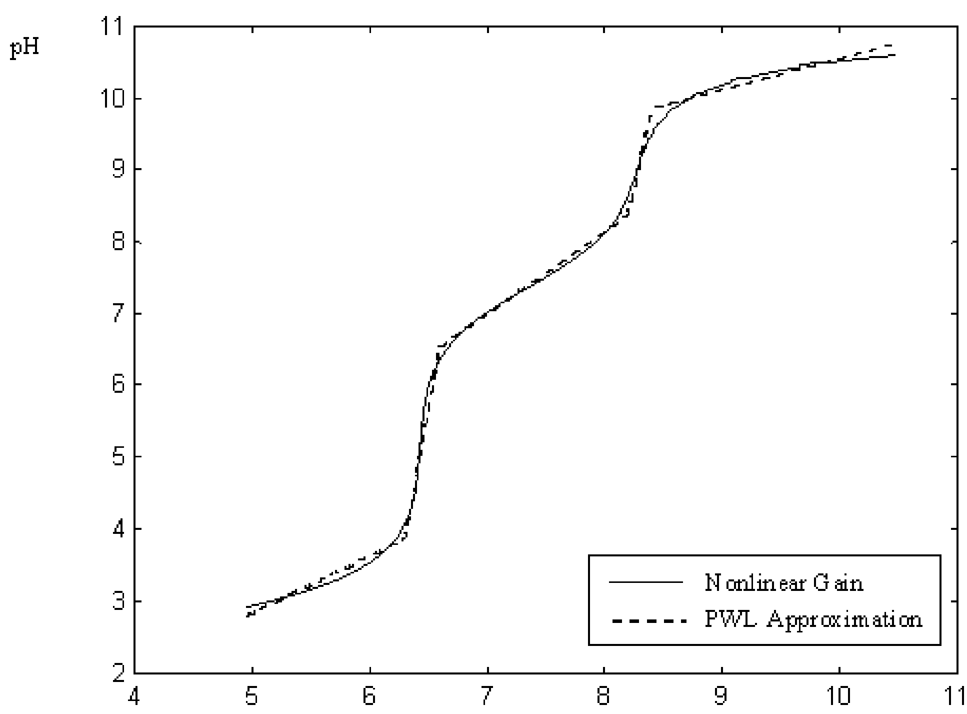

Figure 11. Non-linear gain and PWL approximation.

The resulting four states controller is

$$K = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} = \begin{bmatrix} -24.7648 & 24.1648 & 12.0824 & -0.0176 & 0 \\ 40.0916 & -40.6916 & -20.0458 & 0.0293 & 0 \\ 49.9772 & -49.9772 & -25.5886 & 0.0365 & 0 \\ 0 & 0 & 0 & -0.0010 & 39.9349 \\ 859.5248 & -859.5248 & -429.7624 & 0.6275 & 0 \end{bmatrix}$$

which allows the value $\|T_{zw}\|_\infty = 0.3987$ for the objective function.

For the non-linear gain of the controller, the limits of the region are given by the vector

$$\beta^K = \begin{bmatrix} -0.7786 \\ -0.5686 \\ -0.0823 \\ 0.2470 \\ 0.5161 \end{bmatrix}$$

and the set of parameters is

$$E_K = [-0.3979 \ 1.2921 \ -1.1805 \ 0.7672 \ -0.7443 \ 2.1627].$$

5.3. *Simulation results*

In this section the controller performance will be analysed by simulation. The output is asked to follow a given set-point along the whole operative region. To satisfy the continuity constraint for the set-point variation, a polynomial approximation was used to fit the step-like changes.

In the first stage, the PWL plant is controlled using two H_∞ controllers, corresponding to sectors 1 and 2. The results are presented in figures 12 and 13 for the pH and the manipulated variable respectively. From these plots, it is clear that no admissible performance is obtained for a unique controller in the complete operative region. Note that the controller for sector 2 responds with a desirable speed, but it requires an excessive control action. Moreover, in a portion of the plot, the manipulated variable u reaches saturation at zero. On the other hand, the controller of sector 1 is too slow.

Figures 14 and 15 show the simulation results for the PWL controller when it is applied to the PWL model of the plant and to the non-linear real model of the process. Note that the performance (for example the closed-loop time constant) is almost constant along all the operating region. The manipulated variable is smooth enough to ensure good performance when applied to the real process.

5.4. *Robustness analysis*

Before developing the robustness analysis, the uncertainty is characterized. To perform it, a set of data is obtained by stationary and dynamic simulations. Let us

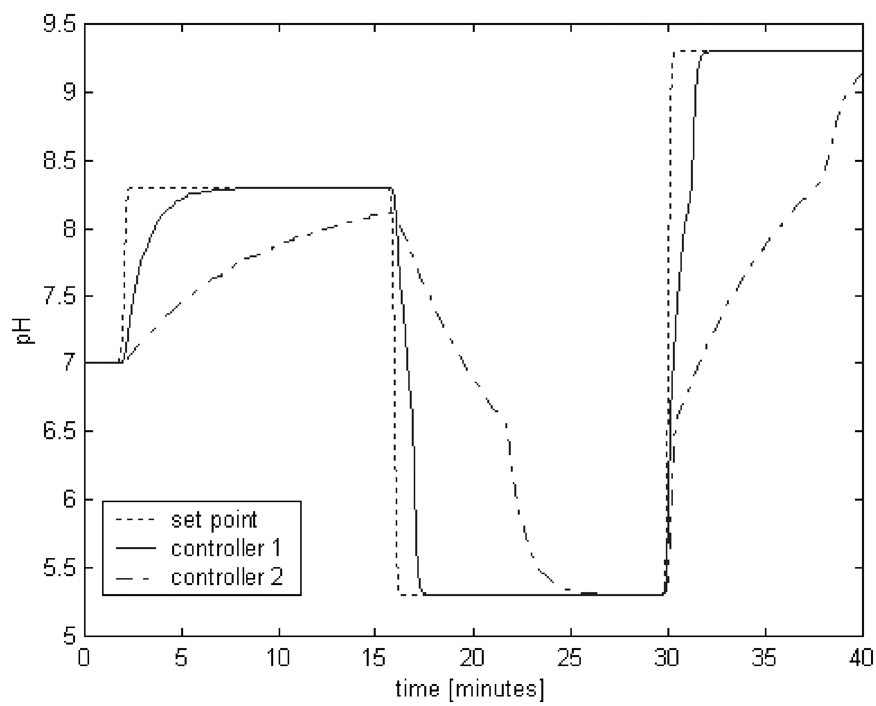

Figure 12. pH reference and process output for PWL plant and two linear controllers designed for sector 1 and sector 2.

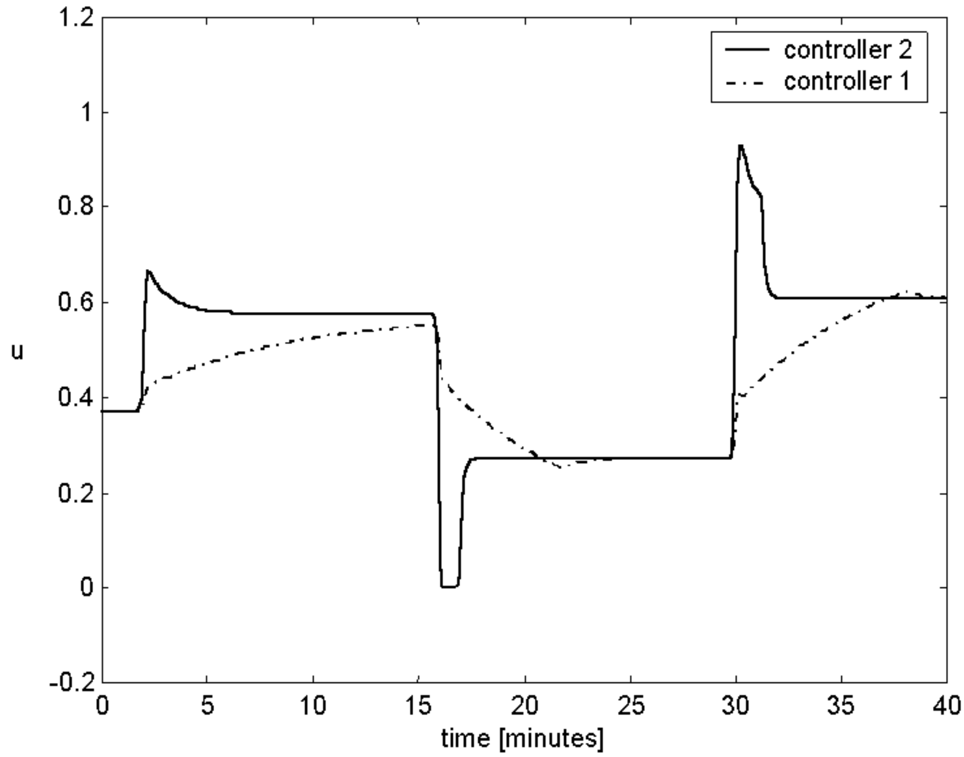

Figure 13. Manipulated variable for PWL plant and two linear controllers (for sector 1 and sector 2).

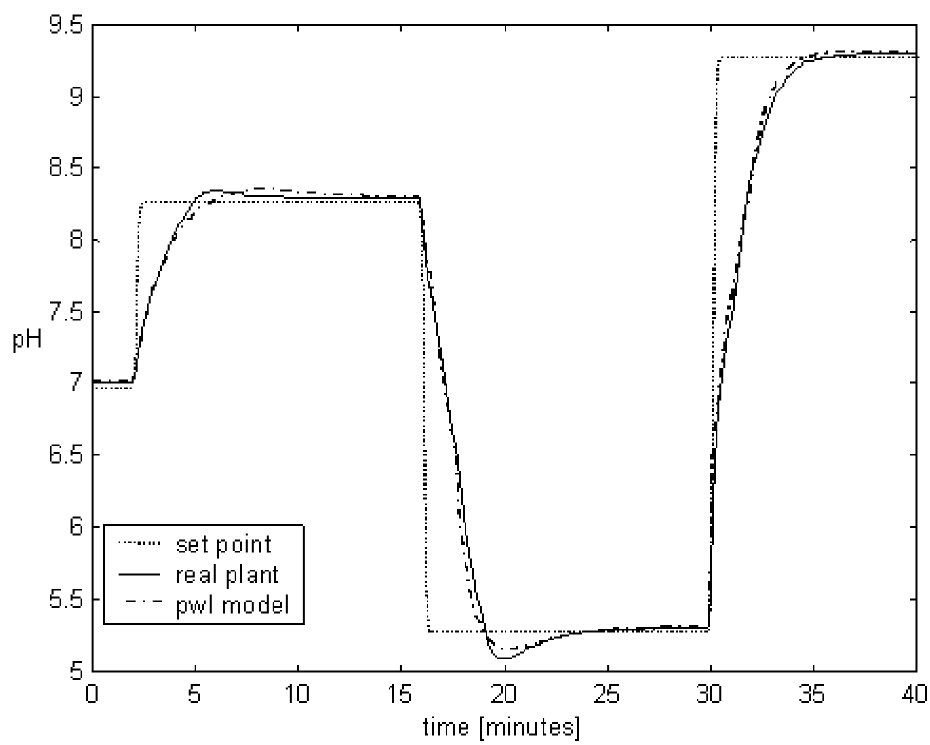

Figure 14. pH reference and process outputs for non-linear plant/PWL controller, and PWL plant/PWL controller.

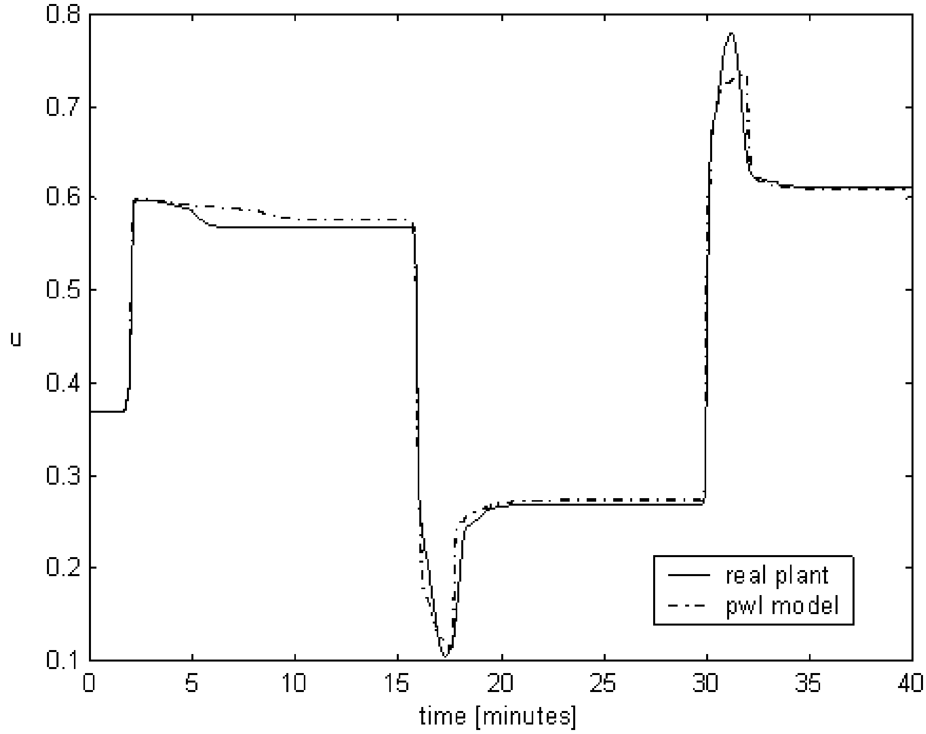

Figure 15. Manipulated variable for non-linear and PWL plant and PWL controller.

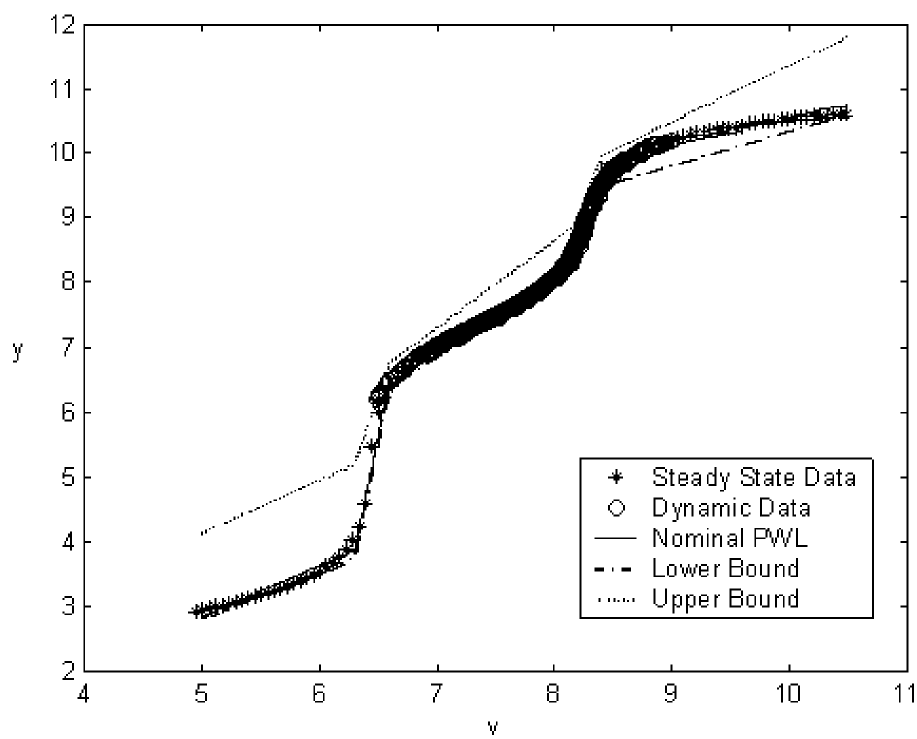

Figure 16. Uncertainty bounds.

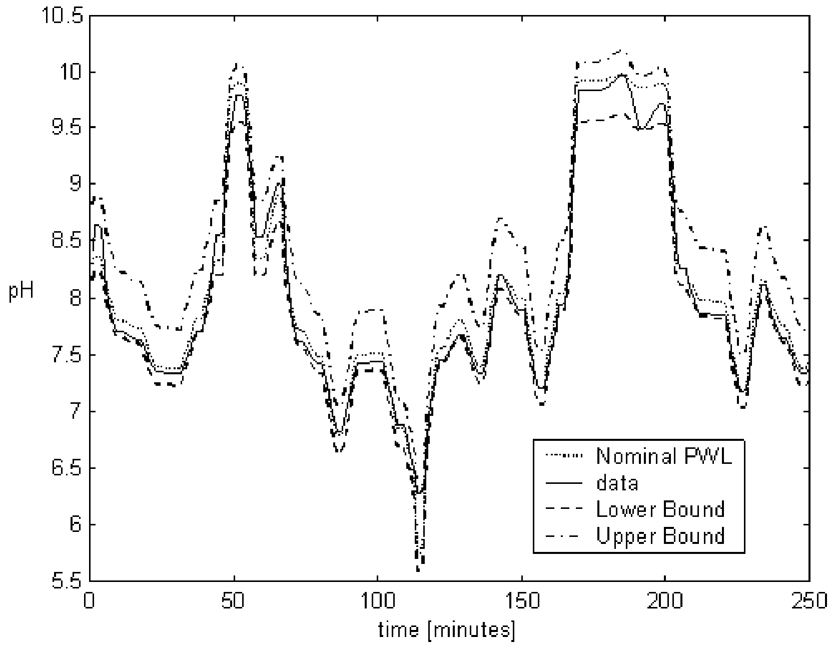

Figure 17. Data validation of uncertainty bounds using dynamic data.

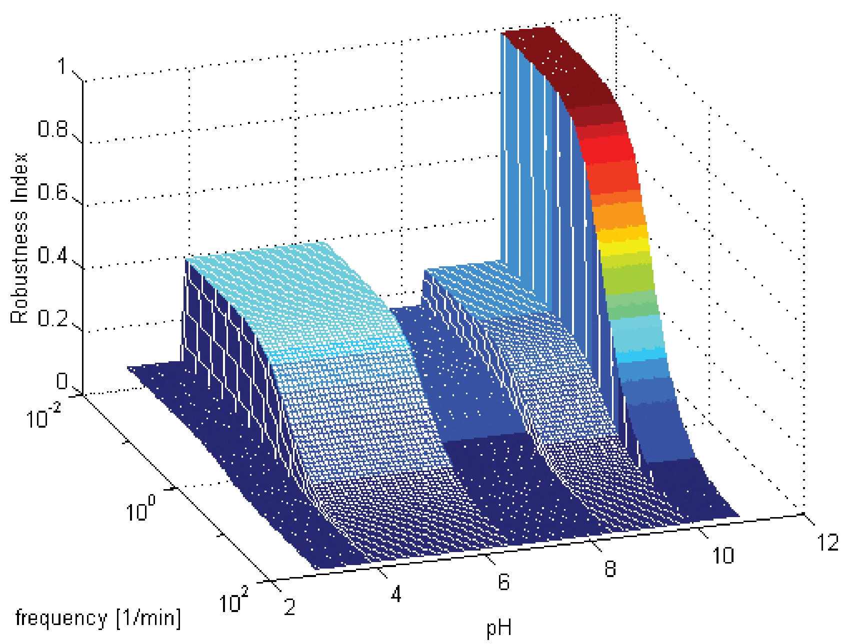

Figure 18. Robust stability measure.

call v_i and y_i the input and output of the linear block, respectively. Using these data, uncertainty bounds are obtained for the static non-linear gain block. The lower bound is computed by solving the linear optimization problem (Lussón Cervantes *et al.* 2003 *b*)

$$\min_{\lambda_l^j, E_l} \sum \lambda_l^j$$

$$\text{s.t.} \quad E_l \Lambda_\beta(v_i) + \lambda_l^j \leq f_i - E\Lambda_\beta(v_i), \quad \forall v_i \in \aleph^j, \quad \lambda_l^j \geq 0.$$

To compute the upper bound, the problem to be solved is

$$\min_{\lambda_u^j, E_u} \sum \lambda_u^j$$

$$\text{s.t.} \quad -E_u \Lambda_\beta(v_i) + \lambda_u^j \leq -f_i + E\Lambda_\beta(v_i),$$
$$\forall v_i \in \aleph^j, \quad \lambda_u^j \geq 0.$$

These uncertainty bounds are enough to include all the uncertainties in the model. The resulting bound parameters are given by the vectors

$$E_l = [-0.0462 \quad -0.0702 \quad 0.0503 \quad 0.0312 \quad -1.1837 \quad 1.2717]$$

and

$$E_u = [1.2595 \quad 0.0379 \quad -3.7355 \quad 3.8937 \quad -2.3124 \quad 2.5556].$$

Figure 16 shows these bounds as an input–output relation, while figure 17 shows the validation of these bounds in a dynamic simulation. From these figures it is clear that these gains are enough to ensure that the real process is between them. Using this description for uncertainty, the robustness analysis is performed. To do this, $|M|$ is computed for all frequencies and all operating conditions (given for steady state values of pH). From it (see figure 18) it is clear that the closed-loop system is robustly stable.

6. Conclusions

The objective of this work was to present a design methodology for an H_∞ controller for non-linear plants. The first step in the design procedure consists of modelling the process as a Wiener system, formed by a linear dynamic block in cascade with a PWL gain. Then, the resulting controller shares this structure, where the dynamic part is a classical lineal H_∞ compensator. Moreover, the controller PWL gain is obtained through a transformation of the partition and the gain of the plant. Results for stability and robustness of the proposed controller are included. Simulations show that the use of the proposed controller gives excellent results.

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