k-tuple colorings of the cartesian product of graphs[☆]

Flavia Bonomo

CONICET and DC, FCEN, Universidad de Buenos Aires, Argentina

Ivo Koch

Instituto de Industria, Universidad Nacional de General Sarmiento, Argentina

Pablo Torres

Universidad Nacional de Rosario and CONICET, Argentina

Mario Valencia-Pabon

Université Paris-13, Sorbonne Paris Cité LIPN, CNRS UMR7030, Villetaneuse, France

Abstract

A k-tuple coloring of a graph G assigns a set of k colors to each vertex of G such that if two vertices are adjacent, the corresponding sets of colors are disjoint. The k-tuple chromatic number of G, $\chi_k(G)$, is the smallest t so that there is a k-tuple coloring of G using t colors. It is well known that $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$. In this paper, we show that there exist graphs G and H such that $\chi_k(G \square H) > \max\{\chi_k(G), \chi_k(H)\}$ for $k \geq 2$. Moreover, we also show that there exist graph families such that, for any $k \geq 1$, the k-tuple chromatic number of their cartesian product is equal to the maximum k-tuple chromatic number of its factors.

Keywords: k-tuple colorings, Cartesian product of graphs, Kneser graphs, Cayley graphs, Hom-idempotent graphs.

[☆]Partially supported by MathAmSud Project 13MATH-07 (Argentina–Brazil–Chile–France), UBACyT Grant 20020130100808BA, CONICET PIP 112-201201-00450CO, ANPCyT PICT 2012-1324 (Argentina), and by the French ANR JCJC CombPhysMat2Tens grant.

Email addresses: `fbonomo@dc.uba.ar` (Flavia Bonomo), `ivo.koch@gmail.com` (Ivo Koch), `ptorres@fceia.unr.edu.ar` (Pablo Torres), `valencia@lipn.univ-paris13.fr` (Mario Valencia-Pabon)

1. Introduction

A classic *coloring* of a graph G is an assignment of colors (or natural numbers) to the vertices of G such that any two adjacent vertices are assigned different colors. The smallest number t such that G admits a coloring with t colors (a t*-coloring*) is called the *chromatic number* of G and is denoted by $\chi(G)$. Several generalizations of the coloring problem have been introduced in the literature, in particular, cases in which each vertex is assigned not only a color but a set of colors, under different restrictions. One of these variations is the k*-tuple coloring* introduced independently by Stahl [11] and Bollobás and Thomason [3]. A k-tuple coloring of a graph G is an assignment of k colors to each vertex in such a way that adjacent vertices are assigned distinct colors. The k*-tuple coloring problem* consists into finding the minimum number of colors in a k-tuple coloring of a graph G, which we denote by $\chi_k(G)$.

The *cartesian product* $G\Box H$ of two graphs G and H has vertex set $V(G) \times V(H)$, two vertices being joined by an edge whenever they have one coordinate equal and the other adjacent. This product is commutative and associative up to isomorphism. There is a simple formula expressing the chromatic number of a cartesian product in terms of its factors:

$$\chi(G\Box H) = \max\{\chi(G), \chi(H)\}. \tag{1}$$

The identity (1) admits a simple proof first given by Sabidussi [10].

The *Kneser graph* $K(m, n)$ has as vertices all n-element subsets of the set $[m] = \{1, \ldots, m\}$ and an edge between two subsets if and only if they are disjoint. We will assume in the rest of this work that $m \geq 2n$, otherwise $K(m, n)$ has no edges. The Kneser graph $K(5, 2)$ is the well known *Petersen Graph*. Lovász [9] showed that $\chi(K(m, n)) = m - 2n + 2$. The value of the k-tuple chromatic number of the Kneser graph is the subject of an almost 40-year-old conjecture of Stahl [11] which asserts that: if $k = qn - r$ where $q \geq 0$ and $0 \leq r < n$, then $\chi_k(K(m, n)) = qm - 2r$. Stahl's conjecture has been confirmed for some values of k, n and m [11, 12].

An *homomorphism* from a graph G into a graph H, denoted by $G \to H$, is an edge-preserving map from $V(G)$ to $V(H)$. It is well known that an ordinary graph coloring of a graph G with m colors is an homomorphism from G into the complete graph K_m. Similarly, an n-tuple coloring of a graph G with m colors is an homomorphism from G into the Kneser graph $K(m, n)$. A graph G is said *hom-idempotent* if there is an homomorphism $G\Box G \to G$. We denote by $G \not\to H$ if there exists no homomorphism from G to H. The *clique number* of a graph G, denoted by $\omega(G)$, is the maximum size of a *clique* in G (i.e., a complete subgraph of G). Clearly, for any graphs G and H, we have that $\chi(G) \geq \omega(G)$ (and so, $\chi_k(G) \geq \chi_k(K_{\omega(G)}) = k\omega(G)$) and, if there is an homomorphism from G to H then, $\chi(G) \leq \chi(H)$ and, moreover, $\chi_k(G) \leq \chi_k(H)$.

In this paper, we show that the analogous of equality (1) for k-tuple colorings of graphs does not hold in general. In fact, we show that for some values of $k \geq 2$, there are Kneser graphs $K(m,n)$ for which $\chi_k(K(m,n) \Box K(m,n)) > \chi_k(K(m,n))$. Surprisingly, there exist some Kneser graphs $K(m,n)$ for which the difference $\chi_k(K(m,n) \Box K(m,n)) - \chi_k(K(m,n))$ can be as large as desired, even when $k = 2$. We also show that there are families of graphs for which equality (1) holds for k-tuple colorings of graphs for any $k \geq 1$. As far as we know, our results are the first ones concerning the k-tuple chromatic number of cartesian product of graphs.

2. Cartesian products of Kneser graphs

We start this section with some upper and lower bounds for the k-tuple chromatic number of Kneser graphs.

Lemma 2.1. *Let G be a graph and let $k > 0$. Then, $\chi_k(G \Box G) \leq k\chi(G)$.*

Proof. Clearly, $\chi_k(G \Box G) \leq k\chi(G \Box G)$. However, by equality (1) we know that $\chi(G \Box G) = \chi(G)$, and thus the lemma holds. $\Box$

Notice that there are cases where $\chi_k(G \Box G) = k\chi(G)$ (see Theorem 3.1). Moreover, it is not difficult to verify that $\chi_2(C_5 \Box C_5) < 2\chi(C_5)$. In fact, $\chi_2(C_5 \Box C_5) = 5$ and $\chi(C_5) = 3$.

Corollary 2.2. $\chi_k(K(m,n) \Box K(m,n)) \leq k\chi(K(m,n)) = k(m-2n+2)$.

We can obtain a trivial lower bound for the k-tuple chromatic number of the graph $K(m,n) \Box K(m,n)$ in terms of the clique number of $K(m,n)$. In fact, notice that $\omega(K(m,n) \Box K(m,n)) = \omega(K(m,n)) = \lfloor \frac{m}{n} \rfloor$. Thus, we have that $\chi_k(K(m,n) \Box K(m,n)) \geq k\omega(K(m,n)) = k\lfloor \frac{m}{n} \rfloor$. In Lemma 2.13, we compute a better lower bound for $\chi_k(K(m,n) \Box K(m,n))$.

Larose et al. [8] showed that no connected Kneser graph $K(m,n)$ is hom-idempotent, that is, for any $m > 2n$, there is no homomorphism from $K(m,n) \Box K(m,n)$ to $K(m,n)$.

Lemma 2.3 ([8])**.** *Let $m > 2n$. Then, $K(m,n) \Box K(m,n) \not\to K(m,n)$.*

Concerning the k-tuple chromatic number of some Kneser graphs, Stahl [11] showed the following results.

Lemma 2.4 ([11])**.** *If $1 \le k \le n$, then $\chi_k(K(m,n)) = m - 2(n-k)$.*

Lemma 2.5 ([11])**.** *$\chi_k(K(2n+1,n)) = 2k + 1 + \lfloor \frac{k-1}{n} \rfloor$, for $k > 0$.*

Lemma 2.6 ([11])**.** *$\chi_{rn}(K(m,n)) = rm$, for $r > 0$ and $m \ge 2n$.*

By using Lemma 2.6 we have the following result.

Lemma 2.7. *Let $m > 2n$. Then, $\chi_n(K(m,n)\Box K(m,n)) > \chi_n(K(m,n))$.*

Proof. By Lemma 2.6 when $r = 1$, we have that $\chi_n(K(m,n)) = m$. If $\chi_n(K(m,n)\Box K(m,n)) = m$, then there exists an homomorphism from the graph $K(m,n)\Box K(m,n)$ to $K(m,n)$ which contradicts Lemma 2.3. $\Box$

By Lemma 2.4, Lemma 2.7 and by using Corollary 2.2, we have that,

Corollary 2.8. *Let $n \ge 2$. Then, $2n+2 \le \chi_n(K(2n+1,n)\Box K(2n+1,n)) \le 3n$. In particular, when $n = 2$, we have that $\chi_2(K(5,2)\Box K(5,2)) = 6$.*

In the case $k = 2$ we have by Lemma 2.7, Lemma 2.4 and by Corollary 2.2, the following result.

Corollary 2.9. *Let $q > 0$. Then, $q+4 \le \chi_2(K(2n+q,n)\Box K(2n+q,n)) \le 2q+4$.*

By Corollary 2.9, notice that in the case when $k = n = 2$ and $q \ge 1$, we must have that $\chi_2(K(q+4,2)\Box K(q+4,2)) > q+4$, otherwise there is a contradiction with Lemma 2.3. This provides a gap of one unity between the 2-tuple chromatic number of the graph $K(q+4,2)\Box K(q+4,2)$ and the graph $K(q+4,2)$ when $q = 1$. Moreover, such a gap increase for $q > 1$ as we will see in Theorem 2.16. In the following, we will prove that, for some Kneser graphs, such a gap can be as large as desired. In order to do this, we need the following technical tools.

A *stable set* $S \subseteq V$ is a subset of pairwise non adjacent vertices of G. The *stability number* of G, denoted by $\alpha(G)$, is the largest cardinality of a stable set in G. Let $m \ge 2n$. An element $i \in [m]$ is called a *center* of a stable set S of the Kneser graph $K(m,n)$ if it lies in each n-set in S.

Lemma 2.10 (Erdős-Ko-Rado [5])**.** *If $m > 2n$, then $\alpha(K(m,n)) = \binom{m-1}{n-1}$. A stable set of $K(m,n)$ with size $\binom{m-1}{n-1}$ has a center i, for some $i \in [m]$.*

Lemma 2.11 (Hilton-Milner [7])**.** *If $m \ge 2n$, then the maximum size of a stable set in $K(m,n)$ with no center is equal to $1 + \binom{m-1}{n-1} - \binom{m-n-1}{n-1}$.*

A graph $G = (V, E)$ is *vertex transitive* if its automorphism group acts transitively on V, that is, for any pair of distinct vertices of G there is an automorphism mapping one to the other one. It is well known that Kneser graphs are vertex transitive graphs.

Lemma 2.12 (No-Homomorphism Lemma, Albertson-Collins [1]). *Let G, H be graphs such that H is vertex transitive and $G \to H$. Then,*

$$\alpha(G)/|V(G)| \geq \alpha(H)/|V(H)|.$$

Lemma 2.13. *Let $m > 2n$. Then, $\chi_k(K(m,n)\square K(m,n)) \geq k\frac{\binom{m}{n}^2}{\alpha(K(m,n)\square K(m,n))}$.*

Proof. Let $t = \chi_k(K(m,n)\square K(m,n))$. Then, $K(m,n)\square K(m,n) \to K(t,k)$ and from the No-Homomorphism Lemma, $\frac{\alpha(K(m,n)\square K(m,n))}{|V(K(m,n)\square K(m,n))|} \geq \frac{\alpha(K(t,k))}{|V(K(t,k))|}$. The result follows from the fact that $\frac{\alpha(K(t,k))}{|V(K(t,k))|} = \frac{k}{t}$. □

Notice that previous lower bound for $\chi_k(K(m,n)\square K(m,n))$ improves the one of $k\lfloor\frac{m}{n}\rfloor$ obtained at the beginning of this section. In fact, Vizing [13] has shown that, for any two graphs G and H, $\alpha(G)\alpha(H)+\min\{|V(G)|-\alpha(H), |V(H)|-\alpha(G)\} \leq \alpha(G\square H) \leq \min\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$. Therefore, by using Vizing's upper bound [13] and Lemma 2.10, we have that $k\frac{\binom{m}{n}^2}{\alpha(K(m,n)\square K(m,n))} \geq k\frac{\binom{m}{n}^2}{\alpha(K(m,n)\binom{m}{n}} = k\frac{m}{n} \geq k\lfloor\frac{m}{n}\rfloor$.

An *edge-coloring* of a graph $G = (V, E)$ is an assignment of colors to the edges of G such that any two incident edges are assigned different colors. The smallest number t such that G admits an edge-coloring with t colors is called the *chromatic index* of G and is denoted by $\chi'(G)$. It is well known that the chromatic index of a complete graph K_n on n vertices is equal to $n-1$ if n is even and n if n is odd (see [2]). Besides, in the case n even each color class i (i.e. the subset of pairwise non incident edges colored with color i) has size $\frac{n}{2}$ and if n is odd each color class has size $\frac{n-1}{2}$. Therefore, using this fact, we obtain the following result.

Lemma 2.14. *Let $m \geq 5$. If m is even then the set of vertices of the Kneser graph $K(m,2)$ can be partitioned into $m-1$ disjoint cliques, each one with size $\frac{m}{2}$ and if m is odd then the set of vertices of the Kneser graph $K(m,2)$ can be partitioned into m disjoint cliques, each one with size $\frac{m-1}{2}$.*

Proof. Notice that there is a natural bijection between the vertex set of $K(m,2)$ and the edge set of the complete graph K_m with vertex set $[m]$: each vertex $\{i,j\}$ in $K(m,2)$ is mapped to the edge $\{i,j\}$ in K_m. Now, if m

is even there is a $(m-1)$-edge coloring of K_m where each color class is a set of pairwise non incident edges with size $\frac{m}{2}$ and if m is odd there is a m-edge coloring of K_m where each color class is a set of pairwise non incident edges with size $\frac{m-1}{2}$. Notice that two edges $e, e' \in K_m$ are non incident edges if and only if $e \cap e' = \emptyset$. Therefore, a color class of the edge-coloring of K_m represents a clique of $K(m, 2)$. $\square$

Now, we are able to obtain an upper bound for the stability number of the graph $K(m, 2)\square K(m, 2)$ as follows.

Lemma 2.15. *Let $m \geq 5$. Then,*

- $\alpha(K(m, 2)\square K(m, 2)) \leq \frac{m(m-1)}{8}(3m-2)$ *if m is even and,*
- $\alpha(K(m, 2)\square K(m, 2)) \leq \frac{m(m-1)}{8}(3m-1)$ *if m is odd.*

Proof. Let m even. First, recall that a stable set X in $K(m, 2)$ has size at most $m-1$ if X has center (see Lemma 2.10) and $|X| \leq 1 + (m-1) - (m-2-1) = 3$ if X has no center (see Lemma 2.11). Besides, observe that the vertex set of $K(m, 2)$ can be partitioned in $m-1$ sets $S_1, \ldots, S_{m-1}$ such that each S_i induces a complete subgraph graph $K_{\frac{m}{2}}$ in $K(m, 2)$, for $i = 1, \ldots, m-1$ (see Lemma 2.14). Consider the subgraph H_i of $K(m, 2)\square K(m, 2)$ induced by $S_i \times V(K(m, 2))$ for $i = 1, \ldots, m-1$. Let I be a stable set in $K(m, 2)\square K(m, 2)$ and $I_i = I \cap H_i$ for $i = 1, \ldots, m-1$. Then, for each $v \in S_i$, $I_i^v = I_i \cap (\{v\} \times V(K(m, 2)))$ is a stable set in $K(m, 2)\square K(m, 2)$ for each $i = 1, \ldots, m-1$. Finally, for each $v \in S_i$, with $1 \leq i \leq m-1$, let $I_{i,2}^v$ be the stable set in $K(m, 2)$ such that $I_i^v = \{v\} \times I_{i,2}^v$.

Now, for a fixed $i \in \{1, \ldots, m-1\}$, assume w.l.o.g. that r ($r \leq \frac{m}{2}$) stable sets $I_{i,2}^1, \ldots, I_{i,2}^r$ of $K(m, 2)$ have distinct center $j_1, \ldots, j_r$, respectively (the case when two of these stable sets have the same center can be easily reduced to this case). Let W be the set of subsets with size two of $\{j_1, \ldots, j_r\}$. Therefore, for all $v \in \{1, \ldots, r\}$, $I_i^v - (\{v\} \times W)$ has size at most $m-1-(r-1) = m-r$ since each center j_v belongs to $r-1$ elements in W. Besides, each element of W belongs to exactly one set $I_{i,2}^v$ for $v \in \{1, \ldots, r\}$, since S_i induces a complete subgraph in $K(m, 2)$ and $\{1, \ldots, r\} \subseteq S_i$. Then, $|I_i^1 \cup \ldots \cup I_i^r| \leq (\sum_{v=1}^r |I_i^v - \{v\} \times W|) + |W| \leq r(m-r) + \frac{r(r-1)}{2}$. Next, each remaining stable set (if exist) $I_{i,2}^{r+1}, \ldots, I_{i,2}^{\frac{m}{2}}$ has no center, then $|I_i^d| \leq 3$ for all $d \in \{r+1, \ldots, \frac{m}{2}\}$. Thus, $|I_i| \leq r(m-r) + \frac{r(r-1)}{2} + 3(\frac{m}{2} - r) = -\frac{r^2}{2} + r(m - \frac{7}{2}) + \frac{3}{2}m$. Since the last expression is non decreasing for $r \in \{1, \ldots, \frac{m}{2}\}$, we have that

$$|I_i| \leq -\frac{m^2}{8} + \frac{m}{2}(m - \frac{7}{2}) + 3\frac{m}{2} = \frac{m}{2}(\frac{3}{4}m - \frac{1}{2})$$

Therefore, $|I_i| \leq \frac{m}{2}(\frac{3}{4}m - \frac{1}{2})$ for every $i = 1, \ldots, m-1$. Since $|I| = \sum_{i=1}^{m-1} |I_i|$, it follows that $|I| \leq \frac{m(m-1)}{2}(\frac{3}{4}m - \frac{1}{2})$ and thus,

$$\alpha(K(m,2)\Box K(m,2)) \leq \frac{m(m-1)}{8}(3m-2)$$

We analyze now the case for m odd, with a similar reasoning. First, recall that a stable set X in $K(m,2)$ has size at most $m-1$ if X has center (see Lemma 2.10) and $|X| \leq 1 + (m-1) - (m-2-1) = 3$ if X has no center (see Lemma 2.11). Besides, observe that the vertex set of $K(m,2)$ can be partitioned in m sets $S_1, \ldots, S_m$ such that each S_i induces a complete subgraph $K_{\frac{m-1}{2}}$ in $K(m,2)$, for $i = 1, \ldots, m$ (see Lemma 2.14). Consider the subgraph H_i of $K(m,2)\Box K(m,2)$ induced by $S_i \times V(K(m,2))$ for $i = 1, \ldots, m$. Let I be a stable set in $K(m,2)\Box K(m,2)$ and $I_i = I \cap H_i$ for $i = 1, \ldots, m$. Then, for each $v \in S_i$, $I_i^v = I_i \cap (\{v\} \times V(K(m,2)))$ is a stable set in $K(m,2)\Box K(m,2)$ for each $i = 1, \ldots, m$. Finally, for each $v \in S_i$, with $1 \leq i \leq m$, let $I_{i,2}^v$ be the stable set in $K(m,2)$ such that $I_i^v = \{v\} \times I_{i,2}^v$.

Now, for a fixed $i \in \{1, \ldots, m\}$, assume w.l.o.g. that r ($r \leq \frac{m-1}{2}$) stable sets $I_{i,2}^1, \ldots, I_{i,2}^r$ of $K(m,2)$ have distinct center $j_1, \ldots, j_r$, respectively (the case when two of these stable sets have the same center can be easily reduced to this case). Let W be the set of subsets with size two of $\{j_1, \ldots, j_r\}$. Therefore, for all $v \in \{1, \ldots, r\}$, $I_i^v - (\{v\} \times W)$ has size at most $m - 1 - (r-1) = m - r$ since each center j_v belongs to $r-1$ elements in W. Besides, each element of W belongs to exactly one set I_i^v for $v \in \{1, \ldots, r\}$, since S_i induces a complete subgraph in $K(m,2)$ and $\{1, \ldots, r\} \subseteq S_i$. Then, $|I_i^1 \cup \ldots \cup I_i^r| \leq (\sum_{v=1}^{r} |I_i^v - \{v\} \times W|) + |W| \leq r(m-r) + \frac{r(r-1)}{2}$.

Next, each remaining stable set (if exist) $I_{i,2}^{r+1}, \ldots, I_{i,2}^{\frac{m-1}{2}}$ has no center, then $|I_i^d| \leq 3$ for all $d \in \{r+1, \ldots, \frac{m-1}{2}\}$. Thus, $|I_i| \leq r(m-r) + \frac{r(r-1)}{2} + 3(\frac{m-1}{2} - r) = -\frac{r^2}{2} + r(m - \frac{7}{2}) + \frac{3}{2}(m-1)$. Since the last expression is non decreasing for $r \in \{0, \ldots, \frac{m-1}{2}\}$, we have that

$$|I_i| \leq -\frac{(m-1)^2}{8} + \frac{m-1}{2}(m - \frac{7}{2}) + \frac{3}{2}(m-1) = \frac{m-1}{2}(\frac{3}{4}m - \frac{1}{4})$$

Therefore, $|I_i| \leq \frac{m-1}{2}(\frac{3}{4}m - \frac{1}{4})$ for every $i = 1, \ldots, m$. Since $|I| = \sum_{i=1}^{m} |I_i|$, it follows that $|I| \leq \frac{m(m-1)}{2}(\frac{3}{4}m - \frac{1}{4})$ and thus,

$$\alpha(K(m,2)\square K(m,2)) \leq \frac{m(m-1)}{8}(3m-1)$$

$\square$

From Lemmas 2.13 and 2.15 we have the following result.

Theorem 2.16. *Let $m \geq 5$. Then,*

- $\chi_k(K(m,2)\square K(m,2)) \geq 2k\frac{m(m-1)}{3m-2}$ *if m is even and,*
- $\chi_k(K(m,2)\square K(m,2)) \geq 2k\frac{m(m-1)}{3m-1}$ *if m is odd.*

In the particular case when $m = 2q+4$, with $q > 0$, and $k = 2$, we have, by Lemma 2.6 and Theorem 2.16, the following result that shows that the difference $\chi_2(K(2q+4,2)\square K(2q+4,2)) - \chi_2(K(2q+4,2))$ can be as large as desired.

Corollary 2.17. *For any integer $q > 0$ and for $k = 2$, we have that,*

$$\chi_2(K(2q+4,2)\square K(2q+4,2)) \geq 2q + \left\lceil \frac{2}{3}q \right\rceil + 5 = \chi_2(K(2q+4,2)) + \left\lceil \frac{2}{3}q \right\rceil + 1.$$

From Lemmas 2.5 and 2.6, Corollary 2.2, and Theorem 2.16, we obtain the results that we summarize in Table 1.

Finally, by applying some known homomorphisms between Kneser graphs, we obtain the following result.

Theorem 2.18. *Let $k > n$ and let $t = \chi_k(K(m,n)\square K(m,n))$, where $m > 2n$. Then, either $t > m + 2(k-n)$ or $t < m + (k-n)$.*

Proof. Suppose that $m + (k-n) \leq t \leq m + 2(k-n)$. Therefore, there exists an homomorphism $K(m,n)\square K(m,n) \to K(t,k)$. Now, Stahl [11] showed that there is an homomorphism $K(m,n) \to K(m-2,n-1)$ whenever $n > 1$ and $m \geq 2n$. Moreover, it is easy to see that there is an homomorphism $K(m,n) \to K(m-1,n-1)$. By applying the former homomorphism $t-(m+(k-n))$ times to the graph $K(t,k)$ we obtain an homomorphism $K(t,k) \to K(2(m+k-n)-t, 2k+m-n-t)$. Finally, by applying $2k+m-t-2n$ times the latter homomorphism to the graph $K(2(m+k-n)-t, 2k+m-n-t)$ we obtain an homomorphism $K(2(m+k-n)-t, 2k+m-n-t) \to K(m,n)$. Therefore, by homomorphism composition, $K(m,n)\square K(m,n) \to K(m,n)$ which contradicts Lemma 2.3. $\square$

G	k	$\chi_k(G)$ [11]	$\chi_k(G\square G) =$
$K(5,2)$	2	5	6
	3	8	9
	4	10	12
	5	13	15
	6	15	18
	7	18	$20-21$
$K(6,2)$	2	6	8
	3	?	12
	4	12	$15-16$
	5	?	$19-20$
	6	18	$23-24$
$K(7,2)$	2	7	$9-10$
	3	?	$13-15$
	4	14	$17-20$
$K(8,2)$	2	8	$11-12$
	3	?	$16-18$
	4	16	$21-24$

Table 1: Summary of results

3. Cases where $\chi_k(G\square H) = \max\{\chi_k(G), \chi_k(H)\}$

Theorem 3.1. *Let G and H be graphs such that $\chi(G) \leq \chi(H) = \omega(H)$. Then, $\chi_k(G\square H) = \max\{\chi_k(G), \chi_k(H)\} = \chi_k(H) = k\omega(H)$.*

Proof. Let $t = \omega(H)$ and let $\{h_1, \ldots, h_t\}$ be the vertex set of a maximum clique K_t in H with size t. Clearly, $\chi_k(G) \leq \chi_k(H) = \chi_k(K_t)$. Let ρ be a k-tuple coloring of H with $\chi_k(H)$ colors. By equality (1), there exists a t-coloring f of $G\square H$. Therefore, the assignment of the k-set $\rho(h_{f((a,b))})$ to each vertex (a,b) in $G\square H$ defines a k-tuple coloring of $G\square H$ with $\chi_k(K_t)$ colors. $\square$

Notice that if G and H are both bipartite, then $\chi_k(G\square H) = \chi_k(G) = \chi_k(H)$. In the case when G is not a bipartite graph, we have the following results.

An *automorphism* σ of a graph G is called a *shift* of G if $\{u, \sigma(u)\} \in E(G)$ for each $u \in V(G)$ [8]. In other words, a shift of G maps every vertex to one of its neighbors.

Theorem 3.2. *Let G be a non bipartite graph having a shift $\sigma \in AUT(G)$,* *and let H be a bipartite graph. Then,* $\chi_k(G\Box H) = \max\{\chi_k(G), \chi_k(H)\} = \chi_k(G)$.

Proof. Let $A \cup B$ be a bipartition of the vertex set of H. Let f be a k-tuple coloring of G with $\chi_k(G)$ colors. Clearly, $\chi_k(G) \geq \chi_k(H)$. We define a k-tuple coloring ρ of $G\Box H$ with $\chi_k(G)$ colors as follows: for any vertex (u,v) of $G\Box H$ with $u \in G$ and $v \in H$, define $\rho((u,v)) = f(u)$ if $v \in A$, and $\rho((u,v)) = f(\sigma(u))$ if $v \in B$. $\square$

We may also deduce the following direct result.

Theorem 3.3. *Let G be an hom-idempotent graph and let H be a subgraph of G. Thus,* $\chi_k(G\Box H) = \max\{\chi_k(G), \chi_k(H)\} = \chi_k(G)$.

Let A be a group and S a subset of A that is closed under inverses and does not contain the identity. The *Cayley graph* $Cay(A,S)$ is the graph whose vertex set is A, two vertices u, v being joined by an edge if $u^{-1}v \in S$. If $a^{-1}Sa = S$ for all $a \in A$, then $Cay(A,S)$ is called a *normal Cayley graph*.

Lemma 3.4 ([6])**.** *Any normal Cayley graph is hom-idempotent.*

Note that all Cayley graphs on Abelian groups are normal, and thus hom-idempotent. In particular, the *circulant* graphs are Cayley graphs on cyclic groups (i.e., cycles, powers of cycles, complements of powers of cycles, complete graphs, etc). By Theorem 3.3 and Lemma 3.4 we have the following result.

Theorem 3.5. *Let $Cay(A,S)$ be a normal Cayley graph and let $Cay(A',S')$ be a subgraph of $Cay(A,S)$, with $A' \subseteq A$ and $S' \subseteq S$. Then,*
$\chi_k(Cay(A,S)\Box Cay(A',S')) = \max\{\chi_k(Cay(A,S)), \chi_k(Cay(A',S'))\}$.

Definition 3.6. *Let G be a graph with a shift σ. We define the* order *of σ as the minimum integer i such that σ^i is equal to the identity permutation.*

Theorem 3.7. *Let G be a graph with a shift σ of minimum odd order $2s+1$ and let C_{2t+1} be a cycle graph, where $t \geq s$. Then,*
$\chi_k(G\Box C_{2t+1}) = \max\{\chi_k(G), \chi_k(C_{2t+1})\}$.

Proof. Let $\{0, \ldots, 2t\}$ be the vertex set of C_{2t+1}, where for $0 \leq i \leq 2t$, $\{i, i+1 \mod (2t+1)\} \in E(C_{2t+1})$. Let G_i be the i^{th} copy of G in $G\Box C_{2t+1}$, that is, for each $0 \leq i \leq 2t$, $G_i = \{(g,i) : g \in G\}$. Let f be a k-tuple coloring of G with $\chi_k(G)$ colors. We define a k-tuple coloring of $G\Box C_{2t+1}$ with $\chi_k(G)$ colors as follows: let σ^0 denotes the identity permutation of the vertices in G. Now, for $0 \leq i \leq 2s$, assign to each vertex $(u, i) \in G_i$ the k-tuple $f(\sigma^i(u))$. For $2s+1 \leq j \leq 2t$, assign to each vertex $(u, j) \in G_j$ the k-tuple $f(u)$ if j is odd, otherwise, assign to (u, j) the k-tuple $f(\sigma^1(u))$. It is not difficult to see that this is in fact a proper k-tuple coloring of $G \Box C_{2t+1}$. $\Box$

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