Minimal Matrices and the corresponding Minimal Curves on Flag Manifolds in Low Dimension.*

Dedicated to the memory of Misha Cotlar.
Teacher and friend.

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Abstract

In general C*-algebras, elements with minimal norm in some equivalence class are introduced and characterized. We study the set of minimal hermitian matrices, in the case where the C*-algebra consists of 3×3 complex matrices, and the quotient is taken by the subalgebra of diagonal matrices. We thoroughly study the set of minimal matrices particularly because of its relation to the geometric problem of finding minimal curves in flag manifolds. For the flag manifold of 'four mutually orthogonal complex lines' in $\mathbb{C}^4$, it is shown that there are infinitely many minimal curves joining arbitrarily close points. In the case of the flag manifold of 'three mutually orthogonal complex lines' in $\mathbb{C}^3$, we show that the phenomenon of multiple minimal curves joining arbitrarily close points does not occur. *Key words:* aproximation, curves, flag manifolds, matrices, minimal.

Contents

1 Introduction

Let $\mathcal{A}$ be a unital C*-algebra, and $1 \in \mathcal{B} \subset \mathcal{A}$ a C*-subalgebra. Let $\mathcal{U}$, and $\mathcal{U}_\mathcal{B}$ be the unitary groups of $\mathcal{A}$, and $\mathcal{B}$ respectively. Let us denote with $\mathcal{A}_{ah}$ and $\mathcal{B}_{ah}$ the sets of antihermitian elements of $\mathcal{A}$ and $\mathcal{B}$ (i.e. the Lie algebras of $\mathcal{U}$, and $\mathcal{U}_\mathcal{B}$ respectively). Finally, denote by $\mathcal{P}$ the homogeneous space $\mathcal{U}/\mathcal{U}_\mathcal{B}$, with the natural left action L_g, $g \in \mathcal{U}$, of $\mathcal{U}$ on $\mathcal{P}$. The space $\mathcal{P}$ is provided with the invariant Finsler metric given by the quotient norm in the Banach space $\mathcal{A}_{ah}/\mathcal{B}_{ah}$ (the tangent space of $\mathcal{P}$ at the base point).

In [2] the following theorem is proven.

Theorem 1.1 *Consider $\rho \in \mathcal{P}$ and $X \in (T\mathcal{P})_\rho$. Suppose that there exists $Z \in \mathcal{A}_{ah}$ which projects to X and is a minimal vector i.e. $||Z|| = ||X||_\rho$. Then the oneparameter curve $\gamma(t)$ given by $\gamma(t) = L_{e^{tZ}}\rho$ has minimal length in the class of all curves in $\mathcal{P}$ joining $\gamma(0)$ to $\gamma(t)$ for each t with $|t| \leq \frac{\pi}{2||Z||}$.*

This theorem shows the relevance of the set of minimal vectors in the study of the space of minimal curves in such homogeneous spaces.

In section 2 we prove a convenient characterization of the set of minimal vectors for general C*-algebras. This result is inspired in [2] and it is of fundamental importance for the work presented in the remaining sections.

In section 3 we present a detailed study of the set $\mathbb{M}$ of minimal vectors in the simplest non-trivial case, i.e. the case where $\mathcal{A}$ is the C*-algebra of 3 × 3 complex matrices, and $\mathcal{B}$ is the subalgebra of diagonal matrices in $\mathcal{A}$. It turns out that $\mathbb{M}$ has a non-trivial structure. However this complexity is not reflected in the set of minimal curves, because different minimal vectors (above the same tangent vector) always give rise to the same minimal curve.

Operator approximation problems consist of finding, for a given operator, the element in some special class nearest to it, when distance is measured with a norm. These problems have been treated in the case of hermitian, positive and unitary approximants using different norms in [3], [4], [5], and others. The survey article [6], is related to matrix nearness and presents explicit formulas, uniqueness results and algorithms for computing or estimating the minimal norm attained, as well as the matrix or matrices sought in different contexts, but in this survey the operator norm is not considered. The problem of finding the minimum of $\|M + D\|$ for a given hermitian matrix $M \in \mathbb{C}^{n\times n}$ among all the diagonal matrices $D \in \mathbb{R}^{n\times n}$, and finding the matrix or matrices D that realize the minimum, is indeed an operator approximation problem. It has a trivial translation to the problem of finding a real diagonal matrix D' such that $M + D' \geq 0$ and $\|M + D'\|$ is minimum. In the $n \times n$ case, some bounds of this minimum were obtained in [1]. In that work the calculation of this minimum is related to the estimation of bounds of the norm of the operator $\mathcal{O} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ that for any $n \times n$ matrix replaces all its diagonal entries by zeroes.

In section 4 we consider the case where $\mathcal{A}$ is the C*-algebra of 4×4 complex matrices, and $\mathcal{B}$ is the subalgebra of diagonal matrices in $\mathcal{A}$. In this case, different minimal vectors (above the same tangent vector) can give rise different minimal curves, furthermore the following unusual phenomenon is shown: there are infinitely many minimal curves joining arbitrarily close points.

2 Minimal vectors

Let us denote with $\mathcal{A}_h$ and $\mathcal{B}_h$, the sets of hermitian elements of $\mathcal{A}$ and $\mathcal{B}$ to introduce the following definition.

Definition 2.1 *We call an element $Z \in \mathcal{A}_{ah}$* minimal *if $\|Z\| \leq \|Z+V\|$, for all $V \in \mathcal{B}_{ah}$. Similarly, in the hermitian case, any $Z \in \mathcal{A}_h$ is said to be* minimal *if $\|Z\| \leq \|Z + V\|$, for all $V \in \mathcal{B}_h$.*

Remark 2.1 *Note that since for any operator, $\|Im(X)\| \leq \|X\|$ (and $\|Re(X)\| \leq \|X\|$) it follows that $Z \in \mathcal{A}_{ah}$ (or $Z \in \mathcal{A}_h$) is* minimal *if and only if $\|Z\| \leq \|Z + B\|$, for all $B \in \mathcal{B}$.*

The following theorem follows ideas in [2]. We write it down in its antihermitian form, and a similar theorem can be shown for the isometric set of hermitian elements.

Theorem 2.2 *An element $Z \in \mathcal{A}_{ah}$ is minimal if and only if there exists a representation ρ of $\mathcal{A}$ in a Hilbert space $\mathcal{H}$ and a unit vector ξ such that $\rho(Z^2)\xi = -\|Z\|^2\xi$, and $\langle\rho(Z)\xi\,, \rho(B)\xi\rangle = 0$ for all $B \in \mathcal{B}$.*

Proof of Theorem 2.2 The ($\Leftarrow$) part is short, for suppose that there exist $\rho, \mathcal{H}, \xi$ as above, then if $B \in \mathcal{B}$,

$$\|Z + B\|^2 \geq \|\rho(Z + B)\xi\|^2 = \|\rho(Z)\xi\|^2 + \|\rho(B)\xi\|^2 \geq \|\rho(Z)\xi\|^2 = -\langle\rho(Z^2)\xi\,, \xi\rangle = \|Z\|^2.$$

Next the ($\Rightarrow$) part. Suppose now that Z is minimal. Denote by $\mathcal{S}$ the closed (real) linear span of $Z^2 + \|Z\|^2I$ and the operators of the form $ZB - B^*Z$ for all possible $B \in \mathcal{B}$. Note that $Z^2 + \|Z\|^2I$ is positive and $ZB - B^*Z$ is hermitian, i.e. $\mathcal{S} \subset \mathcal{A}_h$.

Denote by $\mathcal{C}$ the cone of positive and invertible elements of $\mathcal{A}$. We make a claim.

Claim 2.3 *The minimality condition implies that $\mathcal{S} \cap \mathcal{C} = \emptyset$.*

Proof of the claim. Indeed, since $\mathcal{C}$ is open, there would exist otherwise an $s \in \mathbb{R}$ and some $B \in \mathcal{B}$ such that $s(Z^2 + \|Z\|^2I) + ZB - B^*Z \geq rI$, with $r > 0$. We may suppose that $s > 0$, so that dividing by s we get that for some $B \in \mathcal{B}$, $r > 0$,

$$Z^2 + \|Z\|^2I + ZB - B^*Z \geq rI. \tag{2.1}$$

Also note that $Z^2 + \|Z\|^2I \geq 0$, then for $n \geq 1$,

$$n(Z^2 + \|Z\|^2I) + ZB - B^*Z \geq Z^2 + \|Z\|^2I + ZB - B^*Z \geq rI.$$

Or equivalently, dividing by n,

$$Z^2 + \|Z\|^2I + Z(\frac{1}{n}B) - (\frac{1}{n}B^*)Z \geq r'I.$$

In other words, one can find $B \in \mathcal{B}$ of arbitrarily small norm such that inequality (2.1) holds. This inequality clearly implies that

$$\text{Spec}(Z^2 + ZB - B^*Z) \subset (-\|Z\|^2, +\infty).$$

On the other hand, since B can be chosen of arbitrarily small norm, and Z^2 is non positive, it is clear that one can choose B in order that $\mathrm{Spec}(Z^2 + ZB - B^*Z) \subset (-\infty, \|Z\|^2)$. Therefore there exists $B \in \mathcal{B}_{ah}$ such that $\|Z^2 + ZB - B^*Z\| < \|Z\|^2$. Let us show that this contradicts the minimality of Z, which would prove our claim. Indeed, this is Lemma 5.3 of [2]:

Lemma 2.4 *If $\|Z + B\| \geq \|Z\|$ for all $B \in \mathcal{B}$, then also $\|Z^2 + ZB - B^*Z\| \geq \|Z\|^2$.*

Let us prove this lemma. Consider for $t > 0$, $f(t) = Z^2 + \frac{1}{t}((Z + tB)^*(Z + tB) - Z^2)$. Note that $\|f(t)\| \geq \|Z\|^2$. Otherwise $\|f(t)\| < \|Z\|^2$ and then the convex combination $tf(t) + (1-t)Z^2$ has norm strictly smaller than $\|Z\|^2$ for $0 < t < 1$. Note that

$$tf(t) + (1-t)Z^2 = (Z + tB)^*(Z + tB).$$

That is $\|Z + tB\|^2 = \|(Z + tB)^*(Z + tB)\| < \|Z\|^2$, which contradicts the hypothesis, and the lemma is proven, as well as our claim. □

We have that $\mathcal{S} \cap \mathcal{C} = \emptyset$, with $\mathcal{S}$ a closed (real)linear submanifold of $\mathcal{A}_h$ and $\mathcal{C}$ open and convex in $\mathcal{A}_h$. By the Hahn-Banach theorem, there exists a bounded linear functional φ_0 in $\mathcal{A}_h$ such that

$$\varphi_0(\mathcal{S}) = 0 \ \text{ and } \varphi_0(\mathcal{C}) > 0.$$

The functional φ_0 has a unique hermitian extension to $\mathcal{A}$, let φ be the normalization of this functional. Then clearly φ is a state which vanishes on $\mathcal{S}$. Let $\rho, \mathcal{H}, \xi$ be the GNS triple associated to this state. Note that since $Z^2 + \|Z\|^2 I \in \mathcal{S}$, $\langle \rho(Z^2)\xi, \xi\rangle = \varphi(Z^2) = -\|Z\|^2$, and therefore, by the equality part in the Cauchy-Schwartz inequality, it follows that

$$\rho(Z^2)\xi = -\|Z\|^2\xi.$$

Moreover, $0 = \varphi(ZB - B^*Z + Z^2 + \|Z\|^2 I) = \varphi(ZB - B^*Z)$. Since φ is hermitian, this means $Re(\varphi(ZB)) = 0$ for all $B \in \mathcal{B}$. Putting iB in the place of B, one has that in fact $\varphi(ZB) = 0$ for all $B \in \mathcal{B}$. Then,

$$0 = \langle \rho(ZB)\xi, \xi\rangle = \langle \rho(B)\xi, \rho(Z)\xi\rangle,$$

which concludes the proof Theorem 2.2. □

3 Minimal 3×3 matrices

Our interest in minimal 3×3 matrices arises when studying the flag manifold $\mathcal{P}(3)$. This is the space of triples of mutually orthogonal lines in $\mathbb{C}^3$ (1-dimensional complex subspaces) which is indeed a low dimensional example of a homogeneous space. The group of unitary operators in $\mathbb{C}^3$ acts on the left in $\mathcal{P}(3)$. Consider the *canonical flag* $p_e = (\mathrm{sp}\{e_1\}, \mathrm{sp}\{e_2\}, \mathrm{sp}\{e_3\})$ where $\mathrm{sp}\{e_i\}$ is the complex line spanned by the canonical vector e_i in $\mathbb{C}^3$. The isotropy of p_e is the subgroup of 'diagonal' unitary operators.

Again, our interest here in minimal 3×3 matrices is due to the following theorem, for general homogeneous spaces of the unitary group of a C*-algebra $\mathcal{A}$, which is proven in [2]:

Theorem 3.1 *Let $\mathcal{P}$ be a homogeneous space of the unitary group of a C*-algebra $\mathcal{A}$. Consider $\rho \in \mathcal{P}$ and $X \in (T\mathcal{P})_\rho$. Suppose that there exists $Z \in \mathcal{A}_{ah}$ which is a minimal vector i.e. $||Z|| = ||X||_\rho$. Then the oneparameter curve $\gamma(t)$ given by $\gamma(t) = L_{e^{tZ}}\rho$ has minimal length in the class of all curves in $\mathcal{P}$ joining $\gamma(0)$ to $\gamma(t)$ for each t with $|t| \leq \frac{\pi}{2||Z||}$.*

Namely, minimal curves in $\mathcal{P}(3)$ are given by action of (the class of) exponentials of anti-hermitian minimal 3×3 matrices. To study *anti-hermitian* minimal 3×3 matrices is (isometrically) equivalent to investigate the *hermitian* minimal 3×3 matrices, and in this article we find them *notationaly* simpler to consider.

3.1 A study of 3×3 minimal hermitian matrices

From Definition 2.1 of minimal vectors, we have the corresponding definition for 3×3 minimal hermitian matrices:

Definition 3.1 *We say that a non-zero matrix $M \in M^h_{3\times 3}(\mathbb{C})$ is minimal if, $||M|| = \inf_{D \in Diag_{3\times 3}} ||M + D||$, and such set of minimal matrices shall be denoted with $\mathbb{M}$.*

3.1.1 Some notation

We shall denote with $\mathcal{D}_{3\times3}$ the subset of the diagonal real matrices. Let us denote with $\mathcal{M}$ the quotient space $M^h_{3\times3}(\mathbb{C})/\mathcal{D}_{3\times3}$ with the quotient norm

$$|\,[M]\,| = \inf_{D\in\mathcal{D}_{3\times3}} ||\,M + D\,||$$

where $||\cdot||$ is the usual operator norm.

Remark 3.2 *If $M \in \mathbb{M}$ and $||M|| = \lambda(>0)$, and $Tr(M) = \mu$ then:*

1. *A matrix $M \in M^h_{3\times3}(\mathbb{C})$ is minimal if and only if, $||\,M\,|| = |\,[M]\,|$.*
2. *The diameter 'd' of the spectrum of M is $d = 2\lambda$. Otherwise, if $d < 2\lambda$ we can add a real scalar matrix D to M to produce another matrix $M + D$ with norm, $d/2 < \lambda$, and M would not be minimal.*
3. *The numbers $-\lambda$, λ and μ are the eigenvalues of M.*

3.1.2 The minimality theorem

The following theorem is a direct consequence of the *hermitian equivalent* of Theorem 2.2.

Theorem 3.3 *A matrix $M \in \mathbb{M}$ if and only if there exists a positive matrix $P \in M_{3\times3}(\mathbb{C})$ such that,*

- *$P.M^2 = \lambda^2\,P$, where $||M|| = \lambda$.*
- *The diagonal elements of the product $P.M$ are all zero.*

Proof of Theorem 3.3 This is just a translation to the algebra $M_{3\times3}(\mathbb{C})$ an its representations. □

3.1.3 Auxiliary results on $\mathbb{C}^3$

Definition 3.2 *We say that a vector $(a_1, a_2, a_3) \in \mathbb{C}^3$ is* triangular *if it is unitary and the numbers $|a_1|^2$, $|a_2|^2$, $|a_3|^2$ are the lengths of the three side of a triangle, i.e. each of them is smaller than or equal to the sum of the other two numbers.*

Remark 3.4 *If a vector $(a_1, a_2, a_3) \in \mathbb{C}^3$ is triangular and one of its components is zero, then the other two components have lengths equal to $\frac{\sqrt{2}}{2}$.*

Proposition 3.5 *Any vector $v = (a_1, a_2, a_3) \in \mathbb{C}^3$ is triangular if and only if there exists a unitary vector $w = (b_1, b_2, b_3) \in \mathbb{C}^3$, orthogonal to v and such that, $|a_1| = |b_1|$, $|a_2| = |b_2|$, $|a_3| = |b_3|$. Furthermore, there are at most two such vectors, w and $\tilde{w}$, for any given triangular vector v.*

Proof of the 'if' part ($\Leftarrow$) Let's write $v = (\rho_1\,e^{i\theta_1}, \rho_2\,e^{i\theta_2}, \rho_3\,e^{i\theta_3})$ and $w = (\rho_1\,e^{i\psi_1}, \rho_2\,e^{i\psi_2}, \rho_3\,e^{i\psi_3})$. Since v and w are orthogonal we have that,

$$v \cdot w = \rho_1^2\,e^{i(\theta_1-\psi_1)} + \rho_2^2\,e^{i(\theta_2-\psi_2)} + \rho_3^2\,e^{i(\theta_3-\psi_3)} = 0$$

then we have $\rho_1^2\,e^{i(\theta_1-\psi_1)} = -\rho_2^2\,e^{i(\theta_2-\psi_2)} - \rho_3^2\,e^{i(\theta_3-\psi_3)}$ and considering norms we get, by the triangular inequality, $\rho_1^2 \le \rho_2^2 + \rho_3^2$. Similarly we can get $\rho_2^2 \le \rho_1^2 + \rho_3^2$ and $\rho_3^2 \le \rho_1^2 + \rho_2^2$. Now w is unitary, then v is unitary, hence v is triangular. □

Proof of the 'only if' part ($\Rightarrow$) Let's write again $v = (\rho_1\,e^{i\theta_1}, \rho_2\,e^{i\theta_2}, \rho_3\,e^{i\theta_3})$. We just need to construct $w = (\rho_1\,e^{i\psi_1}, \rho_2\,e^{i\psi_2}, \rho_3\,e^{i\psi_3})$ so that,

$$v \cdot w = \rho_1^2\,e^{i(\theta_1-\psi_1)} + \rho_2^2\,e^{i(\theta_2-\psi_2)} + \rho_3^2\,e^{i(\theta_3-\psi_3)} = 0$$

Let's write $\nu_2 = \theta_2 - \psi_2 - \theta_1 + \psi_1$ and $\nu_3 = \theta_3 - \psi_3 - \theta_1 + \psi_1$. Then we just need to solve the complex equation:

$$\rho_1^2 + \rho_2^2\,e^{i\nu_2} + \rho_3^2\,e^{i\nu_3} = 0$$

for ν_2 and ν_3, which can be decomposed as two real equations,

$$\left\{\begin{array}{rcl} \rho_1^2 + \rho_2^2 \cos(\nu_2) + \rho_3^2 \cos(\nu_3) & = & 0 \\ \rho_2^2 \sin(\nu_2) + \rho_3^2 \sin(\nu_3) & = & 0 \end{array}\right. . \tag{3.2}$$

Now we distinguish two cases in this proof:

The case $\rho_1\,\rho_2\,\rho_3 \neq 0$

These equations can be solved for the unknowns $\cos(\nu_2)$, $\sin(\nu_2)$, $\cos(\nu_3)$ and $\sin(\nu_3)$, at most in two forms (two choices of corresponding signs):

$$\begin{array}{rclcrcl} \sin(\nu_2) & = & \mp\sqrt{1-\frac{(-{\rho_1}^4-{\rho_2}^4+{\rho_3}^4)^2}{4\,{\rho_1}^4\,{\rho_2}^4}} & , & \cos(\nu_2) & = & \frac{-{\rho_1}^4-{\rho_2}^4+{\rho_3}^4}{2\,{\rho_1}^2\,{\rho_2}^2} \\ \sin(\nu_3) & = & \pm\sqrt{1-\frac{(-{\rho_1}^4+{\rho_2}^4-{\rho_3}^4)^2}{4\,{\rho_1}^4\,{\rho_3}^4}} & , & \cos(\nu_3) & = & \frac{-{\rho_1}^4+{\rho_2}^4-{\rho_3}^4}{2\,{\rho_1}^2\,{\rho_3}^2} \end{array} . \tag{3.3}$$

To find ν_2 and ν_3 it is sufficient that two sets of inequalities are satisfied:

$$-1 \leq \frac{-{\rho_1}^4-{\rho_2}^4+{\rho_3}^4}{2\,{\rho_1}^2\,{\rho_2}^2} \leq 1 \qquad \text{and} \qquad -1 \leq \frac{-{\rho_1}^4+{\rho_2}^4-{\rho_3}^4}{2\,{\rho_1}^2\,{\rho_3}^2} \leq 1.$$

In the first set, the inequalities can be written as:

$$-2\,{\rho_1}^2\,{\rho_2}^2 \leq -{\rho_1}^4-{\rho_2}^4+{\rho_3}^4 \quad \text{and} \quad -{\rho_1}^4-{\rho_2}^4+{\rho_3}^4 \leq 2\,{\rho_1}^2\,{\rho_2}^2,$$

which become

$$0 \leq -\left({\rho_1}^2-{\rho_2}^2\right)^2+{\rho_3}^4 = \left(-{\rho_1}^2+{\rho_2}^2+{\rho_3}^2\right)\left({\rho_1}^2-{\rho_2}^2+{\rho_3}^2\right)$$

and

$$\left(-{\rho_1}^2-{\rho_2}^2+{\rho_3}^2\right)\left({\rho_1}^2+{\rho_2}^2+{\rho_3}^2\right) = -\left({\rho_1}^2+{\rho_2}^2\right)^2+{\rho_3}^4 \leq 0.$$

But the vector v is triangular, hence both inequalities are satisfied. Similarly, the second set of inequalities is satisfied, for the vector v is triangular. Hence the vector w can be constructed with the desired properties, at most in two different forms, w and $\tilde{w}$, corresponding to the two choices of corresponding signs for ν_2 and ν_3.

The case $\rho_1\,\rho_2\,\rho_3 = 0$

In this case, the triangular vector v has exactly one of its components equal to zero. The other two components must have the same length $\frac{\sqrt{2}}{2}$ as in Remark 3.4. It is easy then to construct a unitary vector w with the desired properties. It can be observed that in this case w is unique (up to multiplication by unitary complex numbers), i.e. there is no alternate vector $\tilde{w}$ with the desired properties.

This completes the proof of Proposition 3.5. □

Definition 3.3 *We say that and ordered pair (v, w) of triangular vectors, $v = (a_1, a_2, a_3)$ and $w = (b_1, b_2, b_3)$ in $\mathbb{C}^3$ form a* triangular-pair *if they are orthogonal to each other and the equalities $|a_1| = |b_1|$, $|a_2| = |b_2|$ and $|a_3| = |b_3|$ hold.*

Corollary 3.6 *Every triangular vector $v = (a_1, a_2, a_3) \in \mathbb{C}^3$ is the first coordinate of at least one and at most two triangular-pairs, (v, w) and $(v, \tilde{w})$.*

Proof of Corollary 3.6 Immediate from Proposition 3.5. □

3.1.4 Preliminary theorem

Theorem 3.7 *Let M be a hermitian matrix, $M \in M^h_{3\times 3}(\mathbb{C})$ with $||M|| = \lambda > 0$. Then $M \in \mathbb{M}$ if and only if there exists a triangular-pair (v_+, v_-) of eigenvectors of M, v_+ for the eigenvalue λ, and v_- for the eigenvalue $-\lambda$.*

We shall prove the following theorem which, by virtue of Proposition 3.5, is just a restatement of Theorem 3.7:

Theorem 3.8 *Let M be a hermitian matrix, $M \in M^h_{3\times 3}(\mathbb{C})$ with $||M|| = \lambda > 0$. Then $M \in \mathbb{M}$ if and only if there exist two unitary eigenvectors, v_+ for the eigenvalue λ, and v_- for the eigenvalue $-\lambda$, such that their corresponding coordinates have the same size, i.e. for every vector e_i in the canonical base of $\mathbb{C}^3$, it holds that $|v_+ \cdot e_i| = |v_- \cdot e_i|$.*

Proof of the 'if' part ($\Leftarrow$) Consider the matrices P_+ and P_- associated to the orthogonal projections onto the complex lines generated by v_+ and v_- respectively. Consider also the positive matrix $P = P_+ + P_-$. We will show that P satisfies the two conditions that in Theorem 3.3 imply the minimality of M, namely: $P.M^2 = \lambda^2 P$, and the diagonal elements of the product $P.M$ are all zero.

Notice that $M = \lambda P_+ - \lambda P_- + \mu P_\mu$ where P_μ is the orthogonal projection onto the complex line generated by an eigenvector v_μ for the third eigenvalue μ of M. Observe that the first condition above is satisfied because $P.M^2 = \lambda^2 P_+ + \lambda^2 P_- = \lambda^2 P$.

For the second condition observe that $P.M = \lambda P_+ - \lambda P_- = \lambda(P_+ - P_-)$. The diagonal elements of $P.M$ are, $(P.M\, e_i) \cdot e_i$ for each of the three canonical vectors e_i. Now,

$$\begin{aligned}(P.M\, e_i) \cdot e_i &= [\lambda(P_+ - P_-)e_i] \cdot e_i = \\ &= \lambda\,([P_+ e_i] \cdot e_i - [P_- e_i] \cdot e_i) \\ &= \lambda\,[([e_i \cdot v_+]\, v_+) \cdot e_i - ([e_i \cdot v_-]\, v_-) \cdot e_i] \\ &= \lambda \left[(e_i \cdot v_+)\overline{(e_i \cdot v_+)} - (e_i \cdot v_-)\,\overline{(e_i \cdot v_-)}\right] \\ &= \lambda\left[|e_i \cdot v_+|^2 - |e_i \cdot v_-|^2\right] = 0\end{aligned}$$

Then all the diagonal elements of $P.M$ are zero, and the second condition for minimality is satisfied. □

For the proof of the 'only if' part ($\Rightarrow$) we shall consider the two possible cases:

1. M has three simple eigenvalues, λ, $-\lambda$ and μ ($|\mu| < \lambda$).

2. Only one eigenvalue of M is simple.

The 'only if' part ($\Rightarrow$) with simple eigenvalues: λ, $-\lambda$ and μ. Consider unitary eigenvectors v_+, v_- and v_μ for the three eigenvalues λ, $-\lambda$ and μ of M. We will show that for $i \in \{1, 2, 3\}$, it happens that $|v_+ \cdot e_i| = |v_- \cdot e_i|$.

Let P_+, P_- and P_μ be the matrices associated to the orthogonal projections onto the complex lines generated by the corresponding eigenvectors above. As above, $M = \lambda P_+ - \lambda P_- + \mu P_\mu$. Consider any positive matrix P that satisfies the two conditions for the minimality of M of Theorem 3.3. Since $P.M^2 = \lambda^2 P$, it is clear that v_μ is in the kernel of P. The condition that the diagonal elements of the product $P.M$ are all zero means that for each vector e_i of the canonical base in $\mathbb{C}^3$ we have

$$(P.M\, e_i) \cdot e_i = (M\, e_i) \cdot (P\, e_i) = 0 \tag{3.4}$$

Consider the inner product of $\mathbb{C}^3$ induced by P, i.e. $\langle x, y\rangle = x \cdot (Py)$. Denote with $\langle\!\langle x\rangle\!\rangle = \sqrt{\langle x, x\rangle}$ the norm for this inner product.

Equation (3.4) above can be rewritten as,

$$\langle M\, e_i, e_i\rangle = 0 \tag{3.5}$$

Recall that $e_i = (e_i \cdot v_+)\, v_+ + (e_i \cdot v_-)\, v_- + (e_i \cdot v_\mu)\, v_\mu$ and the fact that $\langle x, v_\mu\rangle = 0$ for any $x \in \mathbb{C}^3$.

Expanding equation (3.5) we get

$$\begin{aligned}0 = \langle M\, e_i, e_i\rangle &= \langle M\, e_i, [(e_i \cdot v_+)\, v_+ + (e_i \cdot v_-)\, v_-]\rangle = \\ &= \lambda\,\langle [P_+\, e_i - P_-\, e_i]\,, [(e_i \cdot v_+)\, v_+ + (e_i \cdot v_-)\, v_-]\rangle \\ &= \lambda\,\langle [(e_i \cdot v_+)\, v_+ - (e_i \cdot v_-)\, v_-]\,, [(e_i \cdot v_+)\, v_+ + (e_i \cdot v_-)\, v_-]\rangle \\ &= \lambda\left[|e_i \cdot v_+|^2\, \langle\!\langle v_+\rangle\!\rangle^2 - |e_i \cdot v_-|^2\, \langle\!\langle v_-\rangle\!\rangle^2 + 2\,i\,\mathrm{Im}\left((e_i \cdot v_+)\,\overline{(e_i \cdot v_-)}\langle v_+, v_-\rangle\right)\right]\end{aligned}$$

Then we have that

$$|v_+ \cdot e_i|^2 \, \langle\!\langle v_+ \rangle\!\rangle^2 = |v_- \cdot e_i|^2 \, \langle\!\langle v_- \rangle\!\rangle^2, \tag{3.6}$$

and then

$$\left(\sum_{i=1}^{3} |v_+ \cdot e_i|^2 \right) \langle\!\langle v_+ \rangle\!\rangle^2 = \left(\sum_{i=1}^{3} |v_- \cdot e_i|^2 \right) \langle\!\langle v_- \rangle\!\rangle^2$$

but v_+ and v_- are both unitary so the previous sums are both equal to 1, then $\langle\!\langle v_+ \rangle\!\rangle = \langle\!\langle v_- \rangle\!\rangle$ and from equation (3.6) we have that for $i \in \{1, 2, 3\}$, it happens that $|v_+ \cdot e_i| = |v_- \cdot e_i|$, as we wanted to show. ☐

The 'only if' part (⇒) with only one simple eigenvalue. Without loss of generality, we assume that M is a reflection, i.e. $M^2 = \mathrm{Id}_{3\times 3}$ with $\lambda = 1$ as the simple eigenvalue. Let $v_+ = (c_1, c_2, c_3)$ be a unitary eigenvector for the simple eigenvalue '1'.

Let P_+ be the matrix of the orthogonal projection in the direction of v_+, and let Q be the matrix of the orthogonal projection onto the plane orthogonal to v_+ (the eigenspace for the eigenvalue '−1'). Then $M = P_+ - Q$ and $Q + P_+ = \mathrm{Id}_{3\times 3}$. Let e_i denote the vectors in the canonical base of $\mathbb{C}^3$. For $i \in \{1, 2, 3\}$, we have $e_i = w_i + v_i$ where $w_i = Q\, e_i$ and $v_i = P_+\, e_i$.

Consider, as in the previous case, a positive matrix P that satisfies the two conditions for the minimality of M in Theorem 3.3. Consider also the inner product of $\mathbb{C}^3$ induced by P, $\langle x, y \rangle = x \cdot (Py)$, and $\langle\!\langle x \rangle\!\rangle = \sqrt{\langle x, x \rangle}$ the associated norm. As above, the condition that the diagonal elements of the product $P.M$ are all zero means that for each vector e_i of the canonical base in $\mathbb{C}^3$ we have

$$\langle M\, e_i, e_i \rangle = 0$$

For $i \in \{1, 2, 3\}$, we have $M\, e_i = v_i - w_i$, and

$$\begin{aligned} 0 = \langle M\, e_i, e_i \rangle &= \langle v_i - w_i, v_i + w_i \rangle = \\ &= \langle\!\langle v_i \rangle\!\rangle - \langle\!\langle w_i \rangle\!\rangle - \langle w_i, v_i \rangle + \langle v_i, w_i \rangle \\ &= \langle\!\langle v_i \rangle\!\rangle - \langle\!\langle w_i \rangle\!\rangle + 2\, i \, \mathrm{Im}\left(\langle v_i, w_i \rangle \right) \end{aligned}$$

Then we have that $\langle\!\langle w_i \rangle\!\rangle = \langle\!\langle v_i \rangle\!\rangle$.

Observe that $v_i = P_+\, e_i = (e_i \cdot v_+) v_+ = \bar{c_i}\, v_+$ hence $\langle\!\langle v_i \rangle\!\rangle = |c_i| \, \langle\!\langle v_+ \rangle\!\rangle$.

We claim that $\langle\!\langle v_+ \rangle\!\rangle \neq 0$. In fact, if $\langle\!\langle v_+ \rangle\!\rangle = 0$ then $\langle\!\langle v_i \rangle\!\rangle = 0$ and $\langle\!\langle w_i \rangle\!\rangle = 0$ for $i \in \{1, 2, 3\}$. Hence $\langle\!\langle e_i \rangle\!\rangle = \langle\!\langle v_i + w_i \rangle\!\rangle \leq \langle\!\langle v_i \rangle\!\rangle + \langle\!\langle w_i \rangle\!\rangle = 0$ for $i \in \{1, 2, 3\}$. Then the norm $\langle\!\langle \cdot \rangle\!\rangle$ induced by P is trivial and P could not be a positive matrix.

We can scale the positive matrix P so that $\langle\!\langle v_+ \rangle\!\rangle = 1$. With this assumption, we have that $\langle\!\langle w_i \rangle\!\rangle = \langle\!\langle v_i \rangle\!\rangle = |c_i|$.

Now, observe that $v_+ = c_1\, e_1 + c_2\, e_2 + c_3\, e_3 = c_1\, v_1 + c_2\, v_2 + c_3\, v_3 + c_1\, w_1 + c_2\, w_2 + c_3\, w_3$, but each w_i is orthogonal to v_+, hence

$$v_+ = c_1\, v_1 + c_2\, v_2 + c_3\, v_3, \quad \text{and} \quad c_1\, w_1 + c_2\, w_2 + c_3\, w_3 = 0$$

Then we have $c_3\, w_3 = -c_1\, w_1 - c_2\, w_2$ and

$$|c_3|^2 \, \langle\!\langle w_3 \rangle\!\rangle^2 = |c_1|^2 \, \langle\!\langle w_1 \rangle\!\rangle^2 + |c_2|^2 \, \langle\!\langle w_2 \rangle\!\rangle^2 + 2\, \mathrm{Re}\left(\bar{c_1}\, c_2 \, \langle w_1, w_2 \rangle \right)$$

Recalling that $\langle\!\langle w_i \rangle\!\rangle = \langle\!\langle v_i \rangle\!\rangle = |c_i|$, we have

$$2\, \mathrm{Re}\left(\bar{c_1}\, c_2 \, \langle w_1, w_2 \rangle \right) = |c_3|^4 - |c_1|^4 - |c_2|^4$$

We also get

$$|c_3|^4 - |c_1|^4 - |c_2|^4 \leq |2\, \mathrm{Re}\left(\bar{c_1}\, c_2 \, \langle w_1, w_2 \rangle \right)| \leq 2|c_1| \, |c_2| \, \langle\!\langle w_1 \rangle\!\rangle \, \langle\!\langle w_2 \rangle\!\rangle = 2|c_1|^2 \, |c_2|^2$$

hence,

$$|c_3|^4 - |c_1|^4 - |c_2|^4 \leq 2|c_1|^2 \, |c_2|^2. \tag{3.7}$$

Recalling that $|c_3|^2 + |c_1|^2 + |c_2|^2 = 1$, we get

$$\begin{aligned} |c_3|^2 - |c_1|^2 - |c_2|^2 &= \\ &= \left(|c_3|^2 - |c_1|^2 - |c_2|^2 \right) \left(|c_3|^2 + |c_1|^2 + |c_2|^2 \right) \\ &= |c_3|^4 - \left(|c_1|^2 + |c_2|^2 \right)^2 \\ &= |c_3|^4 - |c_1|^4 - |c_2|^4 - 2|c_1|^2 |c_2|^2 \leq 0 \ \text{(by (3.7))} \end{aligned}$$

Hence $|c_3|^2 \le |c_1|^2 + |c_2|^2$. We can similarly prove that $|c_1|^2 \le |c_2|^2 + |c_3|^2$ and $|c_2|^2 \le |c_1|^2 + |c_3|^2$. Then the vector $v_+ = (c_1, c_2, c_3)$ is triangular and by Proposition 3.5, there exists another vector $v_- = (c_1', c_2', c_3')$ which makes a triangular-pair with v_+. It is clear that v_- belongs to the eigenspace of the eigenvalue -1. This completes the proof of Theorems 3.8 and 3.7. □

3.2 Description of the set of minimal matrices

We shall introduce two definitions, convenient for our next theorem.

Definition 3.4 *Let M be a hermitian matrix, $M \in M^h_{3\times 3}(\mathbb{C})$. We say that M is of* **extremal** *type if there exist:*

1. *$\eta \in [0, 2\pi)$.*
2. *$\lambda > 0$.*
3. *$\mu \in \mathbb{R}$ with $|\mu| \le \lambda$.*

such that M is one of the following three matrices:

$$\begin{pmatrix} \mu & 0 & 0 \\ 0 & 0 & \lambda\, e^{i\,\eta} \\ 0 & \lambda\, e^{-i\,\eta} & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & 0 & \lambda\, e^{-i\,\eta} \\ 0 & \mu & 0 \\ \lambda\, e^{i\,\eta} & 0 & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & \lambda\, e^{-i\,\eta} & 0 \\ \lambda\, e^{i\,\eta} & 0 & 0 \\ 0 & 0 & \mu \end{pmatrix} \tag{3.8}$$

Definition 3.5 *Let M be a hermitian matrix, $M \in M^h_{3\times 3}(\mathbb{C})$. We say that M is of* **non-extremal** *type if there exist:*

1. *Two real numbers η and ξ in $[0, 2\pi)$*
2. *$\lambda > 0$.*
3. *$\mu \in \mathbb{R}$ with $|\mu| \le \lambda$.*
4. *Three non-negative numbers α, β and χ, with: $2\alpha+2\beta+2\chi = 1$, $\alpha+\beta > 0$, $\beta+\chi > 0$ and $\alpha+\chi > 0$.*

such that,

$$M = \mu \begin{pmatrix} 2\alpha & n_{12} & \overline{n_{31}} \\ \overline{n_{12}} & 2\beta & n_{23} \\ n_{31} & \overline{n_{23}} & 2\chi \end{pmatrix} + \lambda \begin{pmatrix} 0 & m_{12} & \overline{m_{31}} \\ \overline{m_{12}} & 0 & m_{23} \\ m_{31} & \overline{m_{23}} & 0 \end{pmatrix} \tag{3.9}$$

where:

$$\left\{ \begin{array}{rcl} n_{12} & = & \dfrac{-2\,\alpha\,\beta \pm i\,\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\chi)(\beta+\chi)}}\, e^{-i\eta} \\ n_{31} & = & \dfrac{-2\,\alpha\,\chi \pm i\,\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\beta)(\beta+\chi)}}\, e^{-i\xi} \\ n_{23} & = & \dfrac{-2\,\beta\,\chi \pm i\,\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\beta)(\alpha+\chi)}}\, e^{-i(\xi-\eta)} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rcl} m_{12} & = & \dfrac{\chi \pm i\,\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\chi)(\beta+\chi)}}\, e^{-i\eta} \\ m_{31} & = & \dfrac{\beta \pm i\,\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\beta)(\beta+\chi)}}\, e^{-i\xi} \\ m_{23} & = & \dfrac{\alpha \pm i\,\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\beta)(\alpha+\chi)}}\, e^{-i(\xi-\eta)} \end{array} \right. \tag{3.10}$$

for one of the two sets of choices of corresponding signs.

Remark 3.9 *For matrices of both types, extremal and non-extremal, the parameters $\lambda > 0$ and μ respectively give the norm of M, $||M|| = \lambda$, and the trace of M, $Tr(M) = \mu$. The proof is just a direct calculation.*

By virtue of Theorem 3.7 a matrix M is minimal if and only if there exists a triangular-pair (v_+, v_-) of eigenvectors of M, v_+ for the eigenvalue λ, and v_- for the eigenvalue $-\lambda$.

Theorem 3.10 (Parametrization) *Let M be a hermitian matrix, $M \in M^h_{3\times 3}(\mathbb{C})$, if M is minimal then one of the following two mutually exclusive cases occur:*

1. *The triangular eigenvector v_+ in the triangular-pair associated to M, has one null component and M is of extremal type.*
2. *The triangular eigenvector v_+ in the triangular-pair associated to M, has no null components and M is of non-extremal type.*

Proof of case (1) in Theorem 3.10: Consider the triangular-pair (v_+, v_-) of eigenvectors of M. The eigenvector v_+ may be written as $v_+ = \big(\rho_1\, e^{i\theta}, \rho_2\, e^{i\eta}, \rho_3\, e^{i\xi}\big)$. In this case $\rho_1\,\rho_2\,\rho_3 = 0$. Now assume that $\rho_1 = 0$. After multiplication by some unitary complex number we can assume that $v_+ = \big(0, \rho_2\, e^{i\eta}, \rho_3\big)$ where $\rho_i > 0$ for $i \in \{2,3\}$ and $\rho_2^2 + \rho_3^2 = 1$. Similarly, we can assume that $v_- = \big(0, \rho_2\, e^{i(\eta+\phi)}, \rho_3\, e^{i\psi}\big)$ for some ϕ and ψ in $[0, 2\pi)$. In fact, since v_+ is triangular, $\rho_2 = \rho_3 = \sqrt{2}/2$, see Remark 3.4 in page 4. Then we may assume that:

$$v_+ = \left(0, \frac{\sqrt{2}}{2}\, e^{i\,\eta}, \frac{\sqrt{2}}{2}\right) \quad \text{and} \quad v_- = \left(0, \frac{\sqrt{2}}{2}\, e^{i(\eta+\phi)}, \frac{\sqrt{2}}{2}\, e^{i\psi}\right)$$

The fact that v_+ and v_- are orthogonal, leads one to conclude that:

$$v_+ = \left(0, \frac{\sqrt{2}}{2}\, e^{i\,\eta}, \frac{\sqrt{2}}{2}\right) \quad \text{and} \quad v_- = \left(0, \pm\frac{\sqrt{2}}{2}\, e^{i\,\eta}, \mp\frac{\sqrt{2}}{2}\right)$$

A unitary eigenvector v_μ associated to the eigenvalue μ is given by $v_\mu = (1, 0, 0)$.

We can write $M = B^*\, D\, B$ where B is the change of base matrix from the canonical base to the orthonormal base $\{v_\mu, v_+, v_-\}$; B^* is the adjoint of B and D is the diagonal matrix with λ, $-\lambda$ and μ in the diagonal (from top to bottom). The product $B^*\, D\, B$ gives the first expression of M as in Definition 3.4.

The second and third possible forms of M (in Definition 3.4) correspond to the cases where $\rho_2 = 0$ or when $\rho_3 = 0$. □

Proof of case 2 in Theorem 3.10: Consider the triangular-pair (v_+, v_-) of eigenvectors of M. After multiplication by some unitary complex number we can assume that $v_+ = \big(\rho_1, \rho_2\, e^{i\eta}, \rho_3\, e^{i\xi}\big)$ where $\rho_i > 0$ for $i \in \{1,2,3\}$ and $\rho_1^2 + \rho_2^2 + \rho_3^2 = 1$. Similarly, we can assume that $v_- = \big(\rho_1, \rho_2\, e^{i(\eta+\phi)}, \rho_3\, e^{i(\xi+\psi)}\big)$ for some ϕ and ψ in $[0, 2\pi)$.

The condition of orthogonality between v_+ and v_- is translated as two real equations:

$$\begin{cases} \rho_1^2 + \rho_2^2\, \cos(\phi) + \rho_3^2\, \cos(\psi) &= 0 \\ \rho_2^2\, \sin(\phi) + \rho_3^2\, \sin(\psi) &= 0 \end{cases} \tag{3.11}$$

These equations have at most two sets of solutions for the unknowns $\cos(\phi)$, $\sin(\phi)$, $\cos(\psi)$ and $\sin(\psi)$, as seen earlier in equations (3.2) and (3.3) in page 5:

$$\begin{aligned} \sin(\phi) &= \mp\sqrt{1 - \tfrac{(-{\rho_1}^4 - {\rho_2}^4 + {\rho_3}^4)^2}{4\,{\rho_1}^4\,{\rho_2}^4}} \quad , \quad \cos(\phi) = \tfrac{-{\rho_1}^4 - {\rho_2}^4 + {\rho_3}^4}{2\,{\rho_1}^2\,{\rho_2}^2} \\ \sin(\psi) &= \pm\sqrt{1 - \tfrac{(-{\rho_1}^4 + {\rho_2}^4 - {\rho_3}^4)^2}{4\,{\rho_1}^4\,{\rho_3}^4}} \quad , \quad \cos(\psi) = \tfrac{-{\rho_1}^4 + {\rho_2}^4 - {\rho_3}^4}{2\,{\rho_1}^2\,{\rho_3}^2} \end{aligned}$$

The nature of these solutions and the fact that v_+ and v_- are triangular vectors lead us to introduce new variables α, β and χ, with $2\alpha + 2\beta + 2\chi = 1$, given by

$$\rho_1^2 = \beta + \chi, \quad \rho_2^2 = \alpha + \chi, \quad \rho_3^2 = \alpha + \beta$$

These α, β and χ are given in the picture of Figure 1, recalling that ρ_1^2, ρ_2^2 and ρ_3^2 represent the three sides of a triangle with perimeter of length one, and possibly degenerate (area zero).

$$|BC| = \rho_1^2 = \beta + \chi, \quad |AC| = \rho_2^2 = \alpha + \chi, \quad |AB| = \rho_3^2 = \alpha + \beta$$

With these new variables, we can write the complex numbers $e^{i\phi}$ and $e^{i\psi}$ from the solutions of equation (3.11) as follows:

$$\begin{aligned} e^{i\phi} &= \frac{\alpha\,(\beta - \chi) - \chi\,(\beta + \chi) \pm i\,\sqrt{2\,\alpha\,\beta\,\chi}}{(\alpha + \chi)\,(\beta + \chi)} \\ e^{i\psi} &= \frac{\alpha\,(\chi - \beta) - \beta\,(\beta + \chi) \mp i\,\sqrt{2\,\alpha\,\beta\,\chi}}{(\alpha + \beta)\,(\beta + \chi)} \end{aligned}$$

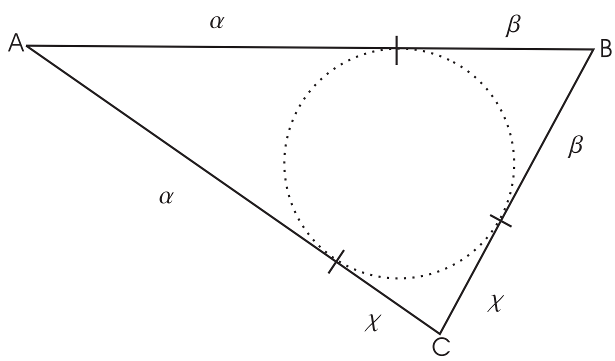


Figure 1: The construction of α, β and χ.

where one of the two combinations of signs gives the correct values for $e^{i\phi}$ and $e^{i\psi}$.

Then the vectors v_+ and v_- can be written as:

$$v_+ = \left(\sqrt{\beta+\chi}, \sqrt{\alpha+\chi}\, e^{i\,\eta}, \sqrt{\alpha+\beta}\, e^{i\,\xi}\right) \tag{3.12}$$

and

$$v_- = \left(\sqrt{\beta+\chi}, \frac{\alpha\,(\beta-\chi)-\chi\,(\beta+\chi)\pm i\sqrt{2\alpha\beta\chi}}{\sqrt{\alpha+\chi}\,(\beta+\chi)}\, e^{i\eta}, \frac{\alpha\,(\chi-\beta)-\beta\,(\beta+\chi)\mp i\sqrt{2\alpha\beta\chi}}{\sqrt{\alpha+\beta}\,(\beta+\chi)}\, e^{i\xi}\right). \tag{3.13}$$

A unitary eigenvector v_μ associated to the eigenvalue μ is given by the conjugated cross-product,

$$v_\mu = \left(\frac{\alpha\;(\beta-\chi)\mp i\;\sqrt{2}\,\sqrt{\alpha\,\beta\,\chi}}{\sqrt{\alpha+\beta}\,\sqrt{\alpha+\chi}\;(\beta+\chi)}\, e^{-i\;(\eta+\xi)}, -\frac{\beta\mp i\;\sqrt{2}\,\sqrt{\alpha\,\beta\,\chi}}{\sqrt{\alpha+\beta}\,\sqrt{\beta+\chi}}\, e^{-i\;\xi}, \frac{\chi\pm i\;\sqrt{2}\,\sqrt{\alpha\,\beta\,\chi}}{\sqrt{\alpha+\chi}\,\sqrt{\beta+\chi}}\, e^{-i\;\eta}\right). \tag{3.14}$$

Again we can write $M = B^*\,D\,B$ where B is the change of base matrix from the canonical base to the orthonormal base $\{v_+, v_-, v_\mu\}$; B^* is the adjoint of B and D is the diagonal matrix with λ, $-\lambda$ and μ in the diagonal (from top to bottom). The product $B^*\,D\,B$ gives the expression of M as in formulas (3.9) and (3.10) with the two possible combinations of signs. □

Remark 3.11 *For minimal matrices of extremal type, the construction of the corresponding numbers α, β and χ as in Figure 1, would lead to one of the three* extremal *points (vertices) $\alpha = 1/2$, $\beta = 1/2$ or $\chi = 1/2$ of the simplex (the equilateral triangle) $2\alpha + 2\beta + 2\chi = 1$ in the first octant of the (α, β, χ)-space.*

Theorem 3.12 (Construction) *Let M be a hermitian matrix, $M \in M^h_{3\times 3}(\mathbb{C})$. If M is of extremal type, or if M is of non-extremal type, with $\lambda \geq |\mu|$, then M is minimal.*

Proof. Let M be a hermitian matrix, $M \in M^h_{3\times 3}(\mathbb{C})$ of extremal or non-extremal type, then by Remark 3.9 the parameters $\lambda > 0$ and μ, correspond to the norm $||M|| = \lambda$ and the trace $\mathrm{Tr}(M) = \mu$ of M. A direct calculation shows that matrices of these two types have both λ and $-\lambda$ as eigenvalues and for them correspond eigenvectors v_+ and v_- which form a triangular-pair. Then by Theorem 3.7 these matrices are minimal. □

Theorem 3.13 (Non-extremals and Extremals) *Let M be a hermitian matrix, $M \in M^h_{3\times 3}(\mathbb{C})$. Then M is minimal if and only if M is of one of the two types: extremal or non-extremal.*

Proof. Immediate from Theorems 3.10 and 3.12. □

3.3 Minimal 3×3 matrices in a class

3.3.1 The algebraic setting of the problem

Any matrix $M \in M^h_{3\times 3}(\mathbb{C})$ can be written as,

$$M = \begin{pmatrix} a & x & \bar{y} \\ \bar{x} & b & z \\ y & \bar{z} & c \end{pmatrix}, \tag{3.15}$$

where $a\,,b\,,c \in \mathbb{R}$ and $x\,,y\,,z \in \mathbb{C}$. Observe that the quotient space $\mathcal{M}$ is homeomorphic to $\mathbb{C}^3 - \{0\}$ because the class of M, $[M] \in \mathcal{M}$, is given by the triple $(x,\ y,\ z)$ of complex numbers. From Remarks 3.2 in page 4, we have that the three real eigenvalues of every minimal matrix $M \in M^h_{3\times 3}(\mathbb{C})$, are: some $\lambda \in (0, +\infty)$, the opposite $-\lambda$ and an intermediate number μ ($|\,\mu\,| \leq \lambda$). This fact imposes some necessary conditions to the coefficients u, v and w of the characteristic polynomial, $\det(M - \Lambda\mathbb{I}) = -\Lambda^3 + u\,\Lambda^2 + v\,\Lambda + w$, of every minimal matrix M:

Claim 3.14 *Let u, v and w be the coefficients of the characteristic polynomial of a hermitian matrix M, $\det(M - \Lambda\mathbb{I}) = -\Lambda^3 + u\,\Lambda^2 + v\,\Lambda + w$: Then M has two eigenvalues of opposite signs if and only if the coefficients u, v and w satisfy:*

$$u\,v + w = 0. \tag{3.16}$$

Proof of claim 3.14 $(\Rightarrow)$ We have two equations, for the two roots λ and $-\lambda$,

$$-\lambda^3 + u\,\lambda^2 + v\,\lambda + w = 0, \quad \text{and} \quad \lambda^3 + u\,\lambda^2 - v\,\lambda + w = 0.$$

Adding and subtracting these equations we get:

$$u\,\lambda^2 + w = 0, \quad \text{and} \quad -\lambda^2 + v = 0. \tag{3.17}$$

Then, the coefficients u, v and w must satisfy the equation $u\,v + w = 0$.

$(\Leftarrow)$ Suppose the equation $u\,v + w = 0$ is satisfied, then

$$\begin{aligned} \det(M - \Lambda\mathbb{I}) &= -\Lambda^3 + u\,\Lambda^2 + v\,\Lambda + w = -\Lambda^3 + u\,\Lambda^2 + v\,\Lambda - u\,v \\ &= -(\Lambda^2 - v)(\Lambda - u), \end{aligned}$$

which means that $\lambda = \sqrt{v}$ and $\lambda = -\sqrt{v}$ are two real eigenvalues of M.

□

For any minimal matrix M, $\lambda = \sqrt{v} > 0$ and, in the notation above, the coefficient u is the trace of the matrix. The condition $|\mu| \leq \lambda$ can be written as $u^2 \leq v$.

For a fixed class $[M_0] \in \mathcal{M}$, of a matrix $M_0 \in M^h_{3\times 3}(\mathbb{C})$, we consider the real variety given by the equation:

$$\Delta := u\,v + w = 0. \tag{3.18}$$

Any minimal matrix M in the class $[M_0]$, must lie in Δ, and it must be a minimum for the function $\lambda^2 (= v)$ over the real variety Δ. Observe that the function λ^2 is itself a polynomial. Suppose now that the matrix M given in (3.15) is minimal. Let us rewrite equation (3.18) in terms of elements of the matrix M.

$$\Delta := (a+b)(a+c)(b+c) - (a+b)|x|^2 - (a+c)|y|^2 - (b+c)|z|^2 - 2\,\mathrm{Re}(x\,y\,z) = 0. \tag{3.19}$$

The diagonal elements $(a, b, c) \in \mathbb{R}^3$ of M, must satisfy this cubic equation for M a minimal matrix. Similarly, from equation (3.17) we get

$$\lambda^2 = |x|^2 + |y|^2 + |z|^2 - ab - ac - bc. \tag{3.20}$$

To simplify the expression of the map Δ, we introduce the following linear change of variables,

$$a = (r + s - t)/2\,, \qquad b = (t + r - s)/2\,, \qquad c = (s + t - r)/2.$$

The equations above change to give a new description of Δ and a new expression for λ^2,

$$\Delta \;:=\; r\,s\,t - r\,|x|^2 - s\,|y|^2 - t\,|z|^2 + 2\,\mathrm{Re}(x\,y\,z) = 0,$$

and the function to minimize,

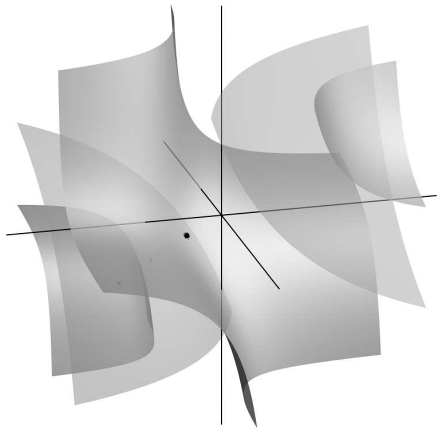
$$\lambda^2(r,s,t) \;=\; \frac{1}{4}(r^2 + s^2 + t^2) - \frac{1}{2}(r\,s + r\,t + s\,t) + |x|^2 + |y|^2 + |z|^2.$$

3.3.2 Finding the minimal matrix of a class $[M]$ in $\mathcal{M}$

To find the minimal matrix (or matrices) in the class $[M_0]$ we just have to minimize $\lambda^2(r, s, t)$ on Δ. We shall consider four cases depending on the triple (x, y, z). Figures representing Δ are shown in each case. Theorem 3.15 below states that only in the fourth case there might be multiple minima in the given class $[M_0]$. Two rounded surfaces, shown in the first three figures, do not belong to Δ, they represent the bounding surfaces $\mu = \pm\lambda$ in between which the (unique) minimum is located.

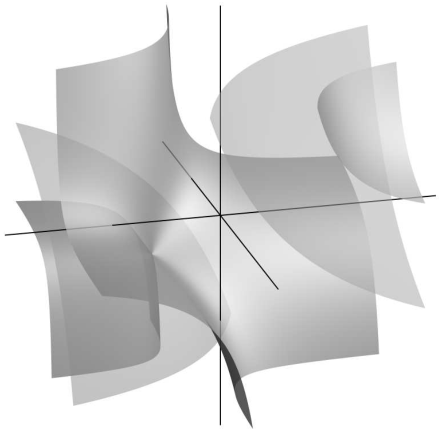
1. When $\mathrm{Im}(x\,y\,z) \neq 0$.

   In this case the surface Δ is regular (a smooth manifold) and the method of Lagrange multipliers can be used to find the unique minimum in the class. In the figure to the right, the middle portion represents the component satisfying $u^2 \leq v$, and the dark point indicates the minimum.



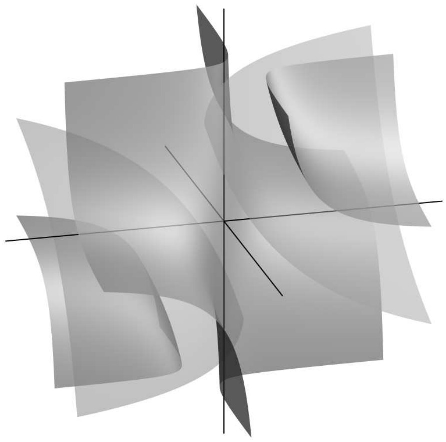
2. When $\mathrm{Im}(x\,y\,z) = 0$ and $\mathrm{Re}(x\,y\,z) \neq 0$.

   In this case the surface Δ is not regular, has one singular point which is the unique minimum in the class. In the figure to the right, two components of Δ touch at the singular point which is the minimum.



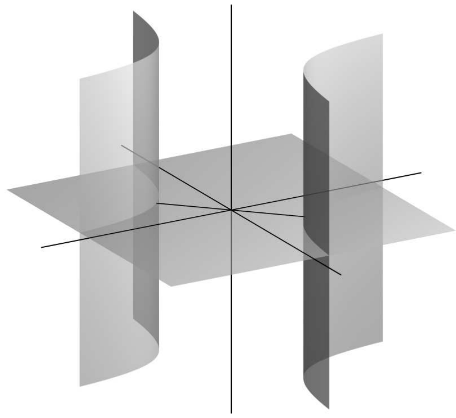
3. When $\mathrm{Im}(x\,y\,z) = 0 = \mathrm{Re}(x\,y\,z)$, and exactly one of the coordinates in the triple (x, y, z) vanishes.

   In this case the surface Δ is regular; the class has a unique minimum at the origin, $(r, s, t) = (0, 0, 0) = (a, b, c)$ as shown in Theorem 3.16 below. Observe that in the figure the vertical "t-axis" lies in Δ.



4. When $\mathrm{Im}(x\,y\,z) = 0 = \mathrm{Re}(x\,y\,z)$, and exactly two of the coordinates in the triple (x, y, z) vanish.

In this case the surface Δ is not regular along two curves, the two branches of the hyperbola shown in the figure, and the class has multiple minima, represented by the segment shown in the figure joining the two branches of the hyperbola.



Theorem of uniqueness

The main purpose of this section is to prove the following uniqueness theorem.

Theorem 3.15 (Uniqueness) *For every class $[M] \in \mathcal{M}$, there is only one minimal matrix, unless two of the coordinates vanish in the triple $(x\,,\,y\,,\,z)$.*

Proof Theorem 3.15: Suppose that there exist two minima M_1 and M_2 in the same class $[M] \in \mathcal{M}$. Necessarily the segment of matrices, $\xi M_1 + (1-\xi)M_2$, for $0 \le \xi \le 1$, is contained in that class. This implies that there exist a segment $(r_0 + \rho\xi, s_0 + \sigma\xi, t_0 + \tau\xi)$, $\xi \in [0, 1]$, contained in the variety $\Delta = 0$, with direction vector $(\rho, \sigma, \tau) \neq (0, 0, 0) \in \mathbb{R}^3$ and along which λ^2 is constant. Let us write equations (3.19) and (3.20) for the matrices with $(r, s, t) = (r_0 + \rho\,\xi, s_0 + \sigma\,\xi, t_0 + \tau\,\xi)$,

$$\Delta = 0 = m_0 + m_1\xi + m_2\xi^2 + m_3\xi^3, \quad \text{and} \quad \lambda^2(a, b, c) = n_0 = n_0 + n_1\xi + n_2\xi^2.$$

Condition $m_0 = 0$ states that the initial point $(a, b, c) = (r_0, s_0, t_0)$ is in $\Delta = 0$, and the coefficient n_0 gives the constant value of λ^2. Then the following equations must hold,

$$\begin{aligned} m_1 &= 0 = && \rho\,|x|^2 + \sigma\,|y|^2 + \tau\,|z|^2 - s_0\,t_0\,\rho - r_0\,t_0\sigma - r_0\,s_0\,\tau. \\ m_2 &= 0 = && t_0\,\rho\,\sigma + r_0\tau\,\sigma + s_0\,\rho\,\tau. \\ m_3 &= 0 = && \rho\,\sigma\,\tau. \\ n_1 &= 0 = && r_0(\rho - \sigma - \tau) - t_0(\rho + \sigma - \tau) - s_0(\rho - \sigma + \tau). \\ n_2 &= 0 = && \left(\rho^2 + \sigma^2 + \tau^2 - 2\sigma\tau - 2\rho\tau - 2\sigma\rho\right)/2. \end{aligned} \tag{3.21}$$

We will assume first that y and z are not equal to zero. The equation $m_3 = 0$ says that one of ρ, σ or τ has to be zero.

- Suppose first that $\rho = 0$, then completing squares in equation (3.21), $n_2 = 0$, we get $\tau = \sigma$ and the equations above reduce to,

  $$m_1 = 0 = \sigma\,(|y|^2 + |z|^2) - r_0\,\sigma(s_0 + t_0), \quad m_2 = 0 = r_0\,\sigma^2, \quad n_1 = 0 = -2\,r_0\,\sigma.$$

  In this case, equations $m_2 = 0$ and $n_1 = 0$ both imply $r_0 = 0$ or $\sigma = 0$. If $r_0 = 0$, then m_1 implies $\sigma = 0$. Then we have a contradiction for we get $(\rho, \sigma, \tau) = (0, 0, 0)$.

- Now suppose that $\sigma = 0$ then completing squares in equation (3.21) we get $\tau = \rho$ and the original set of equations reduce to,

  $$m_1 = 0 = \rho\,(|x|^2 + |z|^2) - \rho\,s_0(t_0 + r_0), \quad m_2 = 0 = s_0\,\rho^2, \quad n_1 = 0 = -2\,s_0\,\rho.$$

  In this case, equations $m_2 = 0$ and $n_1 = 0$ both imply $s_0 = 0$ or $\rho = 0$. If $s_0 = 0$ then $m_1 = \rho(|x|^2 + |z|^2) = 0$ and then $\rho = 0$. Again we have a contradiction for we get $(\rho, \sigma, \tau) = (0, 0, 0)$.

It remains now consider the cases when x and z are non-zero, or when x and y are non-zero. It is clear from the analysis above, and from the symmetries in the problem, that if two minima are in the same class, then two of its coordinates (x, y, z) must be equal to zero. $\square$

Classes with exactly one of its coordinates equal to zero.

For matrices with exactly one of $x,\ y$ or z equal to zero, it is easy to find the minima in their classes, as stated in the theorem that follows. This is in turn, a consequence of the description for the non-extremal matrices (in page 8):

Theorem 3.16 *For a class having only one of its (x, y, z) coordinates equal to zero, the minimal matrix in the class is the one with the zero diagonal.*

Proof of Theorem 3.16. From Section 3.2 the classes of non-extremal type minimum matrices are described by:

$$M_{12} = \frac{\lambda\,\chi - 2\,\mu\,\alpha\,\beta \pm i\,(\lambda+\mu)\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\chi)(\beta+\chi)}}e^{-i\eta}, \quad M_{31} = \frac{\lambda\,\beta - 2\,\mu\,\alpha\,\chi \pm i\,(\lambda+\mu)\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\beta)(\beta+\chi)}}e^{-i\xi}, \quad \text{and}$$

$$M_{23} = \frac{\lambda\,\alpha - 2\,\mu\,\beta\,\chi \pm i\,(\lambda+\mu)\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\beta)(\alpha+\chi)}}e^{-i(\xi-\eta)}.$$

Recall that Definition 3.5 of non-extremal matrices, in page 8, does not admit two of the three parameters α, β or χ equal to zero (the formulas above would not even make sense). Let us now suppose that the x coordinate is zero, i.e. $M_{12} = 0$. Then

$$\lambda\,\chi - 2\,\mu\,\alpha\,\beta = 0, \ \ \text{and,}\ (\lambda+\mu)\sqrt{2\,\alpha\,\beta\,\chi} = 0$$

Recall $\lambda > 0$. If $\mu = -\lambda$ then $\chi = -2\,\alpha\,\beta$; but $\chi \geq 0$ so either α or β is zero and $\chi = 0$. We conclude that either the pair of parameters α, χ or the pair β, χ are both zero, which is not allowed in non-extremal matrices. Now suppose that $\chi = 0$, then $\mu\,\alpha\,\beta = 0$ and the only solution is $\mu = 0$. In this case the minimal matrix is

$$M = \sqrt{2}\,\lambda \begin{pmatrix} 0 & 0 & \sqrt{\beta}\,e^{i(\xi-\eta)} \\ 0 & 0 & \sqrt{\alpha}\,e^{-i\eta} \\ \sqrt{\beta}\,e^{-i(\xi-\eta)} & \sqrt{\alpha}\,e^{i\eta} & 0 \end{pmatrix}$$

A similar analysis in the case $M_{31} = 0$ or the case $M_{23} = 0$ draws the same conclusion.

$\square$

3.4 The topology of the set of minimal matrices

In the previous section, Theorem 3.10 gives a light into a parameterization for the set of minimal matrices $\mathbb{M}$. In this section we shall describe the parameterization and identify $\mathbb{M}$ up to homeomorphism.

First consider the 2-simplex $\Delta = \left\{(\alpha, \beta, \chi) \in \mathbb{R}^3 \mid \alpha+\beta+\chi = \frac{1}{2},\ \alpha \geq 0,\ \beta \geq 0,\ \chi \geq 0\right\}$.

Consider $D = [-1, 1] \times \{1\} \subset \mathbb{R}^2$, and $C = \left\{\lambda\, d \in \mathbb{R}^2 \mid \lambda > 0,\ d \in D\right\}$ the positive cone (open cylinder) generated by D in $\mathbb{R}^2$. Consider also $W = \Delta \times S^1 \times S^1 \times C$, where S^1 is the unit circle in the complex plane, and let $\nabla = W_+ \sqcup W_-$, the disjoint union of two copies of W.

We shall denote the three vertices of Δ as follows:

$$v_\alpha \ = \ (\tfrac{1}{2}, 0, 0) \quad , \qquad v_\beta \ = \ (0, \tfrac{1}{2}, 0) \quad , \qquad v_\chi \ = \ (0, 0, \tfrac{1}{2}) \tag{3.22}$$

Consider in ∇ the smallest equivalence relation '$\sim$' that contains the relations shown in Table 1. Consider now $G = \frac{\nabla}{\sim}$, the quotient space of ∇ under the equivalence relation described above.

Theorem 3.17 (Topology) *The space $\mathbb{M}$ of minimal matrices in $M^h_{3\times 3}(\mathbb{C})$ is homeomorphic to $G = \frac{\nabla}{\sim}$.*

A pictorial representation of $\mathbb{M}$ is shown in Figure 2 via the quotient space $G = \frac{\nabla}{\sim}$.

Proof of Theorem 3.17 The homeomorphism $f : G \to \mathbb{M}$ shall be defined by parts:

For all $\eta,\ \xi \in [0, 2\pi)$, all $\alpha \in \left[0, \frac{1}{2}\right]$, and all $(\mu, \lambda) \in C$.	(E1) $\left(\alpha, \frac{1}{2}-\alpha, 0, e^{i\eta}, e^{i\xi}, \mu, \lambda\right)_+ \sim \left(\alpha, \frac{1}{2}-\alpha, 0, e^{i\eta}, e^{i\xi}, \mu, \lambda\right)_-$ (E2) $\left(\alpha, 0, \frac{1}{2}-\alpha, e^{i\eta}, e^{i\xi}, \mu, \lambda\right)_+ \sim \left(\alpha, 0, \frac{1}{2}-\alpha, e^{i\eta}, e^{i\xi}, \mu, \lambda\right)_-$ (E3) $\left(0, \alpha, \frac{1}{2}-\alpha, e^{i\eta}, e^{i\xi}, \mu, \lambda\right)_+ \sim \left(0, \alpha, \frac{1}{2}-\alpha, e^{i\eta}, e^{i\xi}, \mu, \lambda\right)_-$
For all $\xi,\ \eta_1,\ \eta_2 \in [0, 2\pi)$ and all $(\mu, \lambda) \in C$, taking all possible left and right sign combinations.	(V1) $\left(\frac{1}{2}, 0, 0, e^{i\eta_1}, e^{i(\eta_1+\xi)}, \mu, \lambda\right)_\pm \sim \left(\frac{1}{2}, 0, 0, e^{i\eta_2}, e^{i(\eta_2+\xi)}, \mu, \lambda\right)_\pm$ (V2) $\left(0, \frac{1}{2}, 0, e^{i\eta_1}, e^{i\xi}, \mu, \lambda\right)_\pm \sim \left(0, \frac{1}{2}, 0, e^{i\eta_2}, e^{i\xi}, \mu, \lambda\right)_\pm$ (V3) $\left(0, 0, \frac{1}{2}, e^{i\xi}, e^{i\eta_1}, \mu, \lambda\right)_\pm \sim \left(0, 0, \frac{1}{2}, e^{i\xi}, e^{i\eta_2}, \mu, \lambda\right)_\pm$
For all $(\alpha, \beta, \chi) \in \mathring{\Delta}$, the interior of Δ, all $\eta,\ \xi \in [0, 2\pi)$ and all $\lambda \in (0, \infty)$.	(C1) $\left(\alpha, \beta, \chi, e^{i\eta}, e^{i\xi}, -\lambda, \lambda\right)_+ \sim \left(\alpha, \beta, \chi, e^{i\eta}, e^{i\xi}, -\lambda, \lambda\right)_-$ (C2) $\left(\alpha, \beta, \chi, e^{i\eta}, e^{i\xi}, \lambda, \lambda\right)_+ \sim \left(\alpha, \beta, \chi, e^{i(\eta+\eta')}, e^{i(\xi+\xi')}, \lambda, \lambda\right)_-$, with $e^{i\eta'} = \frac{\chi-2\alpha\beta-i\,2\sqrt{2}\sqrt{\alpha\beta\chi}}{\chi-2\alpha\beta+i\,2\sqrt{2}\sqrt{\alpha\beta\chi}}$ and $e^{i\xi'} = \frac{\beta-2\alpha\chi-i\,2\sqrt{2}\sqrt{\alpha\beta\chi}}{\beta-2\alpha\chi+i\,2\sqrt{2}\sqrt{\alpha\beta\chi}}$

Observations:

- Relations (E1), (E2) and (E3) make identifications of the boundary of one (α, β, χ)-simplex with the boundary of the other, over any point of the factor C.
- Relations (V1), (V2) and (V3) collapse the toruses on each one of the three vertices of the (α, β, χ)-simplexes onto a circle, over any point of the factor C.
- Relations (C1) and (C2) make identifications between the two toruses, over any point along both boundary edges of the factor C.

Table 1: Description of the relations that generate the equivalence relation '$\sim$'.

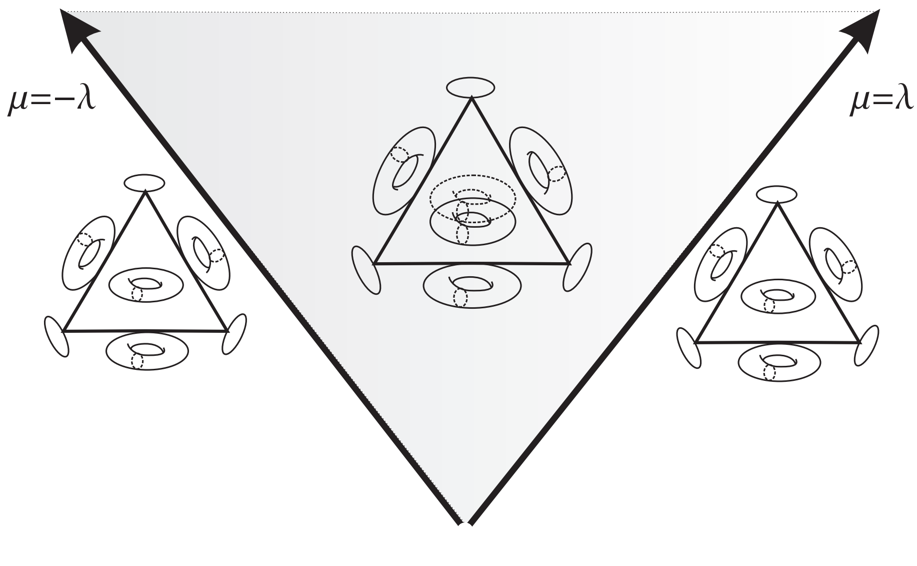


Figure 2: A representation of $\mathbb{M}$ via the quotient space $G = \frac{\nabla}{\sim}$. The extremal matrices correspond to points in the circles on the vertices (extremal points) of the triangles.

- For non-vertex points of Δ, i.e. if $\alpha+\beta>0$, $\beta+\chi>0$ and $\alpha+\chi>0$,

$$f\left(\left[\left(\alpha,\ \beta,\ \chi,\ e^{i\eta},\ e^{i\xi},\ \mu,\ \lambda\right)_{\pm}\right]\right)=\mu\begin{pmatrix}2\alpha & n_{12} & \overline{n_{31}}\\ \overline{n_{12}} & 2\beta & n_{23}\\ n_{31} & \overline{n_{23}} & 2\chi\end{pmatrix}+\lambda\begin{pmatrix}0 & m_{12} & \overline{m_{31}}\\ \overline{m_{12}} & 0 & m_{23}\\ m_{31} & \overline{m_{23}} & 0\end{pmatrix} \tag{3.23}$$

where:

$$\begin{cases} n_{12} &= \dfrac{-2\,\alpha\,\beta\pm i\,\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\chi)(\beta+\chi)}}\,e^{-i\eta}\\ n_{31} &= \dfrac{-2\,\alpha\,\chi\pm i\,\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\beta)(\beta+\chi)}}\,e^{-i\xi}\\ n_{23} &= \dfrac{-2\,\beta\,\chi\pm i\,\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\beta)(\alpha+\chi)}}\,e^{-i(\xi-\eta)}\end{cases} \quad\text{and}\quad \begin{cases} m_{12} &= \dfrac{\chi\pm i\,\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\chi)(\beta+\chi)}}\,e^{-i\eta}\\ m_{31} &= \dfrac{\beta\pm i\,\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\beta)(\beta+\chi)}}\,e^{-i\xi}\\ m_{23} &= \dfrac{\alpha\pm i\,\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\beta)(\alpha+\chi)}}\,e^{-i(\xi-\eta)}\end{cases} \tag{3.24}$$

where the choice of sign '$\pm$' is taken according to the copy W_+ or W_- where $\left(\alpha,\ \beta,\ \chi,\ e^{i\eta},\ e^{i\xi},\ \mu,\ \lambda\right)$ lies.

- For the three vertex points of Δ, v_α, v_β and v_χ,

$$f\left(\left[\left(v_\alpha;\ e^{i\eta},\ e^{i\xi},\ \mu,\ \lambda\right)_{\pm}\right]\right)=\begin{pmatrix}\mu & 0 & 0\\ 0 & 0 & \lambda\,e^{i(\eta-\xi)}\\ 0 & \lambda\,e^{-i(\eta-\xi)} & 0\end{pmatrix} \tag{3.25}$$

$$f\left(\left[\left(v_\beta;\ e^{i\eta},\ e^{i\xi},\ \mu,\ \lambda\right)_{\pm}\right]\right)=\begin{pmatrix}0 & 0 & \lambda\,e^{i\xi}\\ 0 & \mu & 0\\ \lambda\,e^{-i\xi} & 0 & 0\end{pmatrix} \tag{3.26}$$

$$f\left(\left[\left(v_\chi;\ e^{i\eta},\ e^{i\xi},\ \mu,\ \lambda\right)_{\pm}\right]\right)=\begin{pmatrix}0 & \lambda\,e^{-i\eta} & 0\\ \lambda\,e^{i\eta} & 0 & 0\\ 0 & 0 & \mu\end{pmatrix} \tag{3.27}$$

Claim: f is a well defined mapping.

The following sets of equivalent points may need some clarification:

Ei's. Performing the calculations presented in formulas (3.23) and (3.24), the $\pm$ signs can be collected with the factor $\sqrt{\alpha\,\beta\,\chi}$. Then, if $\alpha\,\beta\,\chi=0$, the double choice of signs disappears.

C1. As above, the $\pm$ signs can be collected with the factor $(\mu+\lambda)$. Then, if $\mu=-\lambda$, the double choice of signs disappears.

C2. If $\mu=\lambda$, exactly the indicated pairs of phases are mapped to the same matrix, for the corresponding signs.

Claim: the mapping f is continuous.

From the form of the formulas defining f, we just need to verify three limits, to any of the vertices $V\in\{v_\alpha,\ v_\beta,\ v_\chi\}$, given in formulas (3.22) in page 14:

$$\lim_{(\alpha,\beta,\chi)\to V} f\left[\left(\alpha,\ \beta,\ \chi,\ e^{i\eta},\ e^{i\xi},\ \mu,\ \lambda\right)_{\pm}\right] \quad=\quad f\left(V;\ e^{i\eta},\ e^{i\xi},\ \mu,\ \lambda\right),\ \text{given in (3.25), (3.26) and (3.27)}$$

These limits are readily verified from formulas (3.23) and (3.24) in page 16.

Claim: the mapping f is surjective.

It is clear that all matrices of extremal and non-extremal type are in the image of the mapping f. Then by Theorem 3.13 the map f is surjective.

Claim: the mapping f is injective.

This is the main point in this proof: the identifications given in Table 1 of page 15, determine the injectivity. In fact, for non-extremal type matrices, if two sets of parameters are mapped to the same matrix, then these sets have the same values for α, β, χ, μ and λ. In fact μ and λ are the trace and the norm of the matrix, and the first three are given by the triangular-pair of eigenvectors. The only parameters that may change are the "phases" η and ξ, and these must be taken in different components of ∇, + or −. Equating components, it can be realized that the only way that double phases occur, is that for a minimal matrix M the product of coordinates $M_{12}\, M_{31}\, M_{23}$ is a real number. This implies that the imaginary part of this product is zero, i.e. for the parameterization the following equation holds:

$$0 = \lambda\, \left(\lambda^2 - \mu^2\right)\, \sqrt{\alpha\,\beta\,\chi} \tag{3.28}$$

Then double representation phases exist in two cases:

1. When $\mu = \pm\lambda$ (for non-extremal matrices). This cases are described in C1 and C2 of Table 1.
2. When $\alpha\,\beta\,\chi = 0$. This cases are described with the Ei's of Table 1.

Claim: the mapping f is a homeomorphism.

Consider D the generator of the cylinder C. Lets denote with W_1, the compact subspace of W given by $W_1 = \Delta \times S^1 \times S^1 \times D \subset W$. Let $\nabla_1 = W_{1+} \sqcup W_{1-} \subset \nabla$, and $G_1 = \frac{\nabla_1}{\sim} \subset G$. Let f_1 denote the restriction of f to the subspace G_1. The image of f_1 is the subspace $\mathbb{M}_1$ of minimal matrices in $\mathbb{M}$ with norm equal to one. A well known theorem in General Topology ensures that the bijective and continuous mapping f_1 is a homeomorphism, for its range $\mathbb{M}_1$ is a Haussdorf space and its domain G_1 is compact.

Observe that the mapping $(M_1, k) \mapsto k\, M_1$ from $\mathbb{M}_1 \times (0, \infty)$ to $\mathbb{M}$ is a homeomorphism, for its inverse is just $M \mapsto \left(\frac{M}{||M||}, ||M||\right)$. Then the bijective continuous mapping $g : G_1 \times (0, \infty) \to \mathbb{M}$, given by $(x, k) \mapsto k\, f_1(x)$ is a homeomorphism.

Consider also the mapping $h : G \to G_1 \times (0, \infty)$ given by,

$$h\left(\left[\left(\alpha,\ \beta,\ \chi,\ e^{i\eta},\ e^{i\xi},\ \mu,\ \lambda\right)_\pm\right]\right) = \left(\left[\left(\alpha,\ \beta,\ \chi,\ e^{i\eta},\ e^{i\xi},\ \frac{\mu}{\lambda},\ 1\right)_\pm\right],\ \lambda\right)$$

This mapping h is a homeomorphism for it is the induced quotient map of the trivial homeomorphism

$$\left(\alpha,\ \beta,\ \chi,\ e^{i\eta},\ e^{i\xi},\ \mu,\ \lambda\right)_\pm \mapsto \left(\left(\alpha,\ \beta,\ \chi,\ e^{i\eta},\ e^{i\xi},\ \frac{\mu}{\lambda},\ 1\right)_\pm,\ \lambda\right)$$

Finally the mapping f is a homeomorphism for it is the composition of two homeomorphisms: $f = g \circ h$.

□

3.5 Observations on minimal vectors and minimal curves in $\mathcal{P}(3)$

Commentaries on extremal and non-extremal matrices

Recall from Section 3.1.1, the quotient space $\mathcal{M} = M^h_{3\times3}(\mathbb{C})/\mathcal{D}_{3\times3}$ with the quotient norm

$$|\,[M]\,| = \inf_{D \in \mathcal{D}_{3\times3}} ||\, M + D\, ||$$

where $||\cdot||$ is the usual operator norm. The following proposition is easily verified:

Proposition 3.18 *Let M be an non-extremal matrix with norm $||M|| = \lambda$, let μ_0 be a real number $|\mu_0| \le \lambda$ and let O be any neighborhood of $[M]$ in the quotient space $\mathcal{M}$. Then there exist in O infinitely many classes in paths $[M(t)]$ of extremal matrices, where each $M(t)$ has norm λ and intermediate eigenvalue μ_0 i.e. with spectrum $\{-\lambda,\, \mu_0,\, \lambda\}$.*

Proof of Proposition 3.18 In Formulas (3.25), (3.26) and (3.27), extremal matrices are written as, $f\left(V;\ \ e^{i\eta},\ \ e^{i\xi},\ \ \mu,\ \ \lambda\right)$ where $V \in \{v_\alpha,\ v_\beta,\ v_\chi\}$. The proposition follows from the equality of classes in $\mathcal{M}$,

$$\left[\lim_{(\alpha,\beta,\chi)\to V} f\left(\alpha,\ \beta,\ \chi,\ \ e^{i\eta},\ \ e^{i\xi},\ \ \mu_0,\ \ \lambda\right)_{\pm}\right] \ = \ \left[f\left(V;\ \ e^{i\eta},\ \ e^{i\xi},\ \ \mu,\ \ \lambda\right)\right],$$

for any $V \in \{v_\alpha,\ v_\beta,\ v_\chi\}$, $\eta,\ \xi \in [0, 2\pi)$, $\mu_0,\ \mu \in [-\lambda, \lambda]$ with $\lambda > 0$. □

Let us observe that:

- Each matrix of extremal type contains infinitely many minimal matrices in its class.
- The matrices of non-extremal type are the only minimum in their own classes.
- In the set $\mathbb{M}$ of minimal matrices, the matrices of non-extremal type form an open set.

Then we have the following immediate corollary,

Corollary 3.19 *In the space* $\mathbb{M}$ *of minimal matrices, those of non-extremal type form an open dense set.*

Commentaries on minimal curves in $\mathcal{P}(3)$

The flag manifold $\mathcal{P}(3)$ has minimal curves given by the action of exponentials of minimal matrices, as asserted earlier in Theorem 1.1 in page 1. The question arises, in this context, if the multiplicity of minima in a class could lead to multiple minimal curves starting with the same initial velocity (given by the class in $\mathcal{M}$).

Multiplying by the imaginary unit i an extremal matrix, we get the anti-hermitian version of an extremal matrix. The exponentials of the multiple minima in a fixed class (possible only in this extremal type) produce matrices that differ by a factor in the isotropy of the action, hence the corresponding minimal curves described in Theorem 1.1 are all the same. Hence in $\mathcal{P}(3)$ the minimal curves are unique for a given initial velocity vector X (the class of a minimal matrix).

In conclusion, for close points in $\mathcal{P}(3)$, there are unique minimal curves joining them. In the following section, we shall present the space $\mathcal{P}(4)$ which has infinitely many minimal curves joining arbitrarily close points.

4 An example related to minima in 4×4 hermitian matrices

In this section we shall present a low dimensional Finsler manifold which has *infinitely many minimal curves joining arbitrarily close points.*

The flag manifold $\mathcal{P}(4)$ of 4-tuples of mutually orthogonal lines in $\mathbb{C}^4$

Consider the homogeneous space $\mathcal{P}(4)$, the flag manifold of 4-tuples of mutually orthogonal lines in $\mathbb{C}^4$. The group of unitary operators in $\mathbb{C}^4$ acts on the left in $\mathcal{P}(4)$ by sending each complex line to its image by the unitary operator. Consider the *canonical flag* $p_e = (\mathrm{sp}\{e_1\}, \mathrm{sp}\{e_2\}, \mathrm{sp}\{e_3\}, \mathrm{sp}\{e_4\})$ where $\mathrm{sp}\{e_i\}$ is the complex line spanned by the canonical vector e_i in $\mathbb{C}^4$. The isotropy of p_e is the subgroup of 'diagonal' unitary operators.

Consider the submanifold $\mathcal{P}_d$ of $\mathcal{P}(4)$ given by

$$\mathcal{P}_d = \{(l_1, l_2, l_3, l_4) \in \mathcal{P}(4) \mid \mathrm{sp}\{l_1, l_2\} = \mathrm{sp}\{e_1, e_2\}\}$$

Notice that $\mathcal{P}_d = \mathcal{W} \times \mathcal{W}$ where $\mathcal{W}$ is the flag manifold of couples of mutually orthogonal 1-dimensional complex lines in $\mathbb{C}^2$. Notice also that an ordered pair of mutually orthogonal 1-dimensional complex lines in $\mathbb{C}^2$ is totally determined by the first complex line of the pair, hence $\mathcal{W} = \mathbb{C}P(2)$. Furthermore $\mathbb{C}P(2) = \mathcal{RS}$, the Riemann Sphere, hence $\mathcal{W}{=}\mathcal{RS}$.

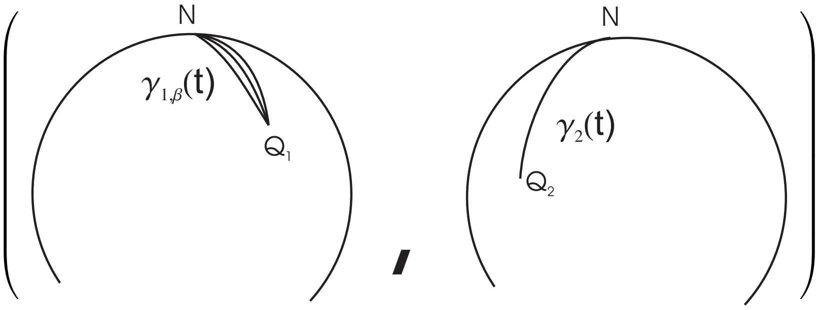
The minimal curves presented in this example shall be constructed in $\mathcal{P}_d$. For a better geometrical view of those curves we shall identify $\mathcal{RS}$, via stereographic projection, with the unit sphere S^2 in $\mathbb{R}^3$, hence we shall make the identification $\mathcal{P}_d = S^2 \times S^2$.

A description of the minimal curves

Let $\mathcal{N} = (N, N) \in \mathcal{P}_d = S^2 \times S^2$ be the point whose coordinates are both the North Pole, $N \in S^2$. Let $\mathcal{Q} = (Q_1, Q_2) \in S^2 \times S^2$ be any point of $\mathcal{P}_d$ such that Q_1 has higher latitude than Q_2 in S^2 (Q_1 is closer to N than Q_2).

We will fix $\mathcal{Q}$ so that Q_2 is above the equator line (and Q_1 is even higher) and present a family of minimal curves $\Gamma_\beta(t) = (\gamma_{1,\beta}(t), \gamma_2(t))$, for $t \in [0, 1]$, all joining $\mathcal{N}$ to $\mathcal{Q}$.

- *The curve $\gamma_2(t)$ in S^2 will trace the smaller arc of the great circle that contains N and Q_2.*
- *The family of curves $\gamma_{1,\beta}(t)$ will vary continuously with the parameter β.*
- *Each of the curves of the family $\gamma_{1,\beta}(t)$ will parameterize the smaller arc of some circle in S^2 that joins N to Q_1; the arcs will not be great circles but for $\beta = 0$.*



A precise description of the minimal curves

To present the curves drawn above we give a more manageable description of $\mathcal{P}(4)$. We consider the unitary subgroup $\mathcal{U} = U(4)$ of the C^*-algebra $\mathcal{A} = M_4(\mathbb{C})$ of 4×4 complex matrices, and denote with $\mathcal{B}$ the subalgebra of diagonal matrices in $\mathcal{A}$. The homogeneous space $\mathcal{P}(4)$ is given by the quotient $\mathcal{U}/\mathcal{D}$, where $\mathcal{D} = \mathcal{U} \cap \mathcal{B}$ is the subgroup of the diagonal unitary matrices. The group $\mathcal{U}$ acts on $\mathcal{P}(4)$ (on the left). The tangent space at 1 (the identity class) is the subspace of anti-hermitian matrices in $\mathcal{A}$ with zeroes on the diagonal.

We construct $\mathcal{P}_d \subset \mathcal{P}(4)$ as follows. First consider the subgroup $SU(2) \times SU(2) \subset \mathcal{U}$ of special unitary matrices build with two, 2×2, blocks on the diagonal. We set $\mathcal{P}_d \subset \mathcal{P}(4)$ as the quotient of $SU(2) \times SU(2)$ by the subgroup $\mathcal{D}$ of diagonal special unitary matrices. This submanifold is in itself a product of two copies of the quotient $\mathcal{W}$ of $SU(2)$ by the subgroup of diagonal matrices in $SU(2)$. For the relations among the different groups here mentioned we suggest [7]. We write $\mathcal{P}_d = \mathcal{W} \times \mathcal{W}$ and a point of $\mathcal{P}_d$ is a class (in a quotient) which in itself has two components which are also classes. We shall use the notation $[U] = ([u_1], [u_2]) \in \mathcal{P}_d = \mathcal{W} \times \mathcal{W}$.

The minimal curves starting at $1 \in \mathcal{P}_d$ are of the form $\gamma(t) = \left[e^{tZ}\right]$ where the matrices Z are anti-hermitian matrices with zero trace in $\mathcal{A}$ built with two blocks of anti-hermitian 2×2 matrices on the diagonal (each one with zero trace).

The minimality of the curves is granted by Theorem 1.1 for the matrices Z shall be minimal vectors according to Theorem 2.2. In fact, we shall consider $Z \in \mathcal{A}_{an}$ of the form

$$Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$$

where Z_1 and Z_2 are anti-hermitian 2×2 matrices of the form

$$Z_1 = \begin{pmatrix} z\, i & r\,(-\sin(\alpha) + i\,\cos(\alpha)) \\ r(\sin(\alpha) + i\,\cos(\alpha)) & -z\, i \end{pmatrix}, \quad (4.29)$$

and

$$Z_2 = \begin{pmatrix} 0 & w \\ -\overline{w} & 0 \end{pmatrix} \quad (4.30)$$

where z, $r, \alpha \in \mathbb{R}$, and $w \in \mathbb{C}$.

The minimality of these matrices Z is assured in the case where $|w|^2 \geq z^2 + r^2$. In such case, $||Z||^2 = |w|^2$ and, in relation to Theorem 2.2, just consider the operator representation ρ of the C^*-algebra $\mathcal{A} = M_4(\mathbb{C})$ on $\mathbb{C}^4$, together with the unit vector $\xi = (0, 0, 0, 1) \in \mathbb{C}^4$.

The two components of the curves in $\mathcal{P}_d$

The curve $\gamma(t) = \left[e^{tZ}\right] = \left(\left[e^{tZ_1}\right], \left[e^{tZ_2}\right]\right)$ in $\mathcal{P}_d$ has two components (in $\mathcal{W}$).

We shall regard the Riemann Sphere $\mathcal{RS}$ as the complex plane $\mathbb{C}$ with the point "∞" added. Consider a matrix u in $SU(2)$

$$u = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \text{ where } a,\, b \in \mathbb{C} \text{ and } |a|^2 + |b|^2 = 1.$$

We consider the mapping L from $SU(2)$ to $\mathcal{RS}$ is given by

$$L(u) = \frac{a}{b}, \text{ if } b \neq 0, \text{ else } L(u) = \infty.$$

It is clear that this mapping induces an explicit diffeomorphism from the quotient of $SU(2)$ by its diagonal matrices to the Riemann Sphere $\mathcal{RS}$.

Consider the unit sphere S^2 in $\mathbb{R}^3$, and let the equatorial plane, $\mathbb{C}$, represent the "finite" part of the Riemann Sphere $\mathcal{RS}$. We set $\varphi : \mathcal{RS} \to S^2$ to be the stereographic projection given as by:

$$\varphi(\zeta) = \left(\frac{2\zeta}{|\zeta|^2+1}, \frac{|\zeta|^2-1}{|\zeta|^2+1}\right) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3, \text{ for } \zeta \in \mathbb{C}, \text{ and } \varphi(\infty) = (0,0,1) = N \in S^2 \subset \mathbb{R}^3.$$

Notice that in the class $b \neq 0$, if $\zeta = L(u) = \frac{a}{b} \in \mathbb{C}$, then $\varphi(\zeta) = \left(2a\bar{b}\,,\, |a|^2 - |b|^2\right)$. If $b = 0$, then $|a| = 1$, hence $\zeta = L(u) = \infty$, then $\varphi(\zeta) = (0\,,\,0\,,\,1)$.

Via a composition of two maps, we define the diffeomorphism Ψ from $\mathcal{W}$ onto S^2: for $[u] = \left[\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}\right] \in \mathcal{W}$ we set,

$$\Psi([u]) = \varphi\left(L(u)\right) = \left(2a\bar{b}\,,\, |a|^2 - |b|^2\right) = \left(2a\bar{b}\,,\, 1 - 2|b|^2\right) \in S^2.$$

Considering the curve $q(t) = e^{tZ_1}$ in $SU(2)$ with Z_1 as in formula (4.29) above, and setting $\lambda = \sqrt{r^2+z^2}$, it can be verified that $L(q(t)) \in \mathcal{RS}$ is given by,

$$\begin{aligned} L(q(t)) &= \frac{z\,(\cos(\alpha) + i\,\sin(\alpha))}{r} + \cot(t\lambda)\frac{\lambda\,(\sin(\alpha) - i\,\cos(\alpha))}{r}, \text{ if } t \notin \left\{\frac{k\pi}{\lambda} \mid k \in \mathbb{Z}\right\} \\ &\text{and,} \\ L(q(t)) &= \infty, \text{ if } t \in \left\{\frac{k\pi}{\lambda} \mid k \in \mathbb{Z}\right\}. \end{aligned} \tag{4.31}$$

Notice then that $L(q(t))$ parameterizes a straight line l_q in $\mathcal{RS}$. Hence the curve

$$\Psi([q(t)]) = \varphi\left(L(q(t))\right)$$

is an arc of a circle in S^2 (not necessarily a great circle) contained in the plane in $\mathbb{R}^3$ that contains both the line l_q, in the equatorial plane, and the North Pole N, in S^2. It can be verified that this plane has unit normal vectors given by:

$$\pm\left(\cos(\beta)\cos(\alpha)\,,\, \cos(\beta)\sin(\alpha)\,,\, \sin(\beta)\right)$$

where $\cos(\beta) = \dfrac{r}{\lambda}$, $\sin(\beta) = \dfrac{z}{\lambda}$, with $\lambda = \sqrt{r^2+z^2}$.

Some observations on the curves $\Psi\left([e^{tZ_1}]\right)$ and $\Psi\left([e^{tZ_2}]\right)$ in $\mathcal{W}$.

Let $\gamma_{1,\beta}(t) = \Psi\left([e^{tZ_1}]\right)$, where $\cos(\beta) = \dfrac{r}{\lambda}$, $\sin(\beta) = \dfrac{z}{\lambda}$, with $\lambda = \sqrt{r^2+z^2}$, and let $\gamma_2(t) = \Psi\left([e^{tZ_2}]\right)$.

- In the constructions above, the curve $\gamma_{1,\beta}(t)$ runs over a great circle in S^2 if and only if $\beta = 0$ (equivalently $z = 0$).
- The curve $\gamma_2(t)$ runs over a great circle in S^2 (Z_2 has parameter $z = 0$).
- The curve $\gamma_{1,\beta}(t)$ varies continuously with the parameter β.

- The curve $\gamma_{1,\beta}(t)$ starts at $N \in S^2$ and returns to that point exactly for $t \in \left\{ \frac{k\pi}{\lambda} \mid k \in \mathbb{Z} \right\}$.
- The curve $\gamma_{1,\beta}(t)$ has constant speed $2\lambda \cos(\beta)$ in S^2.
- The curve $\gamma_2(t)$ has constant speed $2r$ in S^2.

The curves $\Gamma_\beta(t) = \left(\Psi\left([e^{tZ_1}]\right), \Psi\left([e^{tZ_2}]\right)\right)$ in $\mathcal{P}_d$.

Lets give explicit values of the "parameters" z, α, $r \in \mathbb{R}$ and $w \in \mathbb{C}$ that define Z_1 and Z_2 (according to formulas (4.29) and (4.30)), so that for $t \in [0,1]$, the curves $\gamma_{1,\beta}(t) = \Psi\left([e^{tZ_1}]\right)$ and $\gamma_2(t) = \Psi\left([e^{tZ_2}]\right)$ join the point N to Q_1 and Q_2 respectively.

Suppose that the distances from N to Q_1 and Q_2 in S^2 are $2\phi_1$ and $2\phi_2$ respectively (with $\phi_1 < \phi_2$).

By means of some rotation of the sphere S^2 we may suppose that Q_1 is in the plane generated by $\hat{\jmath}$ and $\hat{k}$, as in Figure 3 below, and we have, $Q_1 = (0, \sin(2\phi_1), \cos(2\phi_1))$ and $Q_2 = (\sin(2\phi_2)\cos(\theta_2), \sin(2\phi_2)\sin(\theta_2), \cos(2\phi_2))$.

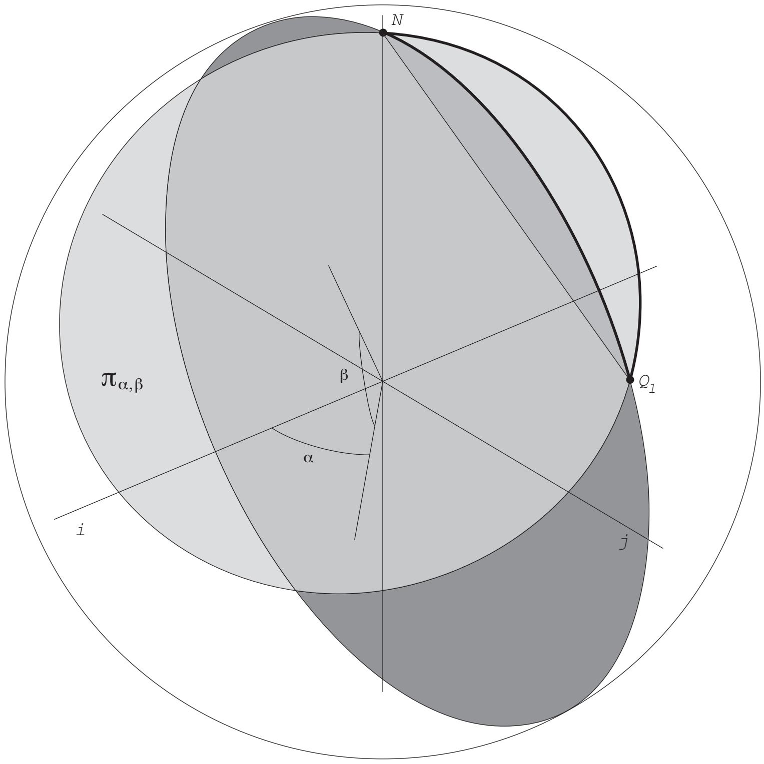


Figure 3: We suppose that Q_1 is in the plane generated by $\hat{\jmath}$ and $\hat{k}$.

For Z_2 we set $w = \phi_2\left(-\sin(\theta_2) + i\,\cos(\theta_2)\right)$ so that $\gamma_2(1) = \Psi\left([e^{Z_2}]\right) = Q_2$.

We have to choose the values z, α, $r \in \mathbb{R}$ and $w \in \mathbb{C}$ that define Z_1. This is equivalent to chose β, α, $\lambda \in \mathbb{R}$ via the change of variables given by the equations

$$\cos(\beta) = \frac{r}{\lambda}, \quad \sin(\beta) = \frac{z}{\lambda}, \text{ with } \lambda = \sqrt{r^2 + z^2}.$$

The parameters α and β are shown in Figure 3, with the only restriction that the vector

$$\vec{n} = (\cos(\beta)\cos(\alpha)\,,\, \cos(\beta)\sin(\alpha)\,,\, \sin(\beta))$$

is orthogonal to a plane $\pi_{\alpha,\beta}$ that contains N and Q_1.

The parameter λ is determined after choosing α and β so that the short arc joining N and Q_1, in the intersection of the plane $\pi_{\alpha,\beta}$ with the sphere S^2 as in Figure 3, has length ℓ equal to $2\lambda\cos(\beta)$, from where the value of λ is drawn.

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