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Effective action for QED_4 through ζ function regularization

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We obtain, through ζ function methods, the one-loop effective action for massive Dirac fields in the presence of a uniform, but otherwise general, electromagnetic background. After discussing renormalization, we compare our ζ function result with Schwinger's proper-time approach. © *2001 American Institute of Physics.*
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I. INTRODUCTION

In QED, the effective one-loop Lagrangian describes the effective nonlinear interaction of the electromagnetic fields due to a single fermion loop. In two dimensions, its general form has been obtained both through proper time and ζ function regularizations.[1,2] In four dimensions, on the other hand, only particular field configurations have been studied.

The $3+1$ dimensional problem of constant electromagnetic fields was first studied by Euler and Heisenberg[3] and independently by Weisskopf.[4] These authors obtained an integral expression for the one-loop effective Lagrangian in the framework of the electron-hole theory. Later on, Schwinger rederived this integral representation in a field-theoretical scenario, by making use of proper time techniques.[5] In all these references, explicit results were derived in some limits, the most famous being the weak-field one. This and other particular field configurations were subsequently studied through the proper-time regularization by a number of authors (see, for example, Refs. 6–10).

More recently, the interest in the subject was renewed, and the Euclidean effective action for constant electromagnetic background configurations was studied through ζ function techniques:[11,12] In Ref. 13 analytic expressions were found for the case of purely magnetic fields in any number of dimensions. In this same reference, the case of equal electric and magnetic fields in four Euclidean dimensions was also studied. A step towards more general field configurations was given in Ref. 14, where the authors obtained the effective Lagrangian as a power series in B/E.

It is the aim of this article to obtain, through ζ function methods, an explicit nonperturbative expression for the full one-loop effective action of quantum electrodynamics in four dimensions in the case of constant, but otherwise arbitrary, electromagnetic fields. To this end, we will work in Euclidean space–time, and define the determinant of the relevant Dirac operator $\not{D}$ through the derivative of the ζ function of $\not{D}^\dagger \not{D}$.

The organization of the article is as follows:

After summarizing some well-known generalities in Sec. II, we devote Sec. III to analytically extending the relevant ζ function to the region $\Re s > -2$. (The main point here is the analytic extension of a Barnes ζ function.) Its value at $s=0$ is also given in this section.

In Sec. IV, a complete analytical expression for the effective action in terms of special functions is given, and the renormalization issue is discussed.

Section V contains a comparison between ζ and proper-time regularizations.

The Appendices A and B contain the derivation of some particular limits for the relevant zeta

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and for the effective action, thus allowing for the comparison with previous work on less general field configurations.

II. GENERALITIES

We study the effective action for massive Dirac particles in the presence of uniform, but otherwise arbitrary, electromagnetic background fields. We work in four-dimensional Euclidean space. Then, the effective action in the one-loop approximation is given by

$$S[A_\mu]=S_{\rm cl}[A_\mu]-\log\mathrm{Det}(\not{D}[A_\mu]), \tag{1}$$

where $S_{\rm cl}[A_\mu]$ is the classical Euclidean action and $\not{D}[A_\mu]=\gamma_\mu(\partial_\mu-ieA_\mu)+im$ is the Euclidean Dirac operator, m being the fermion mass.

Note that, even though $\not{D}$ is not self-adjoint, it is normal; so, the functional determinant appearing in the one-loop correction to the action can be defined through ζ function regularization,[11,12] which leads to

$$S_{\rm eff}[A_\mu]=S_{\rm cl}[A_\mu]+S^{(1)}[A_\mu]=S_{\rm cl}[A_\mu]+\frac{1}{2}\frac{\partial}{\partial s}\zeta(s;\not{D}^\dagger\not{D})\big]_{s=0}. \tag{2}$$

In order to evaluate the one-loop correction $S^{(1)}$ in the previous expression, it is necessary to obtain the spectrum of the operator $\not{D}^\dagger\not{D}$, which is well known in the case of uniform fields.[15] In this particular situation, one can always choose a reference frame such that $F_{03}=-F_{30}=E$ and $F_{12}=-F_{21}=B$, while the remaining components of the field tensor vanish. When doing so, the required zeta function turns out to be

$$\zeta(s;\not{D}^\dagger\not{D})=\mu^4\Omega\frac{ab}{4\pi^2}\Bigg[2\sum_{n_a=1}^{\infty}(2n_a a+c)^{-s}+2\sum_{n_b=1}^{\infty}(2n_b b+c)^{-s} +4\sum_{n_a=1}^{\infty}\sum_{n_b=1}^{\infty}(2n_a a+2n_b b+c)^{-s}+c^{-s}\Bigg]. \tag{3}$$

Here, Ω is the volume of the four-dimensional Euclidean space, $a=e|E|/\mu^2$, $b=e|B|/\mu^2$, $c=m^2/\mu^2$, and μ is a parameter with mass dimension, introduced to render the ζ function dimensionless. Note that the series in Eq. (3) are all convergent for $\Re s>2$, where they define an analytic function of s.

III. ANALYTIC EXTENSION OF THE ζ FUNCTION

In this section, we will perform the analytic extension of the relevant ζ function to a region containing $s=0$. In particular, we will show it to be finite at $s=0$ and give its value at this point.

The first two terms in Eq. (3) can be rewritten in terms of Hurwitz' zeta functions, which are well known to be meromorphic functions with a unique simple pole at $s=1$. On the other hand, the third term is a zeta function of the Barnes' type[16,17] (see also Refs. 18 and 19 and references therein). In order to analytically extend this term, we write it in integral form. After doing so, we get

$$\begin{aligned}\zeta(s;\not{D}^\dagger\not{D})&=\mu^4\Omega\frac{ab}{4\pi^2}\Bigg\{\frac{2}{(2a)^s}\zeta\left(s,\frac{c}{2a}+1\right)+\frac{2}{(2b)^s}\zeta\left(s,\frac{c}{2b}+1\right) +\frac{1}{\Gamma(s)}\int_0^\infty dt\,t^{s-1}\frac{4e^{-2at}e^{-2bt}e^{-ct}}{(1-e^{-2at})(1-e^{-2bt})}+c^{-s}\Bigg\}\\ &=\mathrm{A}(s)+\mathrm{B}(s)+\mathrm{C}(s)+\mathrm{D}(s),\end{aligned} \tag{4}$$

where $\zeta(s,v)$ is Hurwitz' zeta function. This expression (invariant under $a\leftrightarrow b$) is, in principle, well defined for $\Re s>2$. Since the analytic structure of $A(s)$ and $B(s)$ is well known, we will concentrate on the Barnes term $C(s)$, which will be extended to $\Re s>-2$.

To this end, we will use the expansion[20]

$$\frac{1}{e^{at}-e^{-at}}=\frac{1}{2at}+at\sum_{k=1}^{\infty}(-1)^k\frac{1}{(at)^2+(k\pi)^2}, \tag{5}$$

thus obtaining

$$\begin{aligned}C(s)&=2\mu^4\Omega\frac{ab}{4\pi^2}\frac{1}{\Gamma(s)}\left\{\frac{1}{2a}\int_0^\infty dt\,t^{s-2}\frac{e^{-(a+b+c)t}}{e^{bt}-e^{-bt}}\right.\\&\quad\left.+a\int_0^\infty dt\,t^s\frac{e^{-(a+b+c)t}}{e^{bt}-e^{-bt}}\sum_{k=1}^{\infty}(-1)^k\frac{1}{(at)^2+(k\pi)^2}\right\}+a\leftrightarrow b\\&=C_1(s)+C_2(s).\end{aligned} \tag{6}$$

The first term, $C_1(s)$, can be easily seen to be

$$C_1(s)=2\mu^4\Omega\frac{ab}{4\pi^2}\frac{1}{2a}\frac{1}{(s-1)(2b)^{s-1}}\zeta\left(s-1,\frac{a+2b+c}{2b}\right)+a\leftrightarrow b. \tag{7}$$

As all the terms we have analytically extended up to this point, $C_2(s)$ in Eq. (6) involves an integral which diverges at $s=0$. In order to isolate this singularity, we will rewrite this term as

$$\begin{aligned}C_2(s)&=2\mu^4\Omega\frac{ab}{4\pi^2}\frac{1}{\Gamma(s)}a\int_0^\infty dt\,t^s\frac{e^{-(a+b+c)t}}{(e^{bt}-e^{-bt})}\left\{\sum_{k=1}^{\infty}(-1)^k\left[\frac{1}{(at)^2+(k\pi)^2}-\frac{1}{(k\pi)^2}\right]\right.\\&\quad\left.+\sum_{k=1}^{\infty}(-1)^k\frac{1}{(k\pi)^2}\right\}+a\leftrightarrow b\\&=CF_2(s)+CD_2(s).\end{aligned} \tag{8}$$

The integral appearing in $CD_2(s)$ is divergent at $s=0$ but, after performing the sum, this term is easily seen to be

$$CD_2(s)=-\mu^4\Omega\frac{ab}{4\pi^2}\frac{a}{6}\frac{s}{(2b)^{s+1}}\zeta\left(s+1,1+\frac{a+c}{2b}\right)+a\leftrightarrow b. \tag{9}$$

Now, once the difference between brackets is performed, $CF_2(s)$ can be rewritten as

$$CF_2(s)=-2\mu^4\Omega\frac{ab}{4\pi^2}\frac{1}{\Gamma(s)}a^3\sum_{k=1}^{\infty}\frac{(-1)^k}{(k\pi)^2}\int_0^\infty dt\,t^{s+2}\frac{e^{-(a+2b+c)t}}{(1-e^{-2bt})}\frac{1}{(at)^2+(k\pi)^2}+a\leftrightarrow b. \tag{10}$$

As is easily seen, this integral converges for $\Re s>-2$. We have thus obtained an analytic extension for the ζ of the operator as a meromorphic function with only simple poles. Such extension is valid for $\Re s>-2$.

Now, the factor $1/[(at)^2+(k\pi)^2]$ can be written as an integral. In fact,

$$\frac{1}{(at)^2+(k\pi)^2}=\frac{-1}{2ik\pi}\left[\frac{1}{at+ik\pi}-\frac{1}{at-ik\pi}\right]=\frac{1}{k\pi}\int_0^\infty du\, e^{-atu}\sin(k\pi u).$$

When replaced in Eq. (10), this gives

$$\mathrm{CF}_2(s)=-2\mu^4\Omega\frac{ab}{4\pi^2}\frac{1}{\Gamma(s)}a^3\sum_{k=1}^{\infty}\frac{(-1)^k}{(k\pi)^3}\int_0^\infty dt\, t^{s+2}\frac{e^{-(a+2b+c)t}}{(1-e^{-2bt})}\int_0^\infty du\, e^{-atu}\sin(k\pi u)+a\leftrightarrow b$$

or, after interchanging the integrals

$$\mathrm{CF}_2(s)=-2\mu^4\Omega\frac{ab}{4\pi^2}\frac{a^3}{\Gamma(s)}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k\pi)^3}\int_0^\infty du\,\sin(k\pi u)\frac{\Gamma(s+3)}{(2b)^{s+3}}\zeta\left(s+3,\frac{a+2b+c+au}{2b}\right)$$
$$+a\leftrightarrow b.$$

When the ζ function is written in terms of its series development (which is valid for $\Re s>-2$) one has (after interchanging this series and the integral)

$$\mathrm{CF}_2(s)=-2\mu^4\Omega\frac{ab}{4\pi^2}\frac{a^3}{\Gamma(s)}\frac{\Gamma(s+3)}{(2b)^{s+3}}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k\pi)^3}\sum_{l=1}^{\infty}\int_0^\infty du\,\sin(k\pi u)\left(l+\frac{a+c+au}{2b}\right)^{-(s+3)}$$
$$+a\leftrightarrow b.$$

Finally, after performing the remaining integral and making use of the functional relations between incomplete gamma functions,[21] one gets

$$\mathrm{CF}_2(s)=i\mu^4\Omega\frac{ab}{4\pi^2}\frac{\Gamma(s+3)}{\Gamma(s)}a^{-s}\frac{1}{s+2}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k\pi)^{1-s}}\sum_{l=1}^{\infty}\left[i^{s+2}e^{i(k\pi/a)(2bl+a+c)}\Gamma\left(-s\right.\right.$$
$$\left.\left.-1,i\frac{k\pi}{a}(2bl+a+c)\right)-(-i)^{s+2}e^{-i(k\pi/a)(2bl+a+c)}\Gamma\left(-s-1,-i\frac{k\pi}{a}(2bl+a+c)\right)\right]$$
$$+a\leftrightarrow b. \tag{11}$$

The replacement of Eqs. (7), (9) and (11) into Eq. (4) completes the analytic extension of the relevant ζ function. Its value at $s=0$ can be easily computed, which gives

$$\zeta(0;\mathbb{D}^\dagger\mathbb{D})=\frac{\mu^4\Omega}{4\pi^2}\left\{\frac{1}{2}c^2+\frac{a^2+b^2}{3}\right\}. \tag{12}$$

The agreement with the known results for null and equal fields is shown in Appendix A.

IV. THE EFFECTIVE ACTION AND ITS RENORMALIZATION

This section contains the main result in this article, i.e., the one-loop correction to the Euclidean effective action. According to Eq. (2), to obtain such result, one must perform the derivatives at $s=0$ of the various terms in Eq. (4).

We start from $\mathrm{A}(s)$, which contributes with

$$\frac{1}{2}\frac{\partial}{\partial s}\mathrm{A}(s)\Big|_{s=0}=\mu^4\Omega\frac{ab}{4\pi^2}\left\{\log(2a)\left(\frac{1}{2}+\frac{c}{2a}\right)+\log\Gamma\left(\frac{c}{2a}+1\right)-\frac{1}{2}\log(2\pi)\right\}. \tag{13}$$

In a completely analogous way, one has

$$\frac{1}{2}\frac{\partial}{\partial s}\mathrm{B}(s)\Big|_{s=0}=\mu^4\Omega\,\frac{ab}{4\pi^2}\left\{\log(2b)\left(\frac{1}{2}+\frac{c}{2b}\right)+\log\Gamma\left(\frac{c}{2b}+1\right)-\frac{1}{2}\log(2\pi)\right\}. \tag{14}$$

It is also through a direct calculation that one gets

$$\frac{1}{2}\frac{\partial}{\partial s}\mathrm{C}_1(s)\Big|_{s=0}=\mu^4\Omega\,\frac{ab}{4\pi^2}\,\frac{1}{2a}\left\{2b(-1+\log(2b))\zeta\left(-1,1+\frac{a+c}{2b}\right)\right.$$
$$\left.-2b\,\frac{\partial}{\partial s}\Big|_{s=0}\zeta\left(s-1,1+\frac{a+c}{2b}\right)\right\}+a\leftrightarrow b. \tag{15}$$

$$\frac{1}{2}\frac{\partial}{\partial s}\mathrm{CD}_2(s)\Big|_{s=0}=\mu^4\Omega\,\frac{ab}{4\pi^2}\,\frac{a}{24b}\left\{\log(2b)+\Psi\left(1+\frac{a+c}{2b}\right)\right\}+a\leftrightarrow b. \tag{16}$$

As regards $\mathrm{CF}_2(s)$, due to the presence of $\Gamma(s)$ in the denominator, the required derivative reduces to the product $\Gamma(s)\,\mathrm{CF}_2(s)$ at $s=0$, i.e.,

$$\frac{1}{2}\frac{\partial}{\partial s}\mathrm{CF}_2(s)\Big|_{s=0}=-\frac{i}{2}\mu^4\Omega\,\frac{ab}{4\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k\pi}\sum_{l=1}^{\infty}\left[e^{i(k\pi/a)(2bl+a+c)}\Gamma\left(-1,\frac{ik\pi}{a}(2bl+a+c)\right)\right.$$
$$\left.-e^{-i(k\pi/a)(2bl+a+c)}\Gamma\left(-1,-\frac{ik\pi}{a}(2bl+a+c)\right)\right]+a\leftrightarrow b. \tag{17}$$

Summarizing, the Euclidean effective action is given by the sum of the partial contributions in Eqs. (13)–(17), plus

$$\frac{1}{2}\frac{\partial}{\partial s}\mathrm{D}(s)\Big|_{s=0}=-\mu^4\Omega\,\frac{ab}{8\pi^2}\log(c). \tag{18}$$

Notice that even though the result is finite, it depends on the arbitrary parameter μ. However, this effective action still admits a finite renormalization. We will perform it by adopting the criterium (used, for instance, in Ref. 22), that a very massive field does not fluctuate. Thus, we will subtract the one loop correction to the effective action in the limit $m\to\infty$. From Eq. (B6) in Appendix B, the effective action in this limit can be seen to be

$$\mu^4\Omega\,\frac{1}{4\pi^2}\left\{\left[\frac{3}{8}-\frac{1}{4}\log(c)\right]c^2-\frac{1}{6}(b^2+a^2)\log(c)\right\}. \tag{19}$$

After doing this subtraction, all dependence on the parameter μ disappears, and the Euclidean effective action is given by

$$S_{\mathrm{eff}}^{\mathrm{Ren}}[A_\mu]=\frac{\Omega\mu^4}{2e^2}(a^2+b^2)+\mu^4\Omega\,\frac{ab}{4\pi^2}\left\{\frac{1}{8}\log\left(\frac{4ab}{c^2}\right)-\frac{1}{24}\,\frac{(a^2+b^2)}{ab}\log\left(\frac{4ab}{c^2}\right)+\frac{c}{4a}\log\left(\frac{a}{b}\right)\right.$$
$$-\frac{c^2}{16ab}\log\left(\frac{4ab}{c^2}\right)+\log\left(\frac{\Gamma(c/2a+1)}{\sqrt{2\pi}}\right)-\frac{b}{a}\zeta\left(-1,1+\frac{a+c}{2b}\right)-\frac{b}{a}\,\frac{\partial}{\partial s}\Big|_{s=0}\zeta\left(s-1,1\right.$$
$$\left.+\frac{a+c}{2b}\right)-\frac{i}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k\pi}\sum_{l=1}^{\infty}\left[e^{i(k\pi/a)(2bl+a+c)}\Gamma\left(-1,\frac{ik\pi}{a}(2bl+a+c)\right)\right.$$

$$-e^{-i(k\pi/a)(2bl+a+c)}\Gamma\left(-1,\frac{-ik\pi}{a}(2bl+a+c)\right)\Bigg]$$

$$+\frac{a}{24b}\Psi\left(1+\frac{a+c}{2b}\right)-\frac{3}{16}\frac{c^2}{ab}+a\leftrightarrow b\Bigg\}. \tag{20}$$

The renormalization performed amounts to subtracting the zero field effective action (thus redefining the cosmological constant), and renormalizing the classical action. As a result, one gets the following running charge relationship:

$$\frac{1}{e^2}=\frac{1}{e_0^2}+\frac{1}{12\pi^2}\log\frac{\mu^2}{m^2}. \tag{21}$$

Equivalently, for the fine structure constant one has

$$\alpha=\frac{\alpha_0}{1+(\alpha_0/3\pi)\log\mu^2/m^2}. \tag{22}$$

Note that this expression reduces, in the perturbative limit, to the well known result (see, for example, Ref. 23)

$$\alpha=\alpha_0\left(1-\frac{\alpha_0}{3\pi}\log\frac{\mu^2}{m^2}\right). \tag{23}$$

V. COMPARISON WITH THE PROPER TIME RESULT

In Appendix B we show that, in the weak field limit, our result for the ζ regularized effective action coincides, once renormalized, with the Euclidean version of the well known Schwinger's proper time one.

In this section, we will show that this is also the case for arbitrary field strengths. In fact, Schwinger's integral expression for the one loop correction to the effective action is given, after subtracting the divergent terms, by

$$S_{PT}^{(1)}=\mu^4\Omega\Bigg\{\frac{ab}{8\pi^2}\int_0^\infty dt\,t^{s-1}e^{-ct}\coth(bt)\coth(at)-\frac{1}{8\pi^2}\int_0^\infty dt\,t^{s-3}e^{-ct}$$

$$-\frac{a^2+b^2}{24\pi^2}\int_0^\infty dt\,t^{s-1}e^{-ct}\Bigg\}\Bigg|_{s=0}. \tag{24}$$

Now, performing the integrals in the last two terms and comparing with Eq. (4) (with the Hurwitz's zetas written in integral form), the previous expression can be rewritten as

$$S_{PT}^{(1)}=\frac{1}{2}\left\{\Gamma(s)\zeta(s;\mathcal{D}^\dagger\mathcal{D})-\frac{\mu^4\Omega}{4\pi^2}\left(c^{2-s}\Gamma(s-2)+\frac{a^2+b^2}{3}c^{-s}\Gamma(s)\right)\right\}\Bigg|_{s=0}. \tag{25}$$

After developing around $s=0$, it is easy to see that

$$S_{PT}^{(1)}=S_\zeta^{(1)}-\frac{\mu^4\Omega}{4\pi^2}\left[\frac{3}{8}c^2-\left(\frac{c^2}{4}+\frac{a^2+b^2}{6}\right)\log c\right], \tag{26}$$

where $S_\zeta^{(1)}$ is the ζ-regularized one loop correction to the effective action, as defined in Eq. (2), and the remaining terms are precisely the ones we have subtracted through renormalization. So, the exact agreement between both renormalized effective actions is apparent.

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APPENDIX A: THE LIMITS OF NULL AND EQUAL FIELDS

In this section, we will show the agreement of our general ζ function with the results obtained by other authors for some particular cases, i.e., the case of a null electric or magnetic field[13,14] and that of equal electric and magnetic fields.[13]

We will start with the $B\to 0$ limit. It is easy to see that $\lim_{b\to 0}\mathrm{A}(s)=0$. As regards $\lim_{b\to 0}\mathrm{B}(s)$, it can be studied by making use of the asymptotic expansion for Hurwitz' ζ function (see, for example, Ref. 24),

$$\zeta(s,v)=\frac{1}{\Gamma(s)}\left\{v^{1-s}\Gamma(s-1)+\frac{1}{2}v^{-s}\Gamma(s)+\sum_{n=1}^{N}B_{2n}\frac{\Gamma(s+2n-1)}{(2n)!}v^{1-s-2n}\right\}+O(v^{-2N-s-1}), \tag{A1}$$

$$\lim_{b\to 0}\mathrm{B}(s)=\lim_{b\to 0}\left\{\mu^4\Omega\,\frac{ab}{4\pi^2}\,\frac{2}{(2b)^s}\,\frac{\Gamma(s-1)}{\Gamma(s)}\left(\frac{c}{2b}+1\right)^{1-s}\right\}=\frac{\mu^4\Omega}{4\pi^2}\,\frac{a}{s-1}c^{1-s}. \tag{A2}$$

The only contribution to $\mathrm{C}(s)$ in this limit comes from $\mathrm{C}_1(s)$, which gives

$$\lim_{b\to 0}\mathrm{C}(s)=\frac{\mu^4\Omega}{4\pi^2}\,\frac{(2a)^{2-s}}{s-1}\left\{\zeta\left(s-1,\frac{c}{2a}\right)-\left(\frac{c}{2a}\right)^{1-s}\right\}. \tag{A3}$$

Finally, $\mathrm{D}(s)$ vanishes for $b=0$. Then, replacing all these partial results into Eq. (4), one obtains

$$\zeta(s,\mathbb{D}^\dagger\mathbb{D})\rfloor_{b=0}=\frac{\mu^4\Omega}{4\pi^2}\,\frac{(2)^{1-s}}{s-1}a^{2-s}\left\{2\zeta\left(s-1,\frac{c}{2a}\right)-\left(\frac{c}{2a}\right)^{1-s}\right\}, \tag{A4}$$

which is in complete agreement with previous results.[13,14]

Of course, the $E\to 0$ limit gives an analogous expression, which can be obtained by changing $a\to b$ in Eq. (A4).

We will now study the equal fields limit. In this situation, taking $a=b$ in the different terms appearing in the ζ function (4), we have

$$\begin{aligned}\zeta(s;\mathbb{D}^\dagger\mathbb{D})\rfloor_{a=b}=\mu^4\Omega\,\frac{a^2}{4\pi^2}\Bigg\{&\frac{4}{(2a)^s}\zeta\left(s,\frac{c}{2a}+1\right)+c^{-s}+\frac{2^{2-s}a^{-s}}{s-1}\zeta\left(s-1,\frac{3}{2}+\frac{c}{2a}\right)\\&-\frac{1}{6}(2a)^{-s}\,\zeta\left(s+1,\frac{3}{2}+\frac{c}{2a}\right)-i\,2\,a^{-s}(s+1)s\\&\times\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k\pi)^{1-s}}\sum_{l=1}^{\infty}\left[i^{s+2}e^{ik\pi(2l+1+c/a)}\Gamma\left(-s-1,ik\pi\left(2l+1+\frac{c}{a}\right)\right)\right.\end{aligned}$$

$$-(-i)^{s+2}e^{-ik\pi(2l+1+c/a)}\Gamma\left(-s-1,-ik\pi\left(2l+1+\frac{c}{a}\right)\right)\Bigg]\Bigg\}. \tag{A5}$$

In order to compare this expression with the result in Ref. 13, we use the functional relations between incomplete gamma functions once more, thus getting

$$\begin{aligned}\zeta(s;\mathbb{D}^{\dagger}\mathbb{D})\rfloor_{a=b}=\mu^4\Omega\frac{a^2}{4\pi^2}\Bigg\{&\frac{4}{(2a)^s}\zeta\left(s,\frac{c}{2a}+1\right)+c^{-s}+\frac{2^{2-s}a^{-s}}{s-1}\zeta\left(s-1,\frac{3}{2}+\frac{c}{2a}\right)\\&-i\,2\,a^{-s}s\sum_{k=1}^{\infty}\frac{(-1)^k}{(k\pi)^{1-s}}\sum_{l=1}^{\infty}\Bigg[i^{s+2}e^{ik\pi(2l+1+c/a)}\Gamma\left(-s,ik\pi\left(2l+1+\frac{c}{a}\right)\right)\\&-(-i)^{s+2}e^{-ik\pi(2l+1+\frac{c}{a})}\Gamma(-s,-ik\pi(2l+1+c/a))\Bigg]\Bigg\}.\end{aligned} \tag{A6}$$

We now use the integral representation for the incomplete gamma function

$$\Gamma(\alpha,x)=\int_x^{\infty}dt\,e^{-t}t^{\alpha-1}.$$

When doing so, and after interchanging the integral and the sum over l, the last term in Eq.(A6) can be written as

$$\begin{aligned}&(2a)^{-s}s\sum_{k=1}^{\infty}\frac{(-1)^k}{(k\pi)^2}\int_0^{\infty}du\,e^{-u}\left[\zeta\left(s+1,\frac{3}{2}+\frac{c}{2a}-\frac{iu}{2k\pi}\right)+\zeta\left(s+1,\frac{3}{2}+\frac{c}{2a}+\frac{iu}{2k\pi}\right)\right]\\&=2(2a)^{-s}\frac{1}{\Gamma(s)}\sum_{k=1}^{\infty}(-1)^k\int_0^{\infty}dt\,t^s\frac{e^{-(3/2+c/2a)t}}{1-e^{-t}}\frac{1}{(k\pi)^2+(t/2)^2}\end{aligned}$$

where we have used the integral form for the Hurwitz's zeta functions, interchanged the integrals and performed the interior one.

Interchanging now the integral with the sum, and using Eq. (5), we obtain

$$\begin{aligned}&2^{2-s}a^{-s}\frac{1}{\Gamma(s)}\int_0^{\infty}dt\,t^{s-1}\frac{e^{-(3/2+c/2a)t}}{1-e^{-t}}\left[\frac{e^{-t/2}}{1-e^{-t}}-\frac{1}{t}\right]\\&=2^{2-s}a^{-s}\left[\zeta\left(s-1,\frac{c}{2a}+1\right)-\left(\frac{c}{2a}+1\right)\zeta\left(s,\frac{c}{2a}+1\right)-\frac{1}{s-1}\zeta\left(s-1,\frac{3}{2}+\frac{c}{2a}\right)\right].\end{aligned}$$

When replaced in (A6), the final result is

$$\begin{aligned}\zeta(s;\mathbb{D}^{\dagger}\mathbb{D})\rfloor_{a=b}&=\mu^4\Omega\frac{a^2}{4\pi^2}\left\{c^{-s}+2^{2-s}a^{-s}\left[\zeta\left(s-1,\frac{c}{2a}+1\right)-\frac{c}{2a}\zeta\left(s,\frac{c}{2a}+1\right)\right]\right\}\\&=\mu^4\Omega\frac{a^2}{4\pi^2}\left\{c^{-s}+4(2a)^{-s}\left(\zeta\left(s-1,\frac{c}{2a}\right)-\frac{c}{2a}\zeta\left(s,\frac{c}{2a}\right)\right)\right\}.\end{aligned} \tag{A7}$$

This expression coincides with the result obtained in Ref. 13 [see Eqs. (5.2.6) and (5.2.4) in that reference].

APPENDIX B: THE WEAK-FIELD LIMIT

An unavoidable test our effective action must resist is its coincidence with the well known result for weak fields.[3,5] In order to check this is the case, we will develop the different contributions to the effective action [Eqs. (13)–(18)] in powers of the fields over the squared mass. In the cases of Eqs. (13)–(16), such development can be obtained by making use of the well known asymptotic expansions[24] for $\log\Gamma(x)$, $\psi(x)$, and $\zeta(s,x)$ [see also our Eq. (A1)]. When doing so, and retaining terms up to the order of squared fields over mass to the fourth, one gets, after a straightforward though tedious calculation,

$$\frac{1}{2}\frac{\partial}{\partial s}\mathrm{A}(s)\Big|_{s=0} \simeq \mu^4\Omega\,\frac{ab}{4\pi^2}\left\{\frac{1}{6}ac^{-1}+\frac{1}{2}\log(c)+\frac{1}{2a}(\log(c)-1)c\right\}, \tag{B1}$$

$$\frac{1}{2}\frac{\partial}{\partial s}\mathrm{B}(s)\Big|_{s=0} \simeq \mu^4\Omega\,\frac{ab}{4\pi^2}\left\{\frac{1}{6}bc^{-1}+\frac{1}{2}\log(c)+\frac{1}{2b}(\log(c)-1)c\right\}, \tag{B2}$$

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial s}\mathrm{C}_1(s)\Big|_{s=0} \simeq{}& \mu^4\Omega\,\frac{ab}{4\pi^2}\frac{1}{ab}\left\{\left(\frac{1}{4}-\frac{1}{4}\log c+\frac{1}{8}\right)c^2+\left(\frac{1}{2}(a+b)-\frac{1}{2}(a+b)\log c\right)c-\frac{5}{24}(a^2\right.\\ &+b^2)\\ &-\frac{1}{2}ab\log c-\frac{1}{24}(5ba^2+5ab^2+a^3+b^3)c^{-1}\\ &\left.+\left(\frac{1}{24}b^3a+\frac{1}{24}a^3b+\frac{1}{12}b^2a^2+\frac{7}{1440}a^4+\frac{7}{1440}b^4\right)c^{-2}\right\},\end{aligned} \tag{B3}$$

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial s}\mathrm{CD}_2(s)\Big|_{s=0} \simeq{}& \mu^4\Omega\,\frac{ab}{4\pi^2}\frac{1}{24}\left\{\left(\frac{a}{b}+\frac{b}{a}\right)\log c+\left(a+b+\frac{a^2}{b}+\frac{b^2}{a}\right)c^{-1}\right.\\ &\left.-\frac{1}{2}\left(2a^2+2b^2+\frac{a^3}{b}+\frac{b^3}{a}+\frac{4}{3}ba\right)c^{-2}\right\}.\end{aligned} \tag{B4}$$

As regards $\frac{1}{2}(\partial/\partial s)\mathrm{CF}_2(s)\rfloor_{s=0}$, one has to use the asymptotic expansions for the incomplete Γ function and for the Hurwitz' zeta functions [Eq. (A1)]. After doing so, one obtains

$$\frac{1}{2}\frac{\partial}{\partial s}\mathrm{CF}_2(s)\Big|_{s=0} \simeq \mu^4\Omega\,\frac{ab}{4\pi^2}\,\frac{7}{1440}\left(\frac{a^3}{b}+\frac{b^3}{a}\right)c^{-2}. \tag{B5}$$

By summing up the contributions in Eqs. (B1)–(B5), plus the one coming from $\frac{1}{2}(\partial/\partial s)\mathrm{D}(s)\rfloor_{s=0}$, the one-loop correction to the effective action is seen to reduce, in this weak-field limit, to

$$S^{(1)}=\mu^4\Omega\,\frac{1}{4\pi^2}\left\{\left[\frac{3}{8}-\frac{1}{4}\log(c)\right]c^2-\frac{1}{6}(b^2+a^2)\log(c)+\left[\frac{7}{90}(ab)^2-\frac{1}{90}(a^2+b^2)^2\right]c^{-2}\right\}. \tag{B6}$$

Now, renormalizing according to the criterium discussed in Sec. IV, one is left with

$$S_{\mathrm{eff}}=\frac{\Omega}{2}(B^2+E^2)+\frac{\Omega e^4}{8\pi^2m^4}\left[\frac{7}{45}(EB)^2-\frac{1}{45}(E^2+B^2)^2\right], \tag{B7}$$

where the definitions of a, b and c given in the paragraph following Eq. (3) were used.

The expression in (B7) is precisely the Euclidean version of the Euler–Heisenberg effective action for weak fields.[3,5]

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