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Discrete duality for 3-valued Łukasiewicz–Moisil algebras

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In 2011, Düntsch and Orłowska obtained a discrete duality for regular double Stone algebras. On the other hand, it is well known that regular double Stone algebras are polinominally equivalent to 3-valued Łukasiewicz–Moisil algebras (or LM_3-algebras). In [R. Cignoli, Injective De Morgan and Kleene algebra, *Proc. Amer. Math. Soc.* **47** (1975) 269–278], LM_3-algebras are considered as a Kleene algebras $\langle L, \vee, \wedge, \sim, 0, 1\rangle$ endowed with a unary operation $\Box : L \to L$, satisfying the properties: $a \vee \sim \Box a = 1$, $\sim a \wedge a = a \wedge \sim \Box a$ and $\Box a \vee \Box b \leq \Box(a \vee b)$. Motivated by this result, in this paper, we determine another discrete duality for LM_3-algebras, extending the discrete duality to De Morgan algebras described in [W. Dzik, E. Orłowska and C. van Alten, Relational representation theorems for general lattices with negations, in *Relations and Kleene Algebra in Computer Science*, Lecture Notes in Computer Science, Vol. 4136 (Springer, Berlin, 2006), pp. 162–176].

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1. Introduction and Preliminaries

A discrete duality is a relationship between classes of algebras and classes of relational systems (frames): If **Alg** is a class of algebras and **Frm** is a class of frames, to establish a discrete duality between these two classes, the following steps are required:

- For every algebra L from **Alg**, we associate a canonical frame $\mathcal{X}(\mathcal{C}(L))$ of the algebra and show that it belongs to **Frm**.
- For every frame X from **Frm**, we associate a complex algebra $\mathcal{C}(\mathcal{X}(L))$, and show that it belongs to **Alg**.

- Prove two representation theorems:
  - * For each $L \in \mathbf{Alg}$ there is an embedding $h : L \hookrightarrow \mathcal{C}(\mathcal{X}(L))$.
  - * For each $X \in \mathbf{Frm}$ there is an embedding $k : X \hookrightarrow \mathcal{X}(\mathcal{C}(X))$.

Canonical frames correspond to dual spaces of algebras in the Priestley style duality [12]; however, they are not endowed with a topology and hence may be thought of as having a discrete topology. Complex algebras of canonical frames correspond to canonical extensions in the sense of Jónsson and Tarski [7].

A discrete duality leads to what is called duality via truth in [10] (see also [9]). Duality via truth amounts to say that the concept of truth associated with an algebraic semantics of a formal language determined by class **Alg** of algebras and the concept of truth associated with its relational (Kripke-style) semantics determined by class **Frm** of relational systems are equivalent, that is the same formulas are true in both of these classes of semantic structures. General principles and applications of discrete duality are briefly presented in [11].

The main purpose of this paper is to give a discrete duality for 3-valued Łukasiewicz–Moisil algebra (LM_3-algebras). To do this we will extend the discrete duality given in [5], for De Morgan algebras.

Let us recall that an algebra $\langle L, \vee, \wedge, \sim, 0, 1\rangle$ is a De Morgan algebra if the reduct $\langle L, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and $\sim$ is a unary operation on L satisfying the following identities: $\sim (x \vee y) = \sim x \wedge \sim y$, $\sim\sim x = x$ and $\sim 0 = 1$.

On the other hand, a Kleene algebra is a De Morgan algebra $\langle L, \vee, \wedge, 0, 1\rangle$ that satisfies the additional condition:

$$x \wedge \sim x \leq y \vee \sim y.$$

Given a relational structure $\langle X, \leq\rangle$ where $X \neq \emptyset$ and $\leq$ is a reflexive, antisymmetric and transitive binary relation on X (i.e. a poset), we will denote by $[\leq]U$ the set $\{x \in X : \forall y, x \leq y \Rightarrow y \in U\}$, for any $U \subseteq X$. Besides, we will denote by $[Y)$ $((Y])$ the set $\{x \in X : \exists y \in Y y \leq x\}$ $(\{x \in X : \exists y \in Y x \leq y\})$, for any $Y \subseteq X$. In particular, if Y is the single set $\{x\}$ we will write $[x)$ instead of $[\{x\})$.

A De Morgan frame is a structure $\langle X, \leq, g\rangle$, where $\langle X, \leq\rangle$ is a poset and $g : X \to X$ is a function which satisfies:

- $g(g(x)) = x$,
- if $x \leq y$, then $g(y) \leq g(x)$.

Let $\langle L, \vee, \wedge, \sim, 0, 1\rangle$ be a De Morgan algebra and let $\mathcal{X}(L)$ be the set of all prime filters of L. It is known that $\langle \mathcal{X}(L), \leq^c, g^c\rangle$ is a De Morgan frame, where $\leq^c$ is $\subseteq$ and $g^c : \mathcal{X}(L) \to \mathcal{X}(L)$ is the involution defined by

$$g^c(S) = \{x \in L : \sim x \notin S\}, \quad \text{for all } S \in \mathcal{X}(L). \tag{1.1}$$

Moreover, if $\langle X, \leq, g\rangle$ is a De Morgan frame, then

$$\langle \mathcal{C}(X), \cup, \cap, \sim^c, \emptyset, X\rangle$$

is a De Morgan algebra, where $\mathcal{C}(X) = \{U \subseteq X : [\leq]U = U\}$ and $\sim^c : \mathcal{C}(X) \to \mathcal{C}(X)$ is defined by

$$\sim^c U = X \backslash g(U), \quad \text{for every } U \in \mathcal{C}(X). \tag{1.2}$$

These results allow us to obtain a discrete duality for De Morgan algebras by defining the embeddings as follows:

- $h : L \to \mathcal{C}(\mathcal{X}(L))$, defined by $h(a) = \{S \in \mathcal{X}(L) : a \in S\}$,
- $k : X \to X(\mathcal{C}(X))$, defined by $k(x) = \{U \in \mathcal{C}(X) : x \in U\}$.

2. Discrete Duality for LM_3-Algebras

In this section, we describe a discrete duality for LM_3-algebras taking into account the one indicated in Sec. 1 for De Morgan algebras.

A LM_3-algebra (see [1, 4, 8]) is an algebra $\langle L, \vee, \wedge, \sim, \Box, 0, 1\rangle$ such that $\langle L, \vee, \wedge, \sim, 0, 1\rangle$ is a Kleene algebra and $\Box$ is an unary operation on L which satisfy the following conditions:

(L1) $a \vee \sim \Box a = 1$,
(L2) $\sim a \wedge a = a \wedge \sim \Box a$,
(L3) $\Box a \vee \Box b \leq \Box(a \vee b)$.

Definition 2.1. A structure $\langle X, \leq, g, R\rangle$ is a LM_3-frame if $\langle X, \leq, g\rangle$ is a De Morgan frame and R is a binary relation on X such that:

(K0) $x \leq g(x)$ or $g(x) \leq x$,
(K1) R is reflexive,
(K2) $(\leq \circ R \circ \leq) \subseteq R$,
(K3) if $(x, y) \in R$, then $x \leq y$ or $g(x) \leq y$.
(K4) $g(x) \in R(x)$, for all $x \in X$.

Definition 2.2. The complex algebra of a LM_3-frame $\langle X, \leq, g, R\rangle$ is a structure

$$\langle \mathcal{C}(X), \cup, \cap, \sim^c, \Box^c, \emptyset, X\rangle,$$

where $\langle \mathcal{C}(X), \cup, \cap, \sim^c, \emptyset, X\rangle$ is a complex algebra of the De Morgan frame $\langle X, \leq, g\rangle$ and for any $U \in \mathcal{C}(X)$, $\Box^c(U) = \{x \in X : R(x) \subseteq U\}$.

Definition 2.3. The canonical frame of a LM_3-algebra $\langle L, \vee, \wedge, \sim, \Box, 0, 1\rangle$ is a structure

$$\langle \mathcal{X}(L), \leq^c, g^c, R^c\rangle$$

where $\langle \mathcal{X}(L), \leq^c, g^c\rangle$ is the canonical frame associated with $\langle L, \vee, \wedge, \sim, 0, 1\rangle$ and R^c is a binary relation on $\mathcal{X}(L)$ defined by

$$(S, T) \in R^c \Leftrightarrow \Box^{-1}(S) \subseteq T.$$

Lemma 2.1. *The canonical frame of a LM_3-algebra is a LM_3-frame.*

Proof. Taking into account Definition 2.3, we only have to prove from (K0) to (K4).

(K0): Let S be a prime filter such that $S \not\subseteq g^c(S)$ and $g^c(S) \not\subseteq S$. Then, there exists $x \in S$ such that $x \notin g^c(S)$ and there exists $y \in g^c(S)$ such that $y \notin S$. As S is a filter, we have to $x \wedge \sim x \in S$ obtaining as a result $y \vee \sim y \in S$. Then, as $y \notin S$, we have $y \notin g^c(S)$ which is a contradiction.

(K1): Let S be a prime filter such that $x \in \Box^{-1}(S)$. So, $\Box x \in S$. Since $\Box x \leq x$, we have that $x \in S$. Thus, $\Box^{-1}(S) \subseteq S$, i.e. $(S, S) \in R^c$.

(K2): Suppose that $(P, F) \in (\leq^c \circ R^c \leq^c)$. Then, there exists $T, S \in \mathcal{X}(L)$ such that $P \subseteq T$, $(T, S) \in R^c$ and $S \subseteq F$. From this last assertion we deduce that $\Box^{-1}(T) \subseteq F$. Since $P \subseteq T$ we infer that $(P, F) \in R^c$.

(K3): Let $S, T \in \mathcal{X}(L)$ such that $(S, T) \in R^c$. Suppose that $S \not\subseteq T$ and $g^c(S) \not\subseteq T$. Then $S \cap g^c(S) \not\subseteq T$, because T is a prime filter. So there exists $a \in S \cap g^c(S)$ and $a \notin T$. Then $\sim a \notin S$. As $\sim a \wedge a = a \wedge \sim \Box a$, and $a \in S$, $\sim \Box a \notin S$. So, $\Box a \in g^c(S)$. Since $\Box a \wedge \sim \Box a = 0$, $\sim \Box a \notin g^c(S)$, i.e. $\Box a \in S$. So, $a \in T$, because $(S, T) \in R^c$, which is a contradiction.

(K4): Let $P \in \mathcal{X}(L)$ such that $(P, g^c(P)) \notin R^c$. Then, there exists $a \in L$ such that $\Box a \in P$ and $a \notin g^c(P)$. So, $\Box a \wedge \sim a = 0 \in P$, which is a contradiction. □

Lemma 2.2. *The complex algebra of a LM_3-frame is a LM_3-algebra.*

Proof. We need to show closure under the operation $\Box^c$, that is, $\Box^c U = [\leq]\Box^c U$. The inclusion $\supseteq$ follows from reflexivity of $\leq$. Assume that $x \in \Box^c U$. Let $y \in X$ such that $x \leq y$. Take any $z \in X$ such that $(y, z) \in R$. Then, from (K2) we infer that $(x, z) \in R$. So, $z \in U$. Then, $x \in [\leq]\Box^c U$. Therefore, $\Box^c U \subseteq [\leq]\Box^c U$. Is clear that $\mathcal{C}(X)$ is a De Morgan algebra. Now, we prove $U \cap U^c \subseteq V \cup \sim^c V$. Suppose that $x \in U$ and $g(x) \notin U$. Taking into account (K0), we can deduce that $g(x) < x$. On the other hand, we take $x \notin V$ such that $x \notin \sim^c V$. Then, $g(x) \in V$, from which turns out that $x \in V$, as $V \in \mathcal{C}(X)$. Therefore, $x \in V \cup \sim^c V$. Now we will prove (L1), (L2) and (L3).

(L1): Suppose that $\sim^c U \cap \Box^c U \neq \emptyset$. Then, there exists $y \in \sim^c U$ such that $R(y) \subseteq U$. As $g(y) \in R(y)$, we have $g(y) \in U$, which is a contradiction. Then, $\sim^c U \cap \Box^c U = \emptyset$. Therefore, $U \cup \sim^c \Box^c U = X$.

(L2): Let $x \in \sim^c U \cap U$ and suppose that $R(g(x)) \subseteq U$. So, by (K4), we have that $g(x) \in U$ which is a contradiction. Conversely, suppose that $x \in U \cap \sim^c \Box^c U$. Since $R(g(x)) \not\subseteq U$, there exists $y \in R(g(x))$ such that $y \notin U$. So, by (K3), we have that $g(x) \leq y$. Therefore, $g(x) \notin U$. So, since $x \in U$, we deduce that $x \in U \cup \sim^c U$.

(L3): It is a direct consequence of the definition of $\Box^c$. □

Now we show that the embedding $h : L \to \mathcal{C}(\mathcal{X}(L))$, defined in Sec. 1, preserves the unary operator $\Box$, that is, the following.

Lemma 2.3. *For any $a \in L$, $h(\Box a) = \Box^c(h(a))$.*

Proof. Let $F \in h(\Box a)$; then $\Box a \in F$. Suppose that $P \in \mathcal{X}(L)$ verifies that $(F, P) \in R^c$. Then, $\Box^{-1}(F) \subseteq P$ and so, $a \in P$. Therefore, $F \in \Box^c(h(a))$ from which we infer that $h(\Box a) \subseteq \Box^c(h(a))$. Conversely, assume that $F \in \Box^c(h(a))$. Then for every $P \in \mathcal{X}(L)$, $(F, P) \in R^c$ implies $P \in h(a)$. Suppose that $\Box a \notin F$. Then $\Box^{-1}(F)$ is a filter and $a \notin \Box^{-1}(F)$. Hence, there is $T \in \mathcal{X}(L)$ such that $a \notin T$ and $\Box^{-1}(F) \subseteq T$. This last assertion allows us to conclude that $(F, T) \in R^c$. From this statement we have that $T \in h(a)$ and so, $a \in T$, which is a contradiction. Therefore, $h(\Box a) = \Box^c(h(a))$. Thus, by virtue of the results established in [5] the proof is completed. □

Lemma 2.4 will show that the order-embedding $k : X \to \mathcal{X}(\mathcal{C}(X))$ defined in Sec. 1, preserves the relation R.

Lemma 2.4. *Let $\langle X, \leq, g, R\rangle$ be a LM_3-frame and let $x, y \in X$. Then*

- *$(x, y) \in R$ if and only if $(k(x), k(y)) \in R^c$.*

Proof. Assume that $(x, y) \in R$ and suppose that $U \in \mathcal{C}(X)$ verifies $\Box^c U \in k(x)$. Then it is easy to see that $y \in U$ and so, $(k(x), k(y)) \in R^c$. Conversely, let $x, y \in X$ be such that $(k(x), k(y)) \in R^c$. Then $\Box^{c^{-1}}(k(x)) \subseteq k(y)$. On the other hand, note that $[\leq](X\backslash(y]) \in \mathcal{C}(X)$ and $y \notin [\leq](X\backslash(y])$. Thus, $[\leq](X\backslash(y]) \notin k(y)$ and so, $[\leq](X\backslash(y]) \notin \Box^{c^{-1}}(k(x))$. Therefore, $\Box^c([\leq](X\backslash(y])) \notin k(x)$ from which we infer that $x \notin \Box^c([\leq](X\backslash(y]))$. Then there is z such that $(x, z) \in R$ and $z \notin [\leq](X\backslash(y])$. From this last assertion there is w such that $z \leq w$ and $w \leq y$, which allow us to infer that $z \leq y$. Hence, by virtue of the reflexivity of $\leq$ and (K2), $(x, y) \in R$ as required. □

Hence, we have a discrete duality between LM_3-algebras and LM_3-frames.

Theorem 2.1. (a) *Every LM_3-algebra is embeddable into the complex algebra of its canonical frame.*
(b) *Every LM_3-frame is embeddable into the canonical frame of its complex algebra.*

3. Conclusions and Further Studies

The discrete dualities developed in this paper provide, on the one hand, a representation theorem for the classes of LM_3-algebras and, on the other hand, they provide the classes of relational systems which enable us an alternative formalization and interpretation of the relevant domains in the logical framework. The representation theorems constituting the discrete dualities show that the formalization in terms of these relational systems is equivalent to the algebraic formalization.

The present paper provides a basis for further work on discrete duality for De Morgan algebras with modal operators (see [2, 3]).

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