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QUASI-SEMI-HOMOMORPHISMS AND GENERALIZED PROXIMITY RELATIONS BETWEEN BOOLEAN ALGEBRAS

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Abstract. In this paper we shall define the notion of quasi-semi-homomorphisms between Boolean algebras, as a generalization of the quasi-modal operators introduced in [3], of the notion of meet-homomorphism studied in [12] and [11], and the notion of precontact or proximity relation defined in [8]. We will prove that the class of Boolean algebras with quasi-semi-homomorphism is a category, denoted by **BoQS**. We shall prove that this category is equivalent to the category **StQB** of Stone spaces where the morphisms are binary relations, called quasi-Boolean relations, satisfying additional conditions. This duality extends the duality for meet-homomorphism given by P. R. Halmos in [12] and the duality for quasi-modal operators proved in [3].

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1. INTRODUCTION

Recall that a *modal algebra* is a Boolean algebra A with an operator $\Box : A \to A$ such that $\Box 1 = 1$, and $\Box(a \wedge b) = \Box a \wedge \Box b$, for all $a, b \in A$. It is well known that the variety of modal algebras is the algebraic semantic of normal modal logics [10, 16]. Modal algebras are dual objects of descriptive general frames, also called modal spaces, i.e., Stone spaces with a relation verifying certain conditions (see [10], and [16]). P. R. Halmos define in [12] the notion of meet-homomorphism (or hemihomomorphism) between Boolean algebras. Recall that a meet-homomorphism between two Boolean algebras A and B, is a function $h : A \to B$ such that $h(1) = 1$, and $h(a \wedge b) = h(a) \wedge h(b)$, for all $a, b \in A$. If $A = B$, then h is a modal operator [10,16]. Let X and Y be the Stone spaces of A and B, respectively. As it follows from [12] and [11], a meet-homomorphism $h : A \to B$ is dually characterized by means of a relation $R \subseteq Y \times X$ such that $R(y)$ is a closed subset of X, for each $y \in Y$, and $h_R(U) = \{y \in Y : R(y) \subseteq U\}$ is a clopen subset of Y, for each clopen $U \subseteq X$. These relations are called Boolean relations in [12], or Boolean correspondences in [11] (see also [16]).

In [3], the notions of quasi-modal operator and quasi-modal algebra were introduced as a generalization of the notion of modal operator and modal algebra, respectively. A quasi-modal operator in a Boolean algebra A is a map Δ that sends each element $a \in A$ to an ideal Δa of A, and satisfies analogous conditions with the modal operator $\Box$ of modal algebras. A quasi-modal algebra is a pair $\langle A, \Delta \rangle$ where A is a Boolean algebra and Δ is a quasi-modal operator. We note that a quasi-modal operator is not an operation, but has many similar properties to modal operators.

In this paper we shall introduce maps between a Boolean algebra A and the set of all ideals of another Boolean algebra B satisfying analogous conditions with the meet-homomorphism between Boolean algebras [12]. We call these maps *quasi-semi-homomorphisms*. One of the main objectives of this paper is to study this class of maps, and their topological representation.

As we will explain below, the quasi-modal operators are closely connected with the proximity or precontact relations defined between Boolean algebras. We recall that a proximity relation defined on a set X is a binary relation $\delta \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$ satisfying certains conditions (see Definition 2). If $U, V \in \mathcal{P}(X)$, then the intuitive meaning of a proximity relation δ is that $U \delta V$ holds, when U is close to V in some sense. A proximity or precontact space, also called a nearness space, is a pair $\langle X, \delta \rangle$, where X is a set and δ is a proximity relation. Since $\mathcal{P}(X)$ is a Boolean algebra, we can introduced an abstract definition of proximity relation in the class of Boolean algebras (see [15] and [4]). In the literature, there exist many classes of Boolean algebras endowed with some type of proximity relations. As examples, we can mention the Boolean contact algebras defined in [9], or the Boolean connection algebras defined in [17]. For other versions of Boolean algebras endowed relations see [5], [8], [7], [19], and [18]. In [8] the notions of *proximity relation* on a Boolean algebra and the *proximity Boolean algebras* were defined as an abstract version of proximity spaces [15]. This class of structures is the most general class of Boolean algebras endowed with a proximity relation. We note that the notion of proximity Boolean algebras is equivalent to the notion of precontact algebras [8].

There exists a strong connection between proximity relations defined in a Boolean algebra and quasi-modal operator. Given a proximity relation δ in a Boolean algebra A, we can prove that the set $\Delta_\delta b = \{a \in A : (a, \neg b) \notin \delta\}$ is an ideal of A. So, we have a map Δ_δ that send elements to ideals of the algebra A. As we shall see, this map is a quasi-modal operator. Conversely, if we have a quasi-modal operator Δ defined in a Boolean algebra A, then the relation $a \delta_\Delta b$ defined by $a \notin \Delta \neg b$, is a proximity relation on A (for the details see Theorem 1). Thus, we have that the notions of proximity relation and quasi-modal operator are interdefinable. Moreover, since the notion of quasi-semi-homomorphism is a generalization of the notion of quasi-modal operator, and this last is equivalent to the notion of proximity relation, we will get that it is possible to introduce a generalization of the notion of proximity relation.

The paper is organized as follows. In Section 2 we start recalling some basic definitions and results on Stone duality for Boolean algebras. In Section 3 we shall introduce the notion of quasi-semi-homomorphism and the notion of generalized proximity relation. Also, we shall prove that the notions of quasi-semi-homomorphism and generalized proximity relation are equivalent, and as consequence of this fact, we have that the notions of quasi-modal operator and proximity relation are equivalent. This fact has strong consequences, because it puts the proximity relations very close to the modal operators. We shall see that the class of Boolean algebras with the quasi-semi-homomorphism form a category denoted by **BoQS**. In Section 4, we shall introduce the notion of generalized quasi-Boolean relation between Stone spaces, and we shall prove some propierties. We shall prove that the class of Stone spaces with the generalized quasi-Boolean relations form a category, simbolized by **StQB**. In Section 5 we shall prove that the categories **StQB** and **BoQS** are dually equivalent. As an application of this duality we will prove a generalization of the result that assert that the Boolean homomorphisms are the minimal elements in the set of all join-homomorphisms between two Boolean algebras (see [11]). In this last section we prove that the minimal elements in the set of all quasi-Boolean relations defined between two Stone spaces is a Boolean relation.

2. Preliminaries

We assume that the reader is familiar with basic concepts of Boolean algebras and topological duality (see [1] or [13]).

We recall that a subset of a topological space X is *clopen* if it is both closed and open, and that X is *zero-dimensional* if the set of clopen subsets of X forms a basis for the topology. We shall denote by $\mathcal{O}(X)$ ($\mathcal{C}(X)$) the set of all open subsets (closed subsets) of X. The closure of a subset Z is denoted by $\mathrm{cl}(Z)$. We shall denote by $\mathrm{Clo}(X)$ the set of all clopen subsets of X. Clearly the notions of Hausdorff and T_0 coincide in the realm of zero-dimensional spaces. A *Stone space* X is *zero-dimensional, compact and Hausdorff* topological space. We note that a Stone space is *totally disconnected*, i.e., given distinct points $x, y \in X$, there is $U \in \mathrm{Clo}(X)$ of X such that $x \in U$ and $y \notin U$. If X is a Stone space, then $\mathrm{Clo}(X)$ is a Boolean algebra under the set theoretical operations.

If $A = \langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra, by $\mathrm{Ul}(A)$ we shall denote the set of all ultrafilters (or proper maximal filters) of A while by $\mathrm{Id}(A)$ and $\mathrm{Fi}(A)$ we shall denote the families of all ideals and filters of A, respectively.

Let X be a Stone space. The map $\varepsilon_X : X \to \mathrm{Ul}(\mathrm{Clo}(X))$ given by

$$\varepsilon_X(x) = \{U \in \mathrm{Clo}(X) : x \in U\}$$

is a bijective and continuous function. Let A be a Boolean algebra and let

$$\beta_A : A \to \mathcal{P}(\mathrm{Ul}(A))$$

the Stone map defined by $\beta_A(a) = \{P \in \mathrm{Ul}(A) : a \in P\}$. Sometimes we will write β instead of β_A. With each Boolean algebra A we can associate a Stone space $\langle \mathrm{Ul}(A), \tau_A \rangle$ whose points are the elements of $\mathrm{Ul}(A)$ and the topology τ_A is determined by the clopen basis $\beta[A] = \{\beta_A(a) : a \in A\}$. If misunderstanding is excluded, we write $\mathrm{Ul}(A)$ instead of $\langle \mathrm{Ul}(A), \tau_A \rangle$. Thus, if X is a Stone space, then $X \cong \mathrm{Ul}(\mathrm{Clo}(X))$, and if A is a Boolean algebra, then $A \cong \mathrm{Clo}(\mathrm{Ul}(A))$.

If A is a Boolean algebra, then there exists a duality between ideals (filters) of A and open (closed) sets of $\mathrm{Ul}(A)$. More precisely, for $I \in \mathrm{Id}(A)$ and $F \in \mathrm{Fi}(A)$. The value of the function $\varphi_A[I] = \{P \in \mathrm{Ul}(A) : I \cap P \neq \varnothing\}$ is an open of $\mathrm{Ul}(A)$, and thus φ_A is an one-to-one mapping between $\mathrm{Id}(A)$ and the set of $\mathcal{O}(\mathrm{Ul}(A))$ of all open subset of $\mathrm{Ul}(A)$. The function ψ_A defined by $\psi_A[F] = \{P \in \mathrm{Ul}(A) : F \subseteq P\}$, is a one-to-one mapping between $\mathrm{Fi}(A)$ and the set $\mathcal{C}(\mathrm{Ul}(A))$ of all closed subset of $\mathrm{Ul}(A)$. We note that $\varphi_A[I] = \bigcup\{\beta(a) : a \in I\}$. If we denote by $\barwedge$ and by $\veebar$ the meet and the join in the set $\mathrm{Id}(A)$, respectively, then $\varphi_A[I_1 \veebar I_2] = \varphi_A[I_1] \cup \varphi_A[I_2]$, and $\varphi_A[I_1 \barwedge I_2] = \varphi_A[I_1] \cap \varphi_A[I_2]$ (see [13] and [16] for further information on Boolean duality).

Let A be a Boolean algebra. The filter (ideal) generated by a subset $Y \subseteq A$ is denoted by $F(Y)$ $(I(Y))$. If $Y = \{a\}$, then we write $F(a) = [a)$ $(I(a) = (a])$. The set complement of a subset $Y \subseteq A$ will be denoted by Y^c or $A - Y$.

3. Quasi-semi-homomorphisms

In this section we introduce the main notion of this paper. We define the notion of quasi-semi-homomorphim as a generalization of the notion of quasi-modal operator [2,3] and the notion of semi-homomorphism between Boolean algebras [11,12].

Definition 1. Let A and B be two Boolean algebras. A *quasi-semi-homomorphism* is a function $\Delta : A \to \mathrm{Id}(B)$ such that it verifies the following conditions for all $a, b \in A$:

Q1 $\Delta(a \wedge b) = \Delta a \cap \Delta b$,
Q2 $\Delta 1 = B$.

In the following $QS[A, B]$ stands for the set of all quasi-semi-homomorphism defined between A and B. If $\Delta_1, \Delta_2 \in QS[A, B]$ we define $\Delta_1 \leq \Delta_2$ by $\Delta_1(a) \subseteq \Delta_2(a)$, for all $a \in A$. This gives an order relation in $QS[A, B]$. We note that when $A = B$, the elements of $QS[A, A] = QS[A]$ are called quasi-modal operators in [3]. A pair $\langle A, \Delta \rangle$, where $\Delta \in QS[A]$ is called a *quasi-modal algebra.*

If $\Delta \in QS[A, B]$, then Δ is monotonic, because if $a \leq b$, then $a = a \wedge b$, and so $\Delta a = \Delta(a \wedge b) = \Delta a \cap \Delta b$, i.e., $\Delta a \subseteq \Delta b$.

Example 1. Let A be a Boolean algebra. The map $I_A : A \to \mathrm{Id}(A)$ given by $I_A(a) = (a]$, for each $a \in A$, is clearly a quasi-semi-homomorphism.

Example 2. Let A and B two Boolean algebras. We recall first that a meet-hemimorphisms or *meet-homomorphism* [11] [12], is a function $h : A \to B$ such that

$h(1) = 1$, and $h(a \wedge b) = h(a) \wedge h(b)$, for all $a, b \in A$. The function h can be extended to a map $\Delta_h : A \to \mathrm{Id}(B)$ of the following form. Put $\Delta_h(a) = (h(a)]$, for each $a \in A$. It is clear that Δ_h verifies the equalities $\Delta_h(a \wedge b) = \Delta_h(a) \cap \Delta_h(b)$ and $\Delta_h(1) = B$. Thus, Δ_h is a quasi-semi-homomorphism. An element $\Delta \in QS[A, B]$ is called a *principal* quasi-semi-homomorphisms if Δa is principal ideal, for each $a \in A$. In other words, for each $a \in A$, there exists $b \in B$ suh that $\Delta a = (b]$. it is clear that if Δ is principal, then the map $h_\Delta : A \to B$ defined by $h(a) = b$ iff $\Delta a = (b]$ is a meet-hemimorphisms. Thus, the class of principal quasi-semi-homomorphisms is equivalent to the class of meet-homomorphisms.

Recall that when $A = B$ and $\Box : A \to A$ is a meet-homomorphism, the pair $\langle A, \Box \rangle$ is called a modal algebra [16]. So, the class of modal algebras can be identified with the class of pairs $\langle A, \Delta \rangle$ where A is a Boolean algebra and Δ is a principal quasi-semi-homomorphism.

The following example is fundamental in the representation theory of quasi-semi-homomorphisms.

Example 3. Let X and Y be two set. Let R be a relation between X and Y. Define a function $\bar{\Delta}_R : \mathcal{P}(Y) \to \mathrm{Id}(\mathcal{P}(X))$, as $\bar{\Delta}_R(U) = (\Delta_R(U)]$, where $\Delta_R(U) = \{x \in X : R(x) \subseteq U\}$, with $U \in \mathcal{P}(Y)$. Then it is easy to see that $\bar{\Delta}_R \in QS[\mathcal{P}(Y), \mathcal{P}(X)]$.

Let A and B be two Boolean algebras. For each $\Delta \in QS[A, B]$, we define the dual quasi-semi-homomorphism $\nabla : A \to \mathrm{Fi}(B)$ by $\nabla a = \neg\Delta\neg a$, where $\neg\Delta x = \{\neg y : y \in \Delta x\}$. We note that $c \in \nabla(a \vee b) = \neg\Delta\neg(a \vee b) = \neg\Delta(\neg a \wedge \neg b)$ iff $\neg c \in \Delta(\neg a \wedge \neg b) = \Delta\neg a \cap \Delta\neg b$ iff $\neg c \in \Delta\neg a$ and $\neg c \in \Delta\neg b$ iff $c \in \nabla a$ and $c \in \nabla b$ iff $c \in \nabla a \cap \nabla b$. Thus the map ∇ verifies the following conditions:

Q3 $\nabla(a \vee b) = \nabla a \cap \nabla b$,
Q4 $\nabla 0 = B$.

Now we introduce a notion that generalizes the notion of proximity relation (also called precontact relation) defined in a Boolean algebra [8] [7] [14].

Definition 2. Let A and B be two Boolean algebras. A *generalized precontact or generalized proximity relation* between A and B is a relation $\delta \subseteq A \times B$ such that

P1 If $a\delta b$, then $a \neq 0$ and $b \neq 0$.
P2 $a\delta(b \vee c)$ iff $a\delta b$ or $a\delta c$.
P3 $(a \vee b)\delta c$ iff $a\delta c$ or $b\delta c$.

When $A = B$, a generalized precontact relation δ is called a *proximity or precontact* relation, and the pair $\langle A, \delta \rangle$ is called a *proximity* or *precontact algebra* [6–8]. An important example of proximity relations are the proximity spaces. There are many other notions of proximity, and we suggest the reader consults the fundamental text by Naimpally and Warrack [15] for more examples, or the paper [18].

Theorem 1. *Let A and B be two Boolean algebras. There exists a bijective correspondence between quasi-semi-homomorphisms between A and B and the generalized proximity relations defined between A and B.*

Proof. Let A and B be two Boolean algebras. If $\delta \subseteq A \times B$ is a generalized proximity relation, then we prove that the subset of A

$$\Delta_\delta b = \{a \in A : (a, \neg b) \notin \delta\},$$

is an ideal of A. Let $b \in B$. As $(0, \neg b) \notin \delta$, we have that $0 \in \Delta_\delta b$. Let $a_1, a_2 \in A$. Suppose that $a_1 \leq a_2$ and $a_2 \in \Delta_\delta b$. Then $(a_2, \neg b) \notin \delta$. As $a_2 = a_1 \vee a_2$, by condition P3 of Definition 2 we have that $(a_2, \neg b) = (a_1 \vee a_2, \neg b) \notin \delta$ iff $(a_1, \neg b) \notin \delta$ and $(a_2, \neg b) \notin \delta$. Thus, $(a_1, \neg b) \notin \delta$, i.e., $a_1 \in \Delta_\delta b$. Suppose that $a_1 \in \Delta_\delta b$ and $a_2 \in \Delta_\delta b$. Then, $(a_1, \neg b) \notin \delta$ and $(a_2, \neg b) \notin \delta$. Again, by condition P3 of Definition 2 we have that $(a_1 \vee a_2, \neg b) \notin \delta$, i.e., $a_1 \vee a_2 \in \Delta_\delta b$. Thus, $\Delta_\delta b \in \mathrm{Id}(A)$, for each $b \in B$. Then the map $\Delta_\delta : B \to \mathrm{Id}(A)$ is well defined.

Let $b_1, b_2 \in B$ and $a \in A$. Then by condition P2 of Definition 2 we have the following equivalences:

$$\begin{array}{rll} a \in \Delta_\delta (b_1 \wedge b_2) & \text{iff} & (a, \neg(b_1 \wedge b_2)) = (a, \neg b_1 \vee \neg b_2) \notin \delta \\ & \text{iff} & (a, \neg b_1) \notin \delta \text{ and } (a, \neg b_2) \notin \delta \\ & \text{iff} & a \in \Delta_\delta b_1 \cap \Delta_\delta b_2. \end{array}$$

By condition P1 of Definition 2 we have that $(b, \neg 1) = (b, 0) \notin \delta$, for all $b \in B$. Thus, $1 \in \Delta_\delta(b)$, for all $b \in B$.

Conversely. Let $\Delta : A \to \mathrm{Id}(B)$ be a quasi-semi-homomorphism. Define the relation

$$\delta_\Delta = \{(a, b) \in A \times B : a \notin \Delta\neg b\}.$$

Let $(a, b) \in \delta_\Delta$. If $a = 0$, then $0 \notin \Delta\neg b$, which is a contradiction because $\Delta\neg b$ is an ideal. If $b = 0$, then $a \notin \Delta\neg 0 = \Delta 1 = A$, which is a contradiction. Thus, $a \neq 0$ and $b \neq 0$.

Let $a, b \in A$, and $c \in B$. Taking into account that $\Delta\neg c$ is and ideal of B, we get the following equivalences: $(a \vee b, c) \in \delta_\Delta$ iff $a \vee b \notin \Delta\neg c$ iff $a \notin \Delta\neg c$ or $b \notin \Delta\neg c$ iff $(a, c) \in \delta_\Delta$or $(b, c) \in \delta_\Delta$.

Let $a \in A$ and $b, c \in B$. Then $(a, b \vee c) \in \delta_\Delta$ iff

$$a \notin \Delta\neg(b \vee c) = \Delta(\neg b \wedge \neg c) = \Delta\neg b \cap \Delta\neg c$$

iff $a \notin \Delta\neg b$ or $a \notin \Delta\neg c$ iff $(a, b) \in \delta_\Delta$ or $(a, c) \in \delta_\Delta$. Thus, $\delta_\Delta$is a generalized proximity relation between A and B. $\square$

By Theorem 1 we have that the notions of proximity relations and quasi-modal operators are interdefinable.

Definition 3. Let $\Delta \in QS[A, B]$. For each $C \subseteq A$ and for each $D \subseteq B$ define

(1) $\Delta C = I(\bigcup_{c \in C} \Delta c)$,

(2) $\nabla C = F(\bigcup_{c\in C} \nabla c)$,
(3) $\Delta^{-1}(D) = \{a \in A : \Delta a \cap D \neq \varnothing\}$,
(4) $\nabla^{-1}(D) = \{a \in A : \nabla a \subseteq D\}$.
(5) If $D = [a)$, we write $\Delta^{-1}(a)$ instead of $\Delta^{-1}([a))$.

In the following lemma we summarize some properties well known in the theory of Boolean algebras with proximity relations (see [8, 14, 19]). For completeness we will give some proofs.

Lemma 1. *Let $\Delta \in QS[A,B]$.*

(1) Then $\Delta^{-1}(F) = \bigcup_{a\in F} \Delta^{-1}(a) \in \mathrm{Fi}(A)$ for each $F \in \mathrm{Fi}(B)$. Moreover, this union is directed.
(2) If $P \in \mathrm{Ul}(B)$, then $\nabla^{-1}(P)^c \in \mathrm{Id}(A)$.
(3) $\Delta I = \bigcup_{a\in I} \Delta a$ for each $I \in \mathrm{Id}(A)$. Moreover, this union is directed.
(4) $\Delta(I_1 \cap I_2) = \Delta(I_1) \cap \Delta(I_2)$ for all $I_1, I_2 \in \mathrm{Id}(A)$.

Proof. (1) Let $F \in \mathrm{Fi}(B)$. It is easy to see $\Delta^{-1}(F) \in \mathrm{Fi}(A)$. Let $a \in A$. Then

$$\begin{aligned} \Delta a \cap F \neq \varnothing \quad &\text{iff} \quad \exists b \in F\,(b \in \Delta a) \\ &\text{iff} \quad \exists b \in F\,([b) \cap \Delta a \neq \varnothing) \\ &\text{iff} \quad \exists b \in F\,(a \in \Delta^{-1}([b)) = \Delta^{-1}(b)) \\ &\text{iff} \quad \exists b \in F\,(a \in \textstyle\bigcup_{b\in F} \Delta^{-1}(b)). \end{aligned}$$

In order to see that this union is directed suppose that $a, b \in F$. Then it is easy to see that $\Delta^{-1}(a) \cup \Delta^{-1}(b) \subseteq \Delta^{-1}(a \vee b)$, and as $a \vee b \in F$, we get that this union is directed.

(2) We prove that $\nabla^{-1}(P)^c \in \mathrm{Id}(A)$, when $P \in \mathrm{Ul}(B)$. Let $a \leq b$ and $a \in \nabla^{-1}(P)^c$. Then $\nabla a \nsubseteq P$, and as $\nabla b \subseteq \nabla a$, because ∇ is anti-monotonic, we have that $b \notin \nabla^{-1}(P)^c$. Thus $\nabla^{-1}(P)^c$ is decreasing. Let $a, b \in \nabla^{-1}(F)^c$. Then $\nabla a \nsubseteq P$ and $\nabla b \nsubseteq P$. Then there exist $p_1 \in \nabla a - P$ and $p_2 \in \nabla b - P$. So, $p_1 \vee p_2 \in \nabla a \vee \nabla b$, and as P is prime, $p_1 \vee p_2 \notin P$. Then, $p_1 \vee p_2 \in \nabla^{-1}(P)^c$. It is clear that $0 \in \nabla^{-1}(P)^c$, because $\nabla 0 = B$. Thus, $\nabla^{-1}(P)^c$ is an ideal of A.

(3) Let $I \in \mathrm{Id}(A)$. We prove that $I(\bigcup\limits_{c\in C} \Delta c) = \bigcup_{a\in I} \Delta a$. It is clear that $\bigcup_{a\in I} \Delta a \subseteq I(\bigcup\limits_{c\in C} \Delta c)$. Let $c \in I(\bigcup\limits_{c\in C} \Delta c)$. Then there exists $a_i \in I$, and there exists $x_i \in \Delta a_i$, with $1 \leq i \leq n$, such that $c \leq x_1 \vee \ldots \vee x_n$. Since $\Delta a_i \subseteq \Delta(a_1 \ldots \vee a_n)$, for $1 \leq i \leq n$, then $x_1 \vee \ldots \vee x_n \in \Delta(a_1 \vee \ldots \vee a_n)$. As $a_1 \vee \ldots \vee a_n \in I$, and $c \in \Delta(a_1 \vee \ldots \vee a_n)$, we get that $c \in \bigcup_{a\in I} \Delta a$. Thus the union is directed.

(4). Let $I_1, I_2 \in \mathrm{Id}(A)$. As Δ is monotonic, $\Delta(I_1 \cap I_2) \subseteq \Delta(I_1) \cap \Delta(I_2)$. Let $c \in \Delta(I_1) \cap \Delta(I_2)$. Then by item (3), there exist $a \in I_1$ and $b \in I_2$ such that $c \in \Delta a \cap \Delta b = \Delta(a \wedge b)$. As $a \wedge b \in I_1 \cap I_2$, we get that $c \in \Delta(I_1 \cap I_2)$. $\square$

Let A, B and C be Boolean algebras. Let $\Delta_1 \in QS[A,B]$ and $\Delta_2 \in QS[B,C]$. We define the composition of Δ_2 with Δ_1. Recall that for each subset D of B, we can consider an ideal $\Delta_2(D) = \bigvee\{\Delta_2 b : b \in D\}$. Then, as for each $a \in A$, we consider the ideal $\Delta_2[\Delta_1(a)] \in \mathrm{Id}(C)$. Then, define the composition of Δ_2 with Δ_1, in symbols $\Delta_2 \circ \Delta_1$, as

$$(\Delta_2 \circ \Delta_1)(a) = \Delta_2[\Delta_1(a)],$$

for each $a \in A$. We need to prove that $\Delta_2 \circ \Delta_1 \in QS[A,C]$. In the following result we use the quasi-semi-homomorphism defined in Example 1.

Lemma 2. *Let A, B and C be Boolean algebras. Let $\Delta_1 \in QS[A,B]$ and $\Delta_2 \in QS[B,C]$. Then:*

(1) $\Delta_2 \circ \Delta_1 \in QS[A,C]$.
(2) $\Delta_1 \circ I_A = \Delta_1$ and $I_B \circ \Delta_2 = \Delta_2$.

Proof. (1) By (4) of Lemma 1 we get that

$$\begin{array}{rclcl}(\Delta_2 \circ \Delta_1)(a \wedge b) & = & \Delta_2[\Delta_1(a \wedge b)] & = & \Delta_2[\Delta_1(a) \cap \Delta_1(b)] \\ & = & \Delta_2[\Delta_1 a] \cap \Delta_2[\Delta_1 b] & = & (\Delta_2 \circ \Delta_1)(a) \cap (\Delta_2 \circ \Delta_1)(b).\end{array}$$

Moreover, $(\Delta_2 \circ \Delta_1)(1) = \Delta_2[\Delta_1 1] = \Delta_2[B] = A$. Thus, $\Delta_2 \circ \Delta_1$ is a quasi-semi-homomorphism.

(2) Let $a \in A$. Then $(\Delta_1 \circ I_A)(a) = \Delta_1[I_A a] = \Delta_1[(a]] = \Delta_1 a$. The proof of the identity $I_B \circ \Delta_2 = \Delta_2$ is similar. □

Thus we can conclude that we have a category, denoted by **BoQS**, whose objects are Boolean algebras and whose morphism are quasi-semi-homomorphisms. In the next section we will prove that the category **BoQS** is dually equivalent to a category whose objects are Stone spaces, and whose morphism are a particular class of binary relations between Stone spaces.

In the following result we will characterize the isomorphisms (or iso-arrow) in the category **BoQS**. This result will be needed later.

Lemma 3. *Let A and B be Boolean algebras and $\Delta \in QS[A,B]$. Then the following conditions are equivalent:*

(1) Δ is an iso-arrow in the category **BoQS**.
(2) There exists an one to one and onto function $h : A \to B$ such that $\Delta a = (h(a)]$, for each $a \in A$.

Proof. (1) $\Rightarrow$ (2) Since Δ is an iso-arrow in the category **BoQS**, there exists $\Pi \in QS[B,A]$ such that $\Delta \circ \Pi = I_B$ and $\Pi \circ \Delta = I_A$, where I_A and I_B are the quasi-semi-homomorphisms defined in Example 1. Let $a \in A$. Then $(\Pi \circ \Delta)(a) = I_A(a) = (a]$. As $(\Pi \circ \Delta)(a) = \Pi[\Delta a] = \bigcup\{\Pi b : b \in \Delta a\}$, there exists $b \in \Delta a$ such that $\Pi b = (a]$. We prove that b is unique. Suppose that there are $b_1, b_2 \in B$ such that $\Pi b_1 = \Pi b_2$. As $\Delta \circ \Pi = I_B$, we get

$$(b_1] = (\Delta \circ \Pi)(b_1) = \Delta[\Pi b_1] = \Delta[\Pi b_2] = (\Delta \circ \Pi)(b_2) = (b_2].$$

So, $b_1 = b_2$. Then for each $a \in A$ there exists a unique $b \in B$ such that $\Pi b = (a]$. So, we can consider the function $h : A \to B$ defined by:

$$h(a) = b \text{ iff } \Pi b = (a],$$

for each $a \in A$. We note that

$$\Pi(h(a)) = (a]. \tag{3.1}$$

Similarly we can prove that there exists a function $k : B \to A$ such that

$$k(b) = a \text{ iff } \Delta a = (b],$$

for each $b \in B$. Also, we note that

$$\Delta(k(b)) = (b]. \tag{3.2}$$

We prove that $k \circ h = Id_A$ and $h \circ k = Id_B$. Let $a \in A$. Then as $\Pi \circ \Delta = I_A$ we get that:

$$((k \circ h)(a)] = \Pi\,[\Delta(k(h(a)] = \Pi\,[\Delta(k \circ h)(a)] \overset{(3.2)}{=} \Pi\,(h(a)]$$
$$= \Pi h(a) \overset{(3.1)}{=} (a]$$

(2) $\Rightarrow$ (1) Assume that there exists an one to one and onto function $h : A \to B$ such that $\Delta a = (h(a)]$, for each $a \in A$. So, there exists an one to one and onto function $g : B \to A$ such that $(g \circ h)(a) = a$ for all $a \in A$, and $(h \circ g)(b) = b$ for all $b \in B$. Consider the quasi-semi-homomorphism $\Pi : B \to \mathrm{Id}(A)$ defined by$\Pi(b) = (g(b)]$. Then we prove that $\Delta \circ \Pi = I_B$ and $\Pi \circ \Delta = I_A$. We prove that $(\Delta \circ \Pi)(b) = (b]$. Let $b, d \in B$ such that $d \in (\Delta \circ \Pi)(b) = \Delta\,[\Pi b] = \bigcup\{\Delta c : c \in \Pi b = (g(b)]\}$. So, there exists $c \in A$ and $d \in B$ such that $c \le g(b)$ and $d \in \Delta c = (h(c)]$. So, $d \le h(c)$, and thus $d \le h(c) \le h(g(b)) = b$, i.e., $c \in (b]$. So, $(\Delta \circ \Pi)(b) \subseteq (b]$. The other inclusion it is left to the reader. Thus, $\Delta \circ \Pi = I_B$. Similarly we can prove that $\Pi \circ \Delta = I_A$. Therefore, Δ is an iso-arrow in the category **BoQS**. □

4. Generalized quasi-Boolean relations

Let X and Y be two topological spaces. Let $R \subseteq X \times Y$ be a relation. We shall say that R is *upper-semi-continuous* (u.s.c) if $\Delta_R(O) = \{x \in X : R(x) \subseteq O\}$ is an open subset of X for every open subset O of Y. We note that $\Delta_R(O)$ is open for each open O of Y iff $\nabla_R(C) = \{x \in X : R(x) \cap C \neq \varnothing\}$ is an closed of X for each closed C of Y. We shall say that R is *point-compact (point-closed)* if $R(x)$ is a compact (closed) subset of Y, for each $x \in X$. Clearly, if Y is a compact space, a relation $R \subseteq X \times Y$ is point-compact iff it is point-closed.

Lemma 4. *Let X and Y be two topological space. Suppose that Y is zero-dimensional space. Let R be a point-compact relation. Then the following conditions are equivalent:*

(1) R is upper-semi-continuous,

(2) $\Delta_R(U) \in \mathcal{O}(X)$, for each $U \in \mathrm{Clo}(Y)$.

Proof. If R is upper-semi-continuous, then $\Delta_R(U) \in \mathrm{Clo}(X)$, for each $U \in \mathrm{Clo}(Y)$, because $\mathrm{Clo}(Y) \subseteq \mathcal{O}(Y)$.

Conversely. Assume that $\Delta_R(U) \in \mathcal{O}(X)$, for each $U \in \mathrm{Clo}(Y)$. Let $O \in \mathcal{O}(Y)$. As Y is zero-dimensional, $\mathrm{Clo}(Y)$ is a basis, so $O = \bigcup\{U_i \in \mathrm{Clo}(Y) : U_i \subseteq O\}$. Since Δ_R is monotonic,

$$\bigcup\{\Delta_R(U_i) : U_i \subseteq O\} \subseteq \Delta_R(\bigcup\{U_i \in \mathrm{Clo}(Y) : U_i \subseteq O\}) = \Delta_R(O).$$

We prove the other inclusion. Let $x \in \Delta_R(\bigcup\{U_i \in \mathrm{Clo}(Y) : U_i \subseteq O\})$, i.e., $R(x) \subseteq \bigcup\{U_i \in \mathrm{Clo}(Y) : U_i \subseteq O\})$. As $R(x)$ is a compact subset of Y, there exists a finite family $\{U_i \subseteq O : 1 \leq i \leq n\}$ such that $R(x) \subseteq U_1 \cup \ldots \cup U_n = U \subseteq O$, i.e., $x \in \Delta_R(U)$. Thus, $\bigcup\{\Delta_R(U_i) : U_i \subseteq O\} = \Delta_R(O)$. Consequently $\Delta_R(O)$ is an open subset of X, because$\Delta_R(U_i) \in \mathrm{Clo}(X)$, for each $U \in \mathrm{Clo}(Y)$. □

Remark 1. By the previous Lemma, when X and Y are Stone spaces, and $R \subseteq X \times Y$, we have that the following conditions are equivalent:

(1) R is a point-compact relation and $\Delta_R(O) \in \mathcal{O}(X)$, for each $U \in \mathcal{O}(Y)$.
(2) R is a point-closed relation and $\Delta_R(U) \in \mathcal{O}(X)$, for each $U \in \mathrm{Clo}(Y)$.

Definition 4. Let X and Y be two Stone spaces. We shall say that a binary relation $R \subseteq X \times Y$ is a *quasi-Boolean relation* if

(1) R is a point-closed relation,
(2) $\Delta_R(U) \in \mathcal{O}(X)$, for each $U \in \mathrm{Clo}(Y)$.

If $\Delta_R(U) \in \mathrm{Clo}(X)$, for each $U \in \mathrm{Clo}(Y)$, then R is called a *Boolean relation* [12], also called a Boolean correspondence in [11]. It is clear that every Boolean relation is a quasi-Boolean relation.

Remark 2. Let X be a Stone space. A pair $\langle X, R\rangle$, where R is a quasi-Boolean relation defined in X is called a *quasi-modal space.* The quasi-modal spaces are the dual objects of the quasi-modal algebras (see [3] and [2]). If R is a Boolean relation, then the pair $\langle X, R\rangle$ is called a *modal space or descriptive general frame* [10, 16]. The modal spaces are the dual of the modal algebras, i.e., pairs $\langle A, \Box\rangle$, where A is a Boolean algebra and $\Box$ is a modal operator.

Given a Stone space X, the map $\varepsilon_X : X \to \mathrm{Ul}(\mathrm{Clo}(X))$ defined by

$$\varepsilon_X(x) = \{U \in \mathrm{Clo}(X) : x \in U\}$$

, is a bijective and continuous function. Thus, for each $P \in \mathrm{Ul}(\mathrm{Clo}(X))$ there exists a unique $x \in X$ such that $\varepsilon_X(x) = P$.

Let X and Y be two Stone spaces. Let $R \subseteq X \times Y$ be a relation. For each $x \in X$ we can consider the set

$$\Delta_R^{-1}(\varepsilon_X(x)) = \{U \in \mathrm{Clo}(X_2) : x \in \Delta_R(U)\} = \{U \in \mathrm{Clo}(X_2) : R(x) \subseteq U\}.$$

We define the relation $R_{\Delta_R} \subseteq \mathrm{Ul}(\mathrm{Clo}(X)) \times \mathrm{Ul}(\mathrm{Clo}(Y))$, as follows:

$$(\varepsilon_X(x), \varepsilon_Y(y)) \in R_{\Delta_R} \text{ iff } \Delta_R^{-1}(\varepsilon_X(x)) \subseteq \varepsilon_Y(y)).$$

In the following Lemma we shall give an equivalent condition to the condition (1) of Definition 4.

Lemma 5. *Let X_1 and X_2 be two Stone spaces. Let $R \subseteq X_1 \times X_2$ be a relation. Suppose that $\Delta_R(U)$ is an open subset of X_1, for each $U \in \mathrm{Clo}(X_2)$. Then the following conditions are equivalent*

(1) $R(x)$ is a closed subset of X_2, for each $x \in X_1$,
(2) $(x, y) \in R$ iff $(\varepsilon_1(x), \varepsilon_2(y)) \in R_{\Delta_R}$.

Proof. (1) implies (2). Let $x, y \in X_1$. It is clear that if $(x, y) \in R$ then $(\varepsilon_1(x), \varepsilon_2(y)) \in R_{\Delta_R}$. Suppose that $y \notin R(x)$. As $R(x)$ is a closed subset of X_2, there exists $U \in \mathrm{Clo}(X_2)$ such that $y \notin U$ and $R(x) \subseteq U$. So, $x \in \Delta_R(U)$. Then $U \in \Delta_R^{-1}(\varepsilon_1(x))$ and $U \notin \varepsilon_2(y)$, i.e., $(\varepsilon_1(x), \varepsilon_2(y)) \notin R_{\Delta_R}$.

(2) implies (1). We prove that $\mathrm{cl}(R(x)) = R(x)$. Suppose that there exists $y \in \mathrm{cl}(R(x))$ but $y \notin R(x)$. Then $(\varepsilon_1(x), \varepsilon_2(y)) \notin R_{\Delta_R}$, i.e., there exists $U \in \mathrm{Clo}(X_2)$ such that $U \in \Delta_R^{-1}(\varepsilon_1(x))$ and $U \notin \varepsilon_2(y)$. Then $x \in \Delta_R(U)$ and $y \notin U$, i.e., $R(x) \subseteq U$ and $y \notin U$. So, $y \notin \mathrm{cl}(R(x))$, which is a contradiction. Thus, $\mathrm{cl}(R(x)) \subseteq R(x)$, and consequently $R(x)$ is a closed subset of X_2. $\square$

Let X and Y be two Stone spaces. By Lemma 5 we have that a relation $R \subseteq X_1 \times X_2$ is a quasi-Boolean relation iff R satisfies the following conditions:

(1) $(x, y) \in R$ iff $(\varepsilon_1(x), \varepsilon_2(y)) \in R_{\Delta_R}$,
(2) $\Delta_R(U) \in \mathcal{O}(X)$, for each $U \in \mathrm{Clo}(Y)$.

We denote by $QB[X, Y]$ the set of all quasi-Boolean relations between two Stone spaces X and Y.

Lemma 6. *Let X and Y be Stone spaces. Let $R \in QB[X, Y]$. Then $R[C]$ is a closed subset of Y for each closed subset C of X.*

Proof. Let C be a closed subset of X. We note that $R[C] = \bigcup\{R(x) : x \in C\}$. It suffices to prove that for any $y \notin R[C]$ there exists $U \in \mathrm{Clo}(Y)$ such that $R[C] \subseteq U$ and $y \notin U$. Take $y \notin R[C]$. Then $y \notin R(x)$ for each $x \in C$. As R is point-closed, for each $x \in C$ there exists $U_x \in \mathrm{Clo}(Y)$ such that $R(x) \subseteq U_x$ and $y \notin U_x$. So, $x \in \Delta_R(U_x)$, for each $x \in C$. Thus, $C \subseteq \bigcup\{\Delta_R(U_x) : x \in X\}$, and as C is compact, there exists $x_1, \ldots x_n \in C$ such that

$$C \subseteq \Delta_R(U_{x_1}) \cup \ldots \cup \Delta_R(U_{x_n}) \subseteq \Delta_R(U_{x_1} \cup \ldots \cup U_{x_n}) = \Delta_R(U),$$

i.e., $R[C] \subseteq U$. Therefore there exists $U \in \mathrm{Clo}(Y)$ such that $y \notin U$ and $R[C] \subseteq U$. $\square$

Lemma 7. *If $R \in QB[X, Y]$, then $\bar{\Delta}_R \in QS[\mathrm{Clo}(Y), \mathrm{Clo}(X)]$.*

Proof. If $R \in QB[X,Y]$, then by Example 3 it is clear that $\bar{\Delta}_R : \mathrm{Clo}(Y) \to \mathrm{Id}(\mathrm{Clo}(X))$ is generalized quasi-semi-homomorphism.
Thus, $\bar{\Delta}_R \in QS[\mathrm{Clo}(Y), \mathrm{Clo}(X)]$. □

Let X, Y and Z be Stone spaces. Let $R \in QB[X,Y]$, and $S \in QB[Y,Z]$. The *composition* of R with S is the relation

$$R \circ S = \{(x,z) \in X \times Z : \exists y \in Y \; [(x,y) \in R \text{ and } (y,z) \in H]\}.$$

We note that $(R \circ S)(x) = S[R(x)] = \bigcup\{S(y) : y \in R(x)\}$.

Lemma 8. *The composition of quasi-Boolean relations is a quasi-Boolean relation.*

Proof. Let X, Y and Z be Stone spaces. Let $R \in QB[X,Y]$ and $S \in QB[Y,Z]$. We prove that $R \circ S \subseteq X \times Z$ is point-closed. Let $x \in X$. As $R(x)$ is a closed subset of Y, by Lemma 6, we get that $S[R(x)] = (S \circ R)(x)$ is a closed subset of Z.

We prove that $(\Delta_R \circ \Delta_S)(U) = \Delta_{R \circ S}(U)$, for each $U \in \mathrm{Clo}(Z)$. Let $x \in (\Delta_R \circ \Delta_S)(U) = \Delta_R(\Delta_S(U))$, i.e., $R(x) \subseteq \Delta_S(U)$. Let $z \in (R \circ S)(x) = S[R(x)] = \bigcup\{S(y) : y \in R(x)\}$. Then there exists $y \in R(x)$ such that $z \in S(y)$. As $R(x) \subseteq \Delta_S(U)$, $y \in \Delta_S(U)$, i.e., $S(y) \subseteq U$. So, $z \in U$. Thus, $(\Delta_R \circ \Delta_S)(U) \subseteq \Delta_{R \circ S}(U)$.

Let $x \in \Delta_{R \circ S}(U)$. Then $(R \circ S)(x) = S[R(x)] = \bigcup\{S(y) : y \in R(x)\} \subseteq U$, i.e., $S(y) \subseteq U$, for all $y \in R(x)$. So, $y \in \Delta_S(U)$ for all $y \in R(x)$, i.e., $R(x) \subseteq \Delta_S(U)$. Then $x \in \Delta_R(\Delta_S(U)) = (\Delta_R \circ \Delta_S)(U)$. Thus, $\Delta_{R \circ S}(U) \subseteq (\Delta_R \circ \Delta_S)(U)$. □

Let $f : X \to Y$ be a function between two Stone Spaces. Consider the relation $f^* \subseteq X \times Y$ defined by

$$f^* = \{(x,y) \in X \times Y : f(x) = y\}.$$

Lemma 9. *Let X and Y be two Stone spaces. If $f : X \to Y$ is a function such that $f^{-1}(U)$ is an open subset of X for each $U \in \mathrm{Clo}(Y)$, then $f^* \in QB[X,Y]$.*

Proof. It is clear that $\Delta_{f^*}(U) = \{x : f^*(x) \subseteq U\} = \left\{x : x \subseteq f^{-1}(U)\right\} = f^{-1}(U)$. Thus, $\Delta_{f^*}(U)$ is an open subset of X for each $U \in \mathrm{Clo}(Y)$. Also, as Y is a Stone Space, we have $f^*(x)$ is a closed subset of Y, for each $x \in X$. Thus, $f^* \in QB[X,Y]$. □

Using the previous lemma we obtain the following result.

Corollary 1. *Let X be a Stone space. Consider the $\varepsilon_X : X \to \mathrm{Ul}(\mathrm{Clo}(X))$. Then the relation $\varepsilon_X^* \subseteq X \times \mathrm{Ul}(\mathrm{Clo}(X))$ given by $(x,P) \in \varepsilon_X^*$ iff $\varepsilon_X(x) = P$ is a generalized quasi-Boolean relation.*

By Lemma 8 we conclude that the Stone spaces with generalized quasi-Boolean relations is a category, denoted by **StQB** where the identity morphism is the identity map Id_X, where X is a Stone space. The careful reader may have realized that the notation of composition of relations reverses the order of the actual composition in the category. We have decided to preserve this usual notation, instead of giving a new one, in order to make the paper more readable.

In the following result we characterized the isomorphisms (or iso-arrow) in the category **StQB**.

Lemma 10. *Let X and Y be Stone spaces and $R \in QB[X,Y]$. Then the following conditions are equivalent:*

(1) R is an iso-arrow in the category **StQB**.
(2) There exists an one-to-one and onto function $f : X \to Y$ such that $R = f^*$, satisfying the condition $f^{-1}(U)$ is an open set for each $U \in \mathrm{Clo}(Y)$. i.e., f is a continuous function between the Stone spaces X and Y.

Proof. (1) $\Rightarrow$ (2) Let $S \in QB[Y,X]$ such that $R \circ S = Id_X^*$ and $R \circ S = Id_Y^*$, where Id_Y^* and Id_X^* are quasi-Boolean relations corresponding to the functions Id_X and Id_Y, respectively (see Lemma 9). Then for every $x \in X$, $(R \circ S)(x) = S(R(x)) = Id_X^*(x) = \{x\}$. Using the fact that $S \circ Id_X^* = S$, and $x \in S(R(x))$, then there exists $y \in R(x)$ such that $S(y) = \{x\}$. We prove that y is unique. Suppose that there are $y_1, y_2 \in Y$ such that $S(y_1) = S(y_2)$. As $S \circ R = Id_Y^*$, we get

$$\{y_1\} = (S \circ R)(y_1) = R(S(y_1)) = R(S(y_2)) = (S \circ R)(y_2) = \{y_2\}.$$

Thus, $y_1 = y_2$. So we conclude that for each $x \in X$ there exists a unique $y \in Y$ such that $S(y) = \{x\}$. Let us denote by $f : X \to Y$ the function defined by

$$f(x) = y \text{ iff } S(y) = \{x\},$$

for each $x \in X$. Similarly we can prove that there exists a function $g : Y \to X$ such that $R(g(y)) = \{y\}$, for each $y \in Y$. As $Id_Y^* \circ S = S$, we get that $(Id_Y^* \circ S)(y) = S(Id_Y^*(y)) = S(\{y\}) = S(y)$, for each $y \in Y$. Then for every $x \in X$ we have that $\{g(f(x))\} = (R \circ S)(g(f(x))) = S(R(g(f(x))) = S(\{f(x)\}) = \{x\}$. So, $g(f(x)) = x$, for each $x \in X$, i.e., $g \circ f$ is the identity function on X.

Changing the roles of f and g, we obtain that $f \circ g$ is the identity function on Y. We conclude that f is a one to one map from X onto Y and g is its inverse. Observe that $R(x) = R(g(f(x))) = \{f(x)\}$. Then $R = f^*$. Similarly, we have that $S = g^*$.

Consider now $U \in \mathrm{Clo}(Y)$. Since R is a quasi-Boolean relation, we have that

$$\begin{aligned} \Delta_R(U) &= \{x \in X : R(x) \subseteq U\} = \left\{x \in X : f^*(x) \subseteq U\right\} \\ &= \{x \in X : \{f(x)\} \subseteq U\} = \{x \in X : f(x) \in U\} \\ &= f^{-1}(U). \qquad = \end{aligned}$$

So, $f^{-1}(U)$ is an open subset of X.

The direction (2) $\Rightarrow$ (1) follows straightforward from Lemma 9 and the definition of iso-arrow. $\square$

5. CATEGORICAL DUALITY

In this section we prove that there exists a category, whose object are Boolean algebras, and whose morphism are quasi-Boolean relations. In order to complete the duality we need to see how to define quasi-Boolean relation from each quasi-semi-homomorphism between two Boolean algebras.

Let $\Delta \in QS[A,B]$. We define a relation $R_\Delta \subseteq \mathrm{Ul}(B) \times \mathrm{Ul}(A)$ by

$$\begin{aligned}(P,Q) \in R_\Delta \quad &\Leftrightarrow \quad \forall a \in A : \Delta a \cap P \neq \varnothing \text{ then } a \in Q \\ &\Leftrightarrow \quad \Delta^{-1}(P) \subseteq Q.\end{aligned}$$

We note that when $A = B$, the relation R_Δ is the relation used in [3] in the representation of quasi-modal algebras.

We give now a equivalent characterization for the relation R_Δ. We recall that, given $\Delta \in QS[A,B]$, the generalized proximity relation $\delta_\Delta \subseteq B \times A$ is defined as $(b,a) \in \delta_\Delta$ iff $b \notin \Delta\neg a$.

Lemma 11. *Let A and B be two Boolean algebras. Let $\Delta \in QS[A,B]$. Let $(P,Q) \in \mathrm{Ul}(B) \times \mathrm{Ul}(A)$. Then the following conditions are equivalent:*

(1) $(Q,P) \in R_\Delta$,
(2) $Q \times P \subseteq \delta_\Delta$

Proof. Let $(P,Q) \in \mathrm{Ul}(B) \times \mathrm{Ul}(A)$. Assume that $(Q,P) \in R_\Delta$. Let $(q,p) \in Q \times P$. If $(q,p) \notin \delta_\Delta$, then $q \in \Delta\neg p \cap Q$, i.e., $\neg p \in \Delta^{-1}(Q) \subseteq P$. So, $\neg p \wedge p = 0 \in P$, which is a contradiction. Thus, $Q \times P \subseteq \delta_\Delta$.

Assume that $Q \times P \subseteq \delta_\Delta$. Let $\Delta a \cap Q \neq \varnothing$. Then there exist $q \in \Delta a$ and $q \in Q$. Suppose that $a \notin P$. Then $\neg a \in P$. So, $(q, \neg a) \in Q \times P \subseteq \delta_\Delta$, i.e., $q \notin \Delta\neg\neg a = \Delta a$, which is a contradiction. Thus, $(Q,P) \in R_\Delta$. □

Remark 3. When $A = B$, the relation given in (2) is the definition used in [14] for the topological representation of some extensions of proximity Boolean algebras.

Lemma 12. *Let $\Delta \in QS[A,B]$. Let $P \in \mathrm{Ul}(B)$ and $I \in \mathrm{Id}(A)$. Then*

$$\Delta I \cap P = \varnothing \Leftrightarrow \exists Q \in \mathrm{Ul}(A)\left[\Delta^{-1}(P) \subseteq Q \text{ and } I \cap Q = \varnothing\right].$$

Proof. Let $P \in \mathrm{Ul}(B)$ and $I \in \mathrm{Id}(A)$. We note that $\Delta I \cap P = \varnothing$ iff $I \cap \Delta^{-1}(P) = \varnothing$. Indeed. Suppose that $\Delta I \cap P = \varnothing$ and suppose that there exists $a \in I \cap \Delta^{-1}(P)$. Then $\Delta a \cap P \neq \varnothing$, i.e., there exist $p \in P$ and $p \in \Delta a$. As $a \in I$, we get that $p \in \Delta I \cap P$, which is a contradiction. Thus $I \cap \Delta^{-1}(P) = \varnothing$. The other direction is similar and left to the reader.

Assume that $I \cap \Delta^{-1}(P) = \varnothing$. Consider the family

$$\mathcal{F} = \left\{H \in \mathrm{Fi}(A) : I \cap H = \varnothing \text{ and } \Delta^{-1}(P) \subseteq H\right\}.$$

As $\Delta^{-1}(P)$ is a filter of A and $\Delta^{-1}(P) \in \mathcal{F}$, then $\mathcal{F} \neq \varnothing$. By Zorn's Lemma, we can take a maximal $Q \in \mathcal{F}$. It remains to show that Q is an ultrafilter of A. Let

$a \in A$. Take the filters $F_a = F(Q \cup \{a\})$ and $F_{\neg a} = F(Q \cup \{\neg a\})$. If $a, \neg a \notin Q$, then $F_a, F_{\neg a} \notin \mathcal{F}$. So, $F_a \cap I \neq \varnothing$ and $F_{\neg a} \cap I \neq \varnothing$. Then there exists $q_1, q_2 \in Q$ such that $q_1 \wedge a \in I$ and $q_2 \wedge \neg a \in I$. Take $q = q_1 \wedge q_2$. As I is an ideal of A, we have that $(q \wedge a) \vee (q \wedge \neg a) = q \wedge (a \vee \neg a) = q \wedge 1 = q \in I$, which is a contradiction. Thus, $Q \in \mathrm{Ul}(A)$. So, $\Delta^{-1}(P) \subseteq Q$ and $Q \cap I = \varnothing$. The other direction it is easy and left to the reader. □

Theorem 2. *Let $\Delta \in QS[A, B]$. Let $a \in A$ and $P \in \mathrm{Ul}(B)$. Then*

(1) $a \in \Delta^{-1}(P) \Leftrightarrow \forall Q \in \mathrm{Ul}(A) : \Delta^{-1}(P) \subseteq Q$ then $a \in Q$,
(2) $a \in \nabla^{-1}(P) \Leftrightarrow \exists Q \in \mathrm{Ul}(A) : Q \subseteq \nabla^{-1}(P)$ and $a \in Q$.

Proof. We prove (1). The proof of (2) follows by duality. Assume that $a \notin \Delta^{-1}(P)$, i.e., $\Delta a \cap P = \varnothing$. By Lemma 12 we get that there exists $Q \in \mathrm{Ul}(A)$ such that $\Delta^{-1}(P) \subseteq Q$ and $a \notin Q$. The other direction is immediate. □

Recall that if I is an ideal of a Boolean algebra B, then

$$\varphi_B[I] = \{P \in \mathrm{Ul}(B) : I \cap P \neq \varnothing\}$$

is an open subset of the Stone space of B.

Theorem 3. *Let A and B two Boolean algebras. Let $\Delta \in QS[A, B]$. Then*

(1) $\varphi_B[\Delta a] = \Delta_{R_\Delta}(\beta_A(a))$, for all $a \in A$.
(2) $R_\Delta \in \mathrm{QB}[\mathrm{Ul}(B), \mathrm{Ul}(A)]$.

Proof. (1) Let $a \in A$. Let $P \in \Delta_{R_\Delta}(\beta_A(a))$. Then $R_\Delta(P) \subseteq \beta_A(a)$. If $P \notin \varphi_B[\Delta a]$, then $\Delta a \cap P = \varnothing$. So, there exists $Q \in R_\Delta(P)$ such that $a \notin Q$. Then, $R_\Delta(P) \nsubseteq \beta_A(a)$, which is a contradiction. Thus, $P \in \varphi_B[\Delta a]$. The other inclusion is easy and left to the reader. Thus, $\Delta_{R_\Delta}(\beta_A(a))$ is an open subset.

(2) By Theorem 2 we deduce that $R_\Delta(P) = \bigcap\{\beta_A(a) : \Delta a \cap P \neq \varnothing\}$. Therefore, $R_\Delta(P)$ is a closed subset for each $P \in \mathrm{Ul}(A)$, i.e., $R_\Delta$is point-closed. □

We recall that if $\Delta_1 \in QS[A, B]$ and $\Delta_2 \in QS[B, C]$, then $\Delta_2 \circ \Delta_1 \in QS[A, C]$. Thus, $R_{\Delta_2 \circ \Delta_1} \subseteq \mathrm{Ul}(C) \times \mathrm{Ul}(A)$.

Lemma 13. *Let A, B and C be Boolean algebras. Let $\Delta_1 \in QS[A, B]$ and $\Delta_2 \in QS[B, C]$. Then $R_{\Delta_2 \circ \Delta_1} = R_{\Delta_2} \circ R_{\Delta_1}$.*

Proof. Let $(P, Q) \in \mathrm{Ul}(C) \times \mathrm{Ul}(B)$ such that $(P, Q) \in R_{\Delta_2 \circ \Delta_1}$. Then $(\Delta_2 \circ \Delta_1)^{-1}(P) \subseteq Q$, i.e., for all $a \in A$ such that $(\Delta_2 \circ \Delta_1)(a) \cap P \neq \varnothing$, then $a \in Q$. We note that as $Q^c = A - Q$ is an ideal, we have that $\Delta_1(Q^c)$ is an ideal. We prove that

$$\Delta_2^{-1}(P) \cap \Delta_1(Q^c) = \varnothing.$$

Otherwise there exists $a \notin Q$ and $b \in B$ such that $\Delta_2 b \cap P \neq \varnothing$ and $b \in \Delta_1 a$. So, there exists $c \in \Delta_2 b \cap P$. Then, $c \in (\Delta_2 \circ \Delta_1)(a) \cap P$, i.e., $a \in (\Delta_2 \circ \Delta_1)^{-1}(P)$. Thus, $a \in Q$, which is a contradiction. Thus, there exists $D \in \mathrm{Ul}(B)$ such that

$\Delta_2^{-1}(P) \subseteq D$ and $\Delta_1^{-1}(D) \subseteq Q$, i.e, $(P, D) \in R_{\Delta_2}$and $(D, Q) \in R_{\Delta_1}$. Therefore, $(P, Q) \in R_{\Delta_2} \circ R_{\Delta_1}$.

To prove the other inclusion, let $(P, Q) \in R_{\Delta_2} \circ R_{\Delta_1}$. Then there exists $D \in \mathrm{Ul}(B)$ such that $\Delta_2^{-1}(P) \subseteq D$ and $\Delta_1^{-1}(D) \subseteq Q$. Let $a \in A$ such that $(\Delta_2 \circ \Delta_1)(a) \cap P = \Delta_2[\Delta_1(a)] \cap P \neq \varnothing$. Then there exists $b \in B$ and there exists $c \in C$ such that $b \in \Delta_1 a$ and $c \in \Delta_2 b \cap P$. So, $b \in \Delta_2^{-1}(P)$. So, $b \in \Delta_1 a \cap D$, and consequently $a \in Q$. Thus, $(\Delta_2 \circ \Delta_1)^{-1}(P) \subseteq Q$, i.e., $(P, Q) \in R_{\Delta_2 \circ \Delta_1}$. □

Define a contravariant functor $\alpha : \mathbf{BoQS} \to \mathbf{StQB}$ by

$$\alpha(A) = \langle \mathrm{Ul}(A), \tau_A \rangle \quad \text{if} \quad A \text{ is a Boolean algebra}$$

$$\alpha(\Delta) = R_\Delta \quad \text{if} \quad \Delta \in QS[A, B].$$

Define a contravariant functor $\eta : \mathbf{StQB} \to \mathbf{BoQS}$ as

$$\eta(X) = \mathrm{Clo}(X) \quad \text{if} \quad X \text{ is a Boolean space}$$

$$\eta(R) = \Delta_R \quad \text{if} \quad R \in QB[X, Y].$$

Since for each Boolean algebra A the map $\beta_A : A \to \mathrm{Clo}(\mathrm{Ul}(A))$ is an isomorphism in **BoQS**, we get that Theorem 3 means that the composite functor $\eta \circ \alpha$ is naturally equivalent to the identity functor, the natural equivalence being given by the isomorphisms β_A. On the other hand, since for each Stone space X, the map ε_X is a homeomorphism from X onto $X(\mathrm{Clo}(X))$, it follows that the relation ε_X^* defined by

$$(x, P) \in \varepsilon_X^* \text{ iff } \varepsilon_X(x) = P$$

is a quasi-Boolean relation, and by Lemma 5 we have that ε_X^* is an isomorphism in **StQB**. It is easy to see ε_X^* is a natural equivalence from the composite functor $\eta \circ \alpha$ to the identity functor from in **StQB**, i.e., $R_{\Delta_R} \circ \varepsilon_X^* = R \circ \varepsilon_Y^*$ for $R \in QS[X, Y]$. Similarly, it is easy to see that β_A is a natural equivalence between the identity functor in **BoQS** and $\alpha \circ \eta$. Thus, we have the following result.

Theorem 4. *The contravariant functors η and α and the natural equivalences ε and β define a dual equivalence between the category of Boolean algebras with quasi-semi-homomorphisms and the category of Stone spaces with quasi-Boolean relations.*

As an application of the above duality we prove a generalization of the result that asserts that the Boolean homomorphisms are the minimal elements in the set of all join-homomorphisms between two Boolean algebras (see [11]). Now we prove that the minimal elements in the set of all quasi-Boolean relations defined between two Stone spaces is a Boolean relation.

Let A and B be two Boolean algebras. Let X and Y be the Stone spaces of A and B, respectively. Let $QS[X, Y]$ the set of all quasi-Boolean relations defined between X and Y endowed with the order given by the inclusion between relations. Let Δ_1

and $\Delta_2 \in QS[A, B]$ and let R_{Δ_1} and $R_{\Delta_2} \in QS[X, Y]$ the associated quasi-Boolean relations. It is clear that $\Delta_1 \leq \Delta_2$ if and only if $R_{\Delta_1} \subseteq R_{\Delta_2}$.

Theorem 5. *Let X and Y be two Stone spaces. An element of $QS[X, Y]$ it is minimal if and only if is a Boolean relation .*

Proof. Let $R \subseteq X \times Y$ be a minimal element $QS[X, Y]$. We prove that R is a Boolean relation. As R is point-closed, we have to see that $\Delta_R(U)$ is a closed subset of X, for each $U \in \mathrm{Clo}(Y)$. Let $x \in \mathrm{cl}(\Delta_R(U))$. Suppose that $x \notin \Delta_R(U)$. Then $R(x) \nsubseteq U$. So there exists $y \in R(x)$ such that $y \notin U$. Define the relation R_U as:

$$R_U(z) = R(z) \cap U^c,$$

for each $z \in X$. It is clear that $R_U(z)$ is a closed subset for each $z \in X$. Thus R_U is point-closed. Moreover, for $V \in \mathrm{Clo}(Y)$ we have that

$$\begin{aligned} \Delta_{R_U}(V) &= \{z \in X : R_U(z) \subseteq V\} = \{z \in X : R(z) \cap U^c \subseteq V\} \\ &= \{z \in X : R(z) \subseteq U \cup V\} = \{z \in X : z \in \Delta_R(U \cup V)\} \\ &= \Delta_R(U \cup V). \end{aligned}$$

Since R is a quasi-Boolean relation, $\Delta_R(U \cup V)$ is an open subset of X. Then R_U is a quasi-Boolean relation. It is clear that $R_U \subset R$. Thus R is not minimal element in $QS[X, Y]$, which is a contradiction. Therefore, $\mathrm{cl}(\Delta_R(U)) = \Delta_R(U)$, i.e., $\Delta_R(U)$ is a closed subset of X. Consequently, R is a Boolean relation. □

Final remarks

In this paper we have proved a generalization of the Halmos's duality [12] [11], and the duality given in [3] for quasi-modal algebras.

There are several possibilities to extend the results given in this work. One possibility is to consider local Boolean algebras with a special class of morphisms. We recall that a local Boolean algebra is a pair of the form $\langle A, I\rangle$, where A is a Boolean algebra and I is an ideal of A, such that $[I) = A$. A local homomorphism between two local algebras $\langle A, I\rangle$ and $\langle B, J\rangle$ is a Boolean homomorphism $h : A \to B$ satisfying the following condition:

(LH): For each $b \in J$ there exists $a \in I$ such that $b \leq h(a)$, i.e., $J \subseteq (h[I]]$.

A meaningful extension of the Stone duality is given by Geogi Dimov in [6]. In this paper it is shown that the category of local Boolean algebras with local homomorphism is dually equivalent to the category of Boolean spaces (= zero-dimensional locally compact Hausdorff spaces) with continuous maps. In a future work we shall study local Boolean algebras with meet-homomorphisms satisfying the condition **(LH)**, and the representation theory by means of Stone spaces with a relation satisfaying certain conditions.

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