Synthese
https://doi.org/10.1007/s11229-018-01963-3

S.I. : PLURALISTIC PERSPECTIVES ON LOGIC

Substructural logics, pluralism and collapse

Eduardo Alejandro Barrio[1,2] · **Federico Pailos**[1,2] · **Damian Szmuc**[1,2]

Received: 10 June 2018 / Accepted: 22 September 2018

Abstract
When discussing Logical Pluralism several critics argue that such an open-minded position is untenable. The key to this conclusion is that, given a number of widely accepted assumptions, the pluralist view collapses into Logical Monism. In this paper we show that the arguments usually employed to arrive at this conclusion do not work. The main reason for this is the existence of certain substructural logics which have the same set of valid inferences as Classical Logic—although they are, in a clear sense, non-identical to it. We argue that this phenomenon can be generalized, given the existence of logics which coincide with Classical Logic regarding a number of metainferential levels—although they are, again, clearly different systems. We claim this highlights the need to arrive at a more refined version of the Collapse Argument, which we discuss at the end of the paper.

Keywords Substructural logics · Cut rule · Collapse Argument · Logical Pluralism

1 Introduction

Logical Pluralism is the thesis that there is a plurality of different logics all of which are correct. The view was systematically discussed and presented in Beall and Restall (2006), which includes material from previous articles written by these authors, and incorporates a number of critiques leveled against their view.

In the wake of Beall and Restall's writings, many voices raised against Logical Pluralism expressing different concerns revolving around their view. Perhaps one of

✉ Eduardo Alejandro Barrio
eabarrio@gmail.com

Federico Pailos
fpailos@hotmail.com

Damian Szmuc
szmucdamian@gmail.com

1 University of Buenos Aires, Puan 480, C1420 Buenos Aires, Argentina

2 IIF-SADAF (CONICET), Bulnes 642, C1176ABL Buenos Aires, Argentina

Published online: 01 October 2018

the most important lines against it is the one represented by the Collapse Argument, discussed among other places in the works by Williamson (1987), Priest (2006), Read (2006) and Keefe (2014). This allegedly knock-down argument is intended to show, according to Caret, "that despite its intention to articulate a radically pluralistic doctrine about logic, the view unintentionally collapses into logical monism" (Caret 2017, p. 2).

The core of the Collapse Argument is sometimes presented in the form of a question, as in Caret (2017, p. 4). Suppose a pluralist comes to believe the premise(s) of a given argument which she knows is deemed valid by one of the logics she regards as correct, although not by all of them. Should she accept the conclusion of such an argument, or not? The answer must be definite, either positive or negative, and therefore the system constituted by collecting all the arguments regarded as valid—in this sense—by the pluralist represents the logic to which the pluralist position collapses.

In this regard, there have been scholars like Read (2006) who argued that this collapse takes us to the strongest logic among those that the pluralist accepts, but there have been those like Bueno and Shalkowski (2009) who claimed that, much to the contrary, the collapse takes us to the weakest of them. Either way, it seems that as tempting as the pluralist idea might appear, in the end of the day such an approach crumbles and we are left with nothing more than Logical Monism.

In this paper we argue that this kind of reasoning needs to be substantially *refined*. The main reason being that the account of rivalry and of identity between logical systems—shared by the pluralist and the anti-pluralist—is not appropriate in its present form. This account assumes that for two logics formulated on the same language to be rivals they need to be non-identical, meaning by this that they need to have different sets of valid inferences. Our aim is to argue that there are, in fact, different logics which have the same set of valid inferences, but which are nevertheless not identical to one another. The difference between such logics, we claim, is due to their having different sets of valid metainferences, i.e. of inferences between inferences. Thus, a pluralist embracing a pair of logics that are different in this sense will not be subject to the present form of the Collapse Argument. Whence, there is a clear need on the anti-pluralist side to refine this kind of argument.

Furthermore, we argue that this phenomenon can be generalized. Indeed, as it is possible to exemplify different logical systems which coincide regarding their valid inferences—but not their valid metainferences—it is possible to have logical systems which coincide regarding their valid inferences and metainferences—but not regarding their valid metametainferences, i.e. their valid inferences between metainferences. This move, indeed, can be subsequently generalized, over and over. In this vein, and hoping to clarify the discussion between the pluralist and the anti-pluralist, we propose novel and fully general identity and rivalry criteria for logics, in order to design what we take to be the strongest and most detailed version of the Collapse Argument against Logical Pluralism.

The paper is structured as follows. We start in Sect. 2 by reviewing the Collapse Argument. The main parts of the paper are Sect. 3, where we discuss the case of a substructural logic which has the same set of valid inferences as Classical Logic, and Sect. 4 where we show how such a phenomenon can be generalized, later proposing what we take to be the most appropriate modifications for the Collapse Argument. Section 5 includes some concluding remarks.

Before turning to the actual presentation of the problems and arguments of the present paper, let us highlight that in what follows our discussion of rival logical systems is meant to be restricted to *logics formulated on the same language*. This is as it should be, and as it is done in the literature revolving around the debate on Logical Pluralism, for e.g. Classical and Intuitionistic Logic to be properly speaking rival systems and for e.g. Classical Logic and the modal logic **S4** not to be rival systems, since the latter is usually understood as an extension of the former. We might highlight this from time to time, but occasionally we will allow ourselves to omit this clarification, hoping to improve in readability.

2 The Collapse Argument

Different formulations of the Collapse Argument were articulated by, e.g Williamson (1987), Read (2006), Priest (2006), Keefe (2014) and others. Priest, for example, lays the ground for such an attempt to refute Logical Pluralism in the following quote:

> Let s be some situation about which we are reasoning; suppose that s is in different classes of situations, say, K_1 and K_2. Should one use the notion of validity appropriate for K_1 or for K_2? We cannot give the answer 'both' here. Take some inference that is valid in K_1 but not K_2, $[\alpha \Rightarrow \beta]$, and suppose that we know (or assume) α holds in s; are we, or are we not entitled to accept that β does? Either we are or we are not: there can be no pluralism about this. (Priest 2006, p. 203)

Thus, a pluralist attitude towards logic, which implies the acceptance of several logical systems at the same time, collapses in practice into the acceptance of a single logical system. Which of the logical systems accepted by the pluralist this collapse leads to (if any) is left open, for this is not determined by pointing out these difficulties. Suppose, for example, that a pluralist accepts Classical Logic (**CL**, for short) and a paraconsistent logic. Were she to believe α and $\neg\alpha$, should she accept β? There needs to be a definite answer in this regard—although different perspectives on the collapse could justify collapsing into different logics, i.e. some alternatives might recommend collapsing to the stronger logic (that is, **CL**) and some might recommend collapsing to the weaker logic (that is, the paraconsistent logic in question).

In fact, in the face of this collapse, there seem to be at least two possibilities. Given a plurality of allegedly accepted logical systems the collapse either takes us to the strongest or to the weakest of these systems. (With the additional difficulty, it should be said, of putting pluralists accepting some incomparable systems—like **CL** and either Abelian or Connexive logic—in a dilemma. In such cases, e.g. the collapse to the strongest logic will bring them to an inconsistent logic, thereby forcing them, according to Stei (2017), to dialetheism.)[1]

[1] Abelian and Connexive logic are cases of the so-called contra-classical logics, i.e. of systems which are neither subsystems nor linguistic expansions of **CL**. In particular, they include as theorems formulae

The view that the former option—i.e. the collapse to the strongest of the logics accepted by the pluralist—is to be favored is, paradigmatically, represented by Read's work, especially by Read (2006, pp. 194–195). The reasons for such an endorsement are clearly summarized by Caret, as follows:

> Any logic that judges the argument from P to Q to be valid will give a direct affirmative answer to the central question: an agent who knows that P is true should infer that Q is true. A logic that judges the argument from P to Q to be invalid, on the other hand, will give no answer to the central question: it is agnostic as to whether an agent who knows that P is true should infer that Q is true. Since these attitudes do not conflict, an agent who endorses both logics should comply with the stronger demand. (Caret 2017, pp. 4–5)

Whereas the latter option—i.e. the collapse to the weakest of the logics accepted by the pluralist—is apparently considered by Bueno and Shalkowski, as can be read in the following quote by Keefe:

> [Beall and Restall] consider and reject a closely related view (...) proposing that validity requires truth-preservation in all cases of all kinds. (...) (Bueno and Shalkowski 2009, p. 300) argue that Beall and Restall's necessity constraint drives them to this view, and that once we consider the full range of alternative logics, this would result in logical nihilism whereby nothing is valid. (Keefe 2014, footnote 7)

Be that as it may, in this paper we are not concerned with the result of the Collapse Argument, but rather with the ingredients necessary by the anti-pluralist to get the argument running. In this regard, as noticed in Stei (2017, p. 3), among the agreed or conceded points between the pluralist and the anti-pluralist we have that logical consequence is global in scope, that logical consequence is normative, and that there is *rivalry* between different correct consequence relations.

There are many subtleties that need to be considered in order to provide a satisfactory account of rivalry between logics.[2] But once these issues are dealt with what remains is a highly intuitive idea. Namely, that there is rivalry between two logical systems if and

Footnote 1 continued
that are classically invalid which—given the Post-completeness of Classical Logic, and their respective consistency—requires them to have something that **CL** has not, and not to have something that **CL** has. In the case of Connexive logic the target axioms, usually referred to as Aristotle's Thesis and Boethius' Thesis, can be schematically stated as $\neg(A \supset \neg A)$ and $(A \supset B) \supset \neg(A \supset \neg B)$ respectively, although there are many versions thereof. In the case of Abelian logic, its hallmark axiom scheme is $((A \supset B) \supset B) \supset A$. For more on Connexive logic, see e.g. McCall (1966), Priest (1999), McCall (2012) and Wansing (2016); for more on Abelian logic see e.g. Meyer and Slaney (1989), Humberstone (2000) and Read (2006).

[2] Mainly, the two logics need to share the same vocabulary, as was previously remarked, and have the same domain of application—see e.g. Stei (2017). In this regard, an anonymous reviewer sharply points out that a particular variant of pluralists that we might call meaning-variance pluralists—among which Quine (1986) could, perhaps, be counted—could deny that there is a genuine rivalry between different correct consequence relations. After all, if a change of logic is a change of meaning, different logics are talking about different things and cannot therefore be seen as disagreeing or having a dispute about a common ground. This is, in fact, something recognized in the literature e.g. by Stei, who claims that "If we understand plurality in terms of the 'different logic, different language' (...) the collapse arguments do not straightforwardly apply" (Stei 2017, pp. 17–18). Notwithstanding these difficulties, Stei recognizes that even for meaning-variance pluralists a certain notion of rivalry is available "as long as there is the applicational conflict as to whether or

only if there is at least one inference that is valid in one logic, but not in the other. Given the nowadays standard account of a logical system as a pair consisting of a language and a Tarskian consequence relation over the formulae of that language, this account of rivalry can be equivalently expressed by saying that there is a rivalry between two logical systems on the same language if and only if they are not *identical*. One of the gravitational centers of the Collapse Argument is, then, the move that equates the agreement between two logical systems and the identity between their set of valid inferences or, equivalently, the disagreement between two logical systems and the difference between their set of valid inferences.

It could be possible, in principle, to challenge this conception by claiming that a coincidence with regard to the set of valid inferences of two logics *need not be a necessary condition* for their identity. Thus, two logics formulated on the same language and having different sets of valid inferences could be, perhaps, judged to be the same logic. But, even if it is interesting to see where this route takes us, it does not look like it will render a very promising (let alone coherent) account of the identity of logical systems. For what it is worth, such an alternative would require a criterion to tell apart logics with different set of valid inferences which are identical, from logics with different set of valid inferences which are nevertheless not identical. In absence of such a criterion—and, for what it is worth, of any example motivating the reasonableness of this option—this seems a very intricate task, if not an impossible one. The upshot of these reflections seems to be, therefore, that for two logics formulated on the same language to be identical it is indeed *necessary* for them to have the same set of valid inferences.[3]

However controversial the rejection of the necessary condition of the Tarskian criterion of identity for logical systems appears to be, it seems that questioning the sufficient condition of such an account is not as contentious. In fact, claiming that a coincidence with regard to the set of valid inferences of two logics *need not be a sufficient condition* for their identity suggests, in itself, a very interesting route for the pluralist to escape the Collapse Argument—in its present form. Abandoning this idea opens the way for a pluralist to endorse different logics with the same valid inferences, without succumbing to Logical Monism. The main reason being that there will be no inference whatsoever according to which these logics disagree, whence the absence of a disagreement that needs to be solved by means of this argument.

Footnote 2 continued
not to accept the conclusion of a given argument." (Stei 2017, p. 18). Be that as it may, for the sake of simplicity we will be taking pluralists in this paper not to be meaning-variance pluralists, i.e. we will take pluralists to consider the logics they embrace to be genuinely rivals to each other.

[3] However, perhaps such a criterion is in the end available. It could be possible, in principle, to identify logical systems which have the same set of theorems or valid inferences, regardless of them having different set of valid inferences. This would render, e.g. that Classical Logic and Graham Priest's Logic of Paradox from Priest (1979)—which, among other classically valid inferences, invalidates Modus Ponens—are after all the same system. Although it is true that in the past scholars have played with this idea, as documented e.g. in Paoli et al. (2008, p. 1), that logics are nowadays studied in connection with *logical consequence* and not with *logical truth* explains why it seems appropriate to consider the Tarskian point of view to be, at least, a necessary condition for an appropriate identity criterion for logical systems. We would like to thank an anonymous reviewer for discussion on this matter.

But is this actually plausible? Are there any interesting examples of non-identical logical systems coinciding with regard to their inferential validities? In the next section we argue that it is, providing an example coming from the philosophical literature on paradoxes. More particularly, from the literature on the substructural approaches to the paradoxes.

3 Why refine?

In this section we show the case of a pair of logics which have the same set of valid inferences, but which can nonetheless be legitimately regarded as non-identical.

Our case study will be represented by **CL** and the non-transitive system **ST**. This last system was presented and advocated by Pablo Cobreros, Paul Egré, David Ripley and Robert van Rooij in a number of recent publications, with the aim of solving paradoxes coming from the semantic, set-theoretic and vagueness corners.[4] When we say that the system in question is *non-transitive* we mean by this that the structural property of Cut fails, i.e. that it cannot be applied unrestrictedly.[5] Its advocates claim that this failure is not as critical as some may think and that, much to the contrary, their preferred system enjoys many virtues that outweigh this peculiarity—among them, the ability to handle non-trivially and in a rather classical manner the Liar, Curry, and all of the aforementioned paradoxes. In any case, it is important to highlight that given these facts, **ST** constitutes a *substructural* logic. That is, a logic where not all the commonly assumed structural properties that characterize a Tarskian logic hold.

In order to show that the collection formed by **CL** and **ST** witnesses the case of two different logics sharing the same valid inferences we need to fix some terminology. For $\mathcal{L}$ a propositional language and Var a countably infinite set of propositional variables, we denote by $\mathbf{FOR}(\mathcal{L})$ the absolutely free algebra of formulae of $\mathcal{L}$, whose universe is $FOR(\mathcal{L})$. In what follows, and merely for matters of simplicity, the logical connectives of the propositional language will be assumed to be those in the set $\{\neg, \wedge, \vee\}$. An inference $\Gamma \Rightarrow \Delta$ on $\mathcal{L}$ is an ordered pair $\langle \Gamma, \Delta \rangle$, where $\Gamma, \Delta \subseteq FOR(\mathcal{L})$. We denote by $SEQ^0(\mathcal{L})$ the set of all inferences on $\mathcal{L}$. With this in mind, we need to explain how inferential validity is defined in these systems.[6] To do this, attention has to be drawn to the Strong Kleene algebra, i.e. the structure

$$\mathbf{K} = \left\langle \left\{ 1, \frac{1}{2}, 0 \right\}, \{ f_{\mathbf{K}}^{\neg}, f_{\mathbf{K}}^{\wedge}, f_{\mathbf{K}}^{\vee} \} \right\rangle$$

4 A non-exhaustive list of some works of this collective where **ST** is discussed is Cobreros et al. (2012), Ripley (2012) and Cobreros et al. (2014).

5 This property is usually expressed by saying that for all $\Gamma, \Delta \subseteq FOR(\mathcal{L})$ and $A \in FOR(\mathcal{L})$

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}.$$

6 Thus, as a general remark, let us point out that for the sake of simplicity our discussion here will be focusing on propositional, and not e.g. first-order, logics. However, all the conceptual points are applicable, without much effort, to systems of the latter sort.

where the functions $f_{\mathbf{K}}^{\neg}$, $f_{\mathbf{K}}^{\wedge}$, $f_{\mathbf{K}}^{\vee}$ are as follows

	$f_{\mathbf{K}}^{\neg}$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$f_{\mathbf{K}}^{\wedge}$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$f_{\mathbf{K}}^{\vee}$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

Moreover, the functions $\supset$ and $\leftrightarrow$ are definable via the usual definitions. That is, $A \supset B =_{def} \neg A \vee B$ and $A \leftrightarrow B =_{def} (A \supset B) \wedge (B \supset A)$.

With the help of the Strong Kleene algebra, valuations can be defined over the entire language, in the following ways. A Strong Kleene valuation (SK-valuation, hereafter) is a homomorphism from $\mathbf{FOR}(\mathcal{L})$ to $\mathbf{K}$. A Boolean valuation is a SK-valuation whose range is $\{1, 0\}$.

This allows, moreover, to define logical validity in **CL** and **ST**, precisely as follows. A Boolean valuation v satisfies an inference $\Gamma \Rightarrow \Delta$ in **CL** if and only if it is not the case that $v(A) = 1$ for all $A \in \Gamma$ and $v(B) = 0$ for all $B \in \Delta$. We denote this fact by $v \vDash_{\mathbf{CL}} \Gamma \Rightarrow \Delta$. As usual, an inference is valid in **CL** if and only if it is satisfied by all Boolean valuations, and we denote this by $\vDash_{\mathbf{CL}} \Gamma \Rightarrow \Delta$. Similarly, a SK-valuation v satisfies an inference $\Gamma \Rightarrow \Delta$ in **ST** if and only if it is not the case that $v(A) = 1$ for all $A \in \Gamma$ and $v(B) = 0$ for all $B \in \Delta$. We write this as $v \vDash_{\mathbf{ST}} \Gamma \Rightarrow \Delta$. Finally, an inference is valid in **ST** if and only if it is satisfied by all SK-valuations, something we symbolize as $\vDash_{\mathbf{ST}} \Gamma \Rightarrow \Delta$.

According to the previously referred authors, the main advantage of working with **ST** lies in, allegedly, keeping **CL** as the underlying inferential framework even in what pertains the problematic paradoxical phenomena. In other words, **CL** and **ST** coincide with regard to their set of valid inferences—as proved e.g. by Girard (1987) and Cobreros et al. (2012).

Fact 1 *For all* $\Gamma, \Delta \subseteq FOR(\mathcal{L})$*:*

$$\vDash_{\mathbf{ST}} \Gamma \Rightarrow \Delta \quad \textit{if and only if} \quad \vDash_{\mathbf{CL}} \Gamma \Rightarrow \Delta$$

Thus, given the present form of the Collapse Argument, a pluralist which holds that both **CL** and **ST** are correct logical systems, seems to be untouched by the anti-pluralist critique. The main reason is that a pluralist that accepts both of these systems will never find herself in a position where there is an inference that is deemed as valid by one system but not by the other. Whence, she will never have to give a controversial answer to a question in the spirit of the one raised by the quotes in the previous section. Therefore, her stance regarding logical validity will never collapse into one of these systems in detriment of the other—or into any system different from these two, for what it is wort. Thus, the case of these two systems requires of a sincere anti-pluralist to face the difficulty of coming up with a way to accommodate them.

One option for the anti-pluralist could be to insist in claiming that the above fact shows nothing more than the identity of **CL** and **ST**.[7] However, there is something

[7] This seems what Cobreros, Egré, Ripley and van Rooij tend to say, from time to time, although without having in mind the discussion about Logical Pluralism

deeply uncomfortable about such a claim: **CL** is prone to trivialization when faced with transparent truth, vague phenomena and much more, while **ST** does not fall into such troubles. Hence, it seems that these systems are not identical, even if this in itself does not suggest a criterion to tell them apart.

Advocates of the sameness of these two logics may claim that this does not disprove the identity of **CL** and **ST**, but highlight that they are different *modes of presenting* the same logic, i.e. Classical Logic.[8] One by means of two-valued models or valuations, the other by means of three-valued models or valuations. But, then again, this will imply that which consequences can be drawn from an arbitrary piece of content or a theory by closing it under Classical Logic—and, as we will show below, which metainferences are valid—essentially depends on which presentation of Classical Logic we choose to work with. However, not only does this sound odd in connection with identity criterion for logical systems, but it also has strange consequences regarding Logical Pluralism and Logical Monism. Individuals may have a Monist or Pluralist attitude about different logical systems, and they may have an independent an orthogonal Monist or Pluralist stance towards the various modes in which these logics can be presented.

In any case, our previous remark—that triviality ensues when adding transparent truth to **CL** and the difference that this makes with regard to **ST**—would not point out the difference between **CL** and **ST** but between the theories closed under each of these logics, or these modes of presentation of the same logic. This line of thought (as suggested by an anonymous reviewer) would have it such that, although **CL** and **ST** are one and the same logic, closing arbitrary pieces of content under them may lead to different outcomes. With this we cannot but disagree, for we see things completely different. In fact, closing arbitrary theories under numerically different logical systems provides a way—a rather indirect way, but a way nonetheless—of evaluating their strength, of ruling which logics draw more or which draw less consequences, etc. As a limit case, if two logics render exactly the same consequences when closing any arbitrary piece of content under them, we think it is fair to assume that, by all extents and purposes, they are the same logic.[9] This is not the case with **CL** and **ST** and that is why we think it is necessary to have a more *direct* way of identifying and differentiating them, by refining the Tarskian identity criterion.

Given these issues, the remaining option for the anti-pluralist is to develop an identity criterion for logical systems which takes her beyond the standard Tarskian account, and which helps to differentiate systems coinciding with regard to their valid inferences. Saying this is the same as saying that the anti-pluralist is in need of way of refining the identity criterion for logical systems and, therefore, the Collapse Argument itself. In the next section, we show how such an anti-pluralist might proceed in doing so.

[8] We would like to thank an anonymous reviewer for stressing this option.

[9] An option along these lines has been both suggested and explored in what pertains to Tarskian logics—particularly, Classical and Intuitionistic Logic—in Woods (2018).

4 Refining

In order to refine the Collapse Argument, the anti-pluralist needs to come up with a new identity criterion for logical systems that can, hopefully, tell **CL** and **ST** apart. In this vein, a versed reader in the recent literature on semantic paradoxes might point to a very promising clue to solving this riddle.

A solution may be found by reflecting upon the set of metainferences that are valid in each of these systems. To put it intuitively, a metainference is an inference between inferences. More formally, a metainference $\Theta \Rightarrow_1 B$ on $\mathcal{L}$ is an ordered pair $\langle \Theta, B \rangle$, where $\Theta \subseteq SEQ^0(\mathcal{L})$ and $B \in SEQ^0(\mathcal{L})$. We denote, analogously, by $SEQ^1(\mathcal{L})$ the set of all metainferences on $\mathcal{L}$. To illustrate this we can say that, from the following, the one on the left is a formula, the one in the middle is an inference, and the one on the right is a metainference.[10]

$$p \vee \neg p \qquad\qquad p, \neg p \Rightarrow q \qquad\qquad \frac{\emptyset \Rightarrow p \quad \emptyset \Rightarrow \neg p \vee q}{\emptyset \Rightarrow q}$$

The idea would be, in a nutshell, that if these logics agree with respect to their valid inferences but disagree with regard to their valid metainferences, then the anti-pluralist may design an identity criterion for logical systems that cashes out this divergence. This would, eventually, help in rehashing the Collapse Argument in such a way that a pluralist embracing **CL** and **ST** cannot escape it.

Thus, let us have a look at how metainferences are determined to be valid or invalid in the systems we are discussing. The answer is, we think, very much straightforward—although we make room for some debate in this regard, below. The following definitions, borrowed from Dicher and Paoli (2018), say that a Boolean valuation v satisfies a metainference $\Theta \Rightarrow_1 B$ in **CL** if and only if $v \nvDash_{\mathbf{CL}} \theta$ for some $\theta \in \Theta$, or $v \vDash_{\mathbf{CL}} B$, which we write $v \vDash_{\mathbf{CL}} \Theta \Rightarrow_1 B$. We consequently say that a metainference is valid in **CL** if and only if it is satisfied by all Boolean valuations, symbolizing it as $\vDash_{\mathbf{CL}} \Theta \Rightarrow_1 B$. Similarly, we say that a SK-valuation v satisfies a metainference $\Theta \Rightarrow_1 B$ in **ST** if and only if $v \nvDash_{\mathbf{ST}} \theta$ for some $\theta \in \Theta$, or $v \vDash_{\mathbf{ST}} B$, denoting it by $v \vDash_{\mathbf{ST}} \Theta \Rightarrow_1 B$. We concomitantly say a metainference is valid in **ST** if and only if it is satisfied by all SK-valuations, which we write as $\vDash_{\mathbf{ST}} \Theta \Rightarrow_1 B$.

It should be noticed that the semantic understanding of metainferential validity that we chose to work with in the above definitions, and in the rest of the paper, is the one referred to as *local* in Humberstone (1996). In this regard, it is interesting to tell some kind of story that explains why we decided to go with this and not with another reading. To this extent, we would like to observe that, as long as metainferences are taken to be inferences of a special kind—i.e. inferences between a collection of inferences, and a single inference—adopting the local criterion allows to apply the same standard for regular inferences (between collections of formulae) and metainferences. When we apply the usual criterion for validity of regular inferences we require that, if the

[10] It shall be noted that, according to our definition, inferences are multiple-conclusion whereas metainferences (and, below, generalized metainferences of any arbitrary level) are single-conclusion. This is completely inessential and, for all that matters, we could have presented things in a unified multiple-conclusion way. If we did not do so, it is just for the sake of keeping this as simple as possible.

premises are satisfied according to the standard for premise-formulae to be satisfied in the logic in question, then at least one of the conclusions are (in the single-conclusion case: the conclusion is) satisfied according to the standard for conclusion-formulae to be satisfied in the said logic. Similarly, when we apply the local criterion for metainferential validity we require that, if the premises are satisfied according to the standard for inferences to be satisfied—i.e. not to be invalidated—in the logic in question, then the conclusion is are satisfied according to the standard for inferences to be satisfied in the target logic.

Logic is a discipline concerned, among other things, with validity. But were we to adopt something other than the local understanding of metainferential validity, this would mean that validity is something that should be studied one way when it concerns formulae and another way when it concerns inferences. Something that, we think, seems rather undesirable. Having a unified stance towards validity, regardless of its relata, seems like a more promising and interesting endeavour and this is why we think the local reading is the way to go here, and in what follows of the paper. This being said, there are in fact alternative semantic notions of metainferential validity in the literature—e.g. in Humberstone (1996) and Dicher and Paoli (2018)—and nothing that we said should be taken to argue that they should be dismissed. But an exploration of the applicability of such notions to the debate on Logical Pluralism, concerning specially substructural logics, will take us too far afield.[11]

In light of these remarks, it can be noticed that there is an actual interesting difference regarding **CL** and **ST**—namely, that concerning the metainferences valid in them. In fact, as previously advertised, the latter is essentially non-transitive, that is to say, it invalidates the Cut rule. Consequently, as pointed out e.g. in Barrio et al. (2015), there are many metainferences which are valid in **CL** but not in **ST**.

Fact 2 *There are some* $\Theta \subseteq SEQ^0(\mathcal{L})$ *and* $B \in SEQ^0(\mathcal{L})$ *such that:*

$$\nvDash_{\mathbf{ST}} \Theta \Rightarrow_1 B \quad \textit{and} \quad \vDash_{\mathbf{CL}} \Theta \Rightarrow_1 B$$

Among others, as a quick inspection of the definitions shows, the following instances of Cut and of the metainference called meta Explosion by Barrio et al. (2015), witness the aforementioned disparity.

$$\frac{p, q \Rightarrow r, s \quad p, q, r \Rightarrow s}{p, q \Rightarrow s} \qquad \frac{\emptyset \Rightarrow p \quad \emptyset \Rightarrow \neg p}{\emptyset \Rightarrow q}$$

Let us highlight, in passing, that the case of logics like **CL** and **ST** exhibits an interesting phenomenon which was unnoticed before. For instance, as long as **ST** accepts Explosion at the inferential level but rejects Meta-Explosion—e.g., a metainference that can be considered a metainferential version of Explosion—at the metainferential level, it can be said that **ST** has a *mixed* policy with regard to some rules.[12] On the

[11] We are thankful with an anonymous reviewer for asking us to clarify our choice in this regard.

[12] Let us emphasize that there is some connection between Explosion and Meta-Explosion—which can be made precise and which we make precise in what follows—as there also is, in general, between an inference and a metainference of a certain form. We do not want to claim, though, that this connection is more than

other hand, the logics on which the debate was focused until now had a *uniform* policy in this regard, i.e. when asked about a certain rule the systems in question always had the same answer, regardless of this rule being formulated at the inferential or the metainferential level. Thus, the pluralist may claim that logics like **ST** allow for a certain pluralism within themselves—this being the source of the possibility to escape the present form of the Collapse Argument.

Apart from this, though, the anti-pluralist may be tempted to claim that **CL** is not identical to **ST** because, even if they coincide regarding their valid inferences, they do not agree with respect to their valid metainferences. Thus, it could be possible to devise a new and extended identity criterion for logics saying that two logics formulated on the same language are *identical* if and only if they have the same set of valid inferences, and the same set of valid metainferences. It could also be possible to extend this conclusion to our conception of when two logics are rivals, claiming that there is a *rivalry* between two logics if and only if they are not identical with regard to the novel identity criterion that we just outlined.

In this regard, **CL** and **ST** will become rival logics and, hence, the anti-pluralist can easily reinstantiate the Collapse Argument—in a slightly *refined* form. Thus, to follow Priest, suppose the pluralist embraces two logics which agree with respect to their inferences but do not agree with regard to their metainferences. Suppose, moreover, that the pluralist believes all the premises of a given metainference that is valid according to one logic but not the other. Is she or is she not entitled to accept the conclusion of the given metainference? There can be no pluralism about this. Whence, collapse ensues.

One may wonder whether there is a genuine asymmetry between the usual form of the Collapse Argument and the present rejoinder. While in its usual form the argument starts from belief in some sentences and asks whether belief in some other sentence is justified—where these beliefs are not relative to any particular logical system, but *simpliciter*—the refined version of the argument just presented starts from beliefs in inferences themselves and asks whether belief in some other inference is justified—where these beliefs, the objection goes, seem to be *relative* to a particular logical system.

However, the entire debate around Logical Pluralism assumes that when two logics are taken to be rival because they contend that a certain inference is valid, the dispute is over the validity *simpliciter* of the said inference and not over its validity relative to the corresponding logics. In fact, concerning this last technical issue both parties can certainly agree. If they genuinely disagree at all, as is again assumed in the literature,

Footnote 12 continued
a *translation* of the corresponding inference (or inference scheme) to a metainference (or metainference scheme). In particular, we do not want to claim that the inference and the metainference are strictly speaking different incarnations of the same idea. We merely point out a certain resemblance to it. This being said, we can transform an (single-conclusion) inference $\Gamma \Rightarrow A$ in its corresponding metainferential form by taking the premise-set of the metainference to be the set of inferences $\{\emptyset \Rightarrow B \mid B \in \Gamma\}$, and the conclusion to be the singleton $\{\emptyset \Rightarrow A\}$. Similarly, we can go from a given metainference to its inferential form, but for the sake of brevity we will postpone this explanation to footnote 4, for it will be subsumed by a general technique that allows to transform what we call generalized metainferences of level n into generalized metainferences of level $n + 1$, and viceversa. We would like to thank an anonymous reviewer for urging us to clarify these matters.

then there is some sense in which it can be meaningfully thought that an inference is valid (or invalid) without that being, in principle, a question relative to any particular logical system. This point can be straightforwardly applied to other more complex forms of inferences such as metainferences—and, as we will discuss below, to generalized metainferences of any arbitrary level. For instance, when scholars disagree about whether or not Cut is a valid metainference, they are not disagreeing about whether Cut is valid in e.g. **CL** or **ST**, but rather about its validity, independent of any technical calculations that can be made within a certain system. For these reasons, we take the modified version of the Collapse Argument to be, properly speaking, a legitimate refinement thereof.[13]

Now, the anti-pluralist may think that, at this point, the need to refine the Collapse Argument is satisfied. Unfortunately, the news are not that good. Technically speaking, it is fairly straightforward to see how it is possible to design a logic that has the same inferential and metainferential validities as **CL**—namely, the logic **TSST** presented e.g. in Barrio et al. (2018)—but which is, nevertheless, not identical to **CL**.

To define validity in **TSST**, we need to make a detour and first define validity for another logic, called **TS** in French (2016). Before moving on, let us mention that **TS** is a highly peculiar logic, for it is *non-reflexive*. Meaning by this, that the structural property of Reflexivity is not unrestrictedly valid in the system in question.[14] French argues that despite this potentially shocking feature, the system he puts forward is very much interesting and fruitful in treating the paradoxes, just as the non-transitive logic **ST**. Once again, like **ST**, the logic **TS** is also a *substructural* logic.

Moving now to the semantics, we say that a SK-valuation v satisfies an inference $\Gamma \Rightarrow \Delta$ in **TS** if and only if it is not the case $v(A) \in \{1, \frac{1}{2}\}$ for all $A \in \Gamma$ and $v(B) \in \{\frac{1}{2}, 0\}$ for all $B \in \Delta$, which we denote by $v \vDash_{\mathbf{TS}} \Gamma \Rightarrow \Delta$. An inference is valid in **TS** if and only if it is satisfied by all SK-valuations, and we write this as $\vDash_{\mathbf{TS}} \Gamma \Rightarrow \Delta$. Moving now to **TSST**, we say that a SK-valuation v satisfies a metainference $\Theta \Rightarrow_1 B$ in **TSST** if and only if $v \nvDash_{\mathbf{TS}} \theta$ for some $\theta \in \Theta$, or $v \vDash_{\mathbf{ST}} B$, which we symbolize as $v \vDash_{\mathbf{TSST}} \Theta \Rightarrow_1 B$. And we, consequently, say that a metainference is valid in **TSST** if and only if it is satisfied by all SK-valuations, denoting it by $\vDash_{\mathbf{TSST}} \Theta \Rightarrow_1 B$.

Interestingly, as shown in Barrio et al. (2018), **TSST** is a fairly classical system, in that it respects all the classically valid metainferences—in addition to respecting all the classically valid inferences.[15]

Fact 3 *For all* $\Theta \subseteq SEQ^0(\mathcal{L})$ *and* $B \in SEQ^0(\mathcal{L})$*:*

$$\vDash_{\mathbf{TSST}} \Theta \Rightarrow_1 B \quad \textit{if and only if} \quad \vDash_{\mathbf{CL}} \Theta \Rightarrow_1 B$$

[13] We would like to thank an anonymous reviewer for bringing this issue up to us.

[14] This property is usually expressed by saying that for all $A \in FOR(\mathcal{L})$: $A \Rightarrow A$.

[15] One may ask, as an anonymous reviewer does, why is it that the latter holds. After all, as defined **TSST** only gives a way to evaluate metainferences. The answer is that it is possible to recast an inference $\Gamma \Rightarrow \Delta$ as a metainference $\emptyset \Rightarrow_1 \{\Gamma \Rightarrow \Delta\}$, i.e. a metainference whose premise-set is empty and whose conclusion-set is the inference that we want to recast. This illustrates why two logics having the same valid metainferences need to coincide, too, with regard to their set of valid inferences. In a similar fashion, a formula A can be redescribed as an inference $\emptyset \Rightarrow A$, which explains why two logical systems having the same set of valid inferences need to have the same set of theorems or valid formulae. This can be applied without loss of generality to all inferences, allowing to read them as metainferences and—as we will show—to all the objects that we will call generalized metainferences, below.

Thus, in presence of a pluralist that accepts both **CL** and **TSST**, not only the Collapse Argument has no force, but even the previously sketched refinement of the Collapse Argument has not. The main reason is that a pluralist accepting both **CL** and **TSST** will never find herself in a position where there is an inference or a metainference deemed valid by one system but not by the other. Whence, she will never have to give a controversial answer to a question in the spirit of the one raised by the quotes in the previous section and, therefore, her stance regarding inferential or metainferential logical validity will never collapse.

Hence, once again, a case like this requires of a sincere anti-pluralist and advocate of the Collapse Argument to step up in order to think a way of accommodating these systems. Once more, this highlights the need to modify both the identity and the rivalry criteria for logical systems, hoping that this will help to refine the Collapse Argument even more, in order to face pluralists embracing logics with the same set of valid metainferences—like **CL** and **TSST**.

This is in fact possible, if we consider not only valid inferences and valid metainferences, but also valid metametainferences—or metainferences of level 2—in giving identity and rivalry criteria for logical systems. To put it intuitively, a metametainference is an inference between metainferences. Speaking more formally, a metametainference $\Xi \Rightarrow^2 C$ on $\mathcal{L}$ is an ordered pair $\langle \Xi, C \rangle$, where $\Xi \subseteq SEQ^1(\mathcal{L})$ and $C \in SEQ^1(\mathcal{L})$. We denote, analogously, by $SEQ^2(\mathcal{L})$ the set of all metametainferences on $\mathcal{L}$. To illustrate this definition, we depict below a metametainference—in fact, an instance of a metametainferential formulation of the Cut rule.

$$\frac{\dfrac{\Sigma \Rightarrow \Pi}{\Gamma \Rightarrow p, \Delta} \quad \dfrac{\Sigma \Rightarrow \Pi}{\Gamma, p \Rightarrow \Delta}}{\dfrac{\Sigma \Rightarrow \Pi}{\Gamma \Rightarrow \Delta}}$$

As expected, there is a proper difference regarding **CL** and **TSST**—namely, that concerning the metametainferences valid in them.

Fact 4 *There are some* $\Xi \subseteq SEQ^1(\mathcal{L})$ *and* $C \in SEQ^1(\mathcal{L})$ *such that:*

$$\not\models_{\mathbf{TSST}} \Xi \Rightarrow_1 C \quad \textit{and} \quad \models_{\mathbf{CL}} \Xi \Rightarrow_1 C$$

In fact, among others, the following instances of the metametainferential formulations of Explosion and Modus Ponens witness the previously referred difference.

$$\frac{\dfrac{\Sigma \Rightarrow \Pi}{\Gamma \Rightarrow p, \Delta} \quad \dfrac{\Sigma \Rightarrow \Pi}{\Gamma \Rightarrow \neg p, \Delta}}{\dfrac{\Sigma \Rightarrow \Pi}{\Gamma \Rightarrow q, \Delta}} \qquad \frac{\dfrac{\Sigma \Rightarrow \Pi}{\Gamma \Rightarrow p, \Delta} \quad \dfrac{\Sigma \Rightarrow \Pi}{\Gamma \Rightarrow p \supset q, \Delta}}{\dfrac{\Sigma \Rightarrow \Pi}{\Gamma \Rightarrow q, \Delta}}$$

In order to cash out this divergence, the novel identity criterion will say that two logics formulated on the same language are *identical* if and only if they have the same set of valid inferences, the same set of valid metainferences, and the same set of valid metametainferences. The accordingly modified *rivalry* criterion would say that two logics formulated on the same language are rivals when they are not identical, in the previously clarified sense.

Once more, it can be noted that there is an actual difference regarding **CL** and **TSST**. A difference, that is, concerning the metametainferences valid in each of them. As pointed out in Barrio et al. (2018), there are many metametainferences which are valid in **CL** but not in **TSST**.

Fact 5 *There are some* $\Xi \subseteq SEQ^1(\mathcal{L})$ *and* $C \in SEQ^1(\mathcal{L})$ *such that:*

$$\not\models_{\mathbf{TSST}} \Xi \Rightarrow_2 C \quad \textit{and} \quad \models_{\mathbf{CL}} \Xi \Rightarrow_2 C$$

Thus, **CL** and **TSST** will become rival logics and, thanks to this, the anti-pluralist can easily reinstantiate the Collapse Argument—in an ever more *refined* form. Hence, suppose the pluralist embraces two logics which agree with respect to their inferences, their metainferences, but do not agree with regard to their metametainferences. Suppose, moreover, that the pluralist believes all the premises of a given metametainference that is valid according to one logic but not the other. Is she or is she not entitled to accept the conclusion of the given metametainference? Once more, there can be no pluralism about this. Whence, collapse ensues.

Interestingly enough, we do not think that this is the end of the story and, at this point, the reader may anticipate why. As a matter of fact, just as we can define inferences, metainferences and metametainferences, it is possible to given an abstract definition of what we call a *generalized* metainference, i.e. a metainference of a given "level" which holds between inferences of the immediately previous levels. This can be made precise in the following way. A generalized metainference $\Omega \Rightarrow^n D$ of level n on $\mathcal{L}$ (for $1 \leq n < \omega$) is an ordered pair $\langle \Omega, D \rangle$, where $\Omega \subseteq SEQ^{n-1}(\mathcal{L})$ and $D \in SEQ^{n-1}(\mathcal{L})$. $SEQ^n(\mathcal{L})$ is the set of all metainferences of level n on $\mathcal{L}$. By looking at the metainferences and metametainferences above (which are, under this redescription, generalized metainferences of level 1 and 2, respectively), the reader can produce further examples of metainferences of arbitrary large levels, just by using her imagination.[16]

Thus, the main reason why the Collapse Argument in its present form has not the force it is supposed to have is the following. For each identity criterion that we might provide—stating that two logical systems on the same language are identical if they have the same valid inferences, the same valid metainferences, ..., and the same valid metainferences of level n—it is possible to design a logic which coincides with **CL** in that regard. The general construction, which outputs such a hierarchy of classical-like logics can be found in Barrio et al. (2018). Given this, then, a pluralist embracing such

[16] Clarification was asked, by an anonymous reviewer, concerning how a generalized metainference of level n is matched with a corresponding generalized metainference of level $n+1$—and, additionally, we might point out, it is interesting to know how to match a generalized metainference of level $n+1$ with its counterpart of level n. The answer is, in a nutshell, through the following translation functions. The latter task, i.e. how to go from $n+1$ to n, is achieved with the help of the $lower$ function described in Barrio et al. (2018) so that:

- $lower(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \supset \bigvee \Delta$
- for $1 \leq n$, $lower(\Gamma \Rightarrow_n A) = \{lower(\gamma) \mid \gamma \in \Gamma\} \Rightarrow_{n-1} lower(A)$

Whereas the former task can be achieved, correspondingly, with the help of one of the inverses of the $lower$ function—highlighting, therefore, that there is no unique way to go from a generalized inference of level n to one of level $n+1$, but that there is always a way to find at least one counterpart of this sort.

a system and **CL** could, in principle, avoid any version of the Collapse Argument that is refined enough to cope with logics that agree *up to that extent*.[17]

Therefore, the anti-pluralist is in need of a much more fine-grained account of the identity between logics and, therefore, of the rivalry between logics. In need of criteria, that is, that allow her to formulate an absolutely general and maximally refined version of the Collapse Argument. Luckily for someone embracing such a position, we think this is in fact possible.

To arrive at such definitions, we should first notice that each logic of the hierarchy described in Barrio et al. (2018), in addition to agreeing with **CL** up to a certain metainferential level n, is such that it disagree with **CL** concerning the metainferential levels $m > n$. This, suggest a very obvious identity criterion for logics, stating that two logics formulated on the same language are *identical* if and only if they have the same set of valid inferences, and the same set of valid generalized metainferences of all level—i.e. if they have the same set of valid inferences and the same set of valid metainferences of level n, for all $n \in \omega$. In this vein, it could be said that there is *rivalry* between two logics if and only if they are not identical with regard to the fully generalized criterion of identity detailed before. As a consequence of this, not only **CL** and **ST** will be judged as rival logics, but also **CL** and **TSST**, as well as **CL** and any logic that does not coincide with it regarding all the inferences and metainferences that it is possible to think about.

Finally, this allows to present the Collapse Argument in its strongest and most general form, as follows. Suppose the pluralist embraces two logics that are not identical, such that there is at least a certain metainferential level n with regard to which their set of valid metainferences are not the same. Suppose, additionally, that the pluralist believes all the premises of a given metainference of level n that is valid according to one logic but not the other. Is she or is she not entitled to accept the conclusion of the given metainference? There *really* can be no pluralism about this. Faced with the Collapse Argument formulated in this way, it very much seems that the pluralist cannot get away using any of the previously discussed strategies.

5 Concluding remarks

In this paper we have revisited the Collapse Argument against Logical Pluralism, which intends to show that such a position boils down to Logical Monism—when properly understood. We have argued that given the existence of substructural logics which coincide with Classical Logic with regard to its valid inferences, the anti-pluralist needs to devise a strategy to refine her argument. In the same vein, we have argued that given the existence of logics which coincide with Classical Logic with regard to its valid metainferences up to a certain level, an appropriate refinement of the Collapse

[17] As should be clear by the fact that, in what follows, we present a form of the Collapse Argument that takes into account this novel identity criterion for logical systems, we do not mean to propose such a criterion as a defence of Logical Pluralism. Instead, our main goal throughout the paper was to highlight that regardless of where one stands concerning the pluralism debate, a refined criterion is a necessary conceptual tool to have a clear understanding of the debate—e.g. in order to differentiate logical systems that, intuitively, ought to be differentiated. Whether one is a Pluralist or a Monist, we think, should be decided by reasons orthogonal to the kind of identity criterion that one adopts.

Argument that is able to deal with these cases with full generality is needed. In order to do this, we proposed an identity criterion for logical systems according to which two logics formulated on the same language are identical if and only if they have the same set of valid inferences and of valid metainferences of all levels. This renders an appropriate criterion of rivalry between logics, which helps design a much stronger version of the argument for the collapse of Logical Pluralism. Whether or not it is possible for the pluralist to escape this version of the argument by means of rejecting some other premises—like the widely assumed normativity of logic, as discussed in Russell (2017)—is a question for another occasion, and one which we hope to explore in future research.

Acknowledgements The material included in this paper has been presented at the Formal Methods in Philosophy workshop held in Munich as a part of a DFG-MINCYT collaboration project between MCMP and the Buenos Aires Logic Group, and at the Kolloquium Logik und Erkenntnistheorie of the Ruhr-Universität Bochum. Thanks to the audiences of these events for their comments and helpful feedback. In addition, we are thankful to thank Francesco Paoli, and the members of the Buenos Aires Logic Group for discussing previous versions of this work. Finally, we would like to express our gratitude to two anonymous reviewers for this journal whose suggestions substantially improved the paper, and to the editors of special issue, Teresa Kouri Kissel and Colin Caret for their assistance through the editorial process.

References

Barrio, E., Pailos, F., & Szmuc, D. (2018). *A hierarchy of classical and paraconsistent logics*. Manuscript.

Barrio, E., Rosenblatt, L., & Tajer, D. (2015). The logics of strict-tolerant logic. *Journal of Philosophical Logic*, *44*(5), 551–571.

Beall, J., & Restall, G. (2006). *Logical Pluralism*. Oxford: Oxford University Press.

Bueno, O., & Shalkowski, S. (2009). Modalism and Logical Pluralism. *Mind*, *118*(470), 295–321.

Caret, C. R. (2017). The collapse of Logical Pluralism has been greatly exaggerated. *Erkenntnis*, *82*(4), 739–760.

Cobreros, P., Egré, P., Ripley, D., & van Rooij, R. (2012). Tolerant, classical, strict. *Journal of Philosophical Logic*, *41*(2), 347–385.

Cobreros, P., Egré, P., Ripley, D., & van Rooij, R. (2014). Reaching transparent truth. *Mind*, *122*(488), 841–866.

Dicher, B., & Paoli, F. (2018). ST, LP, and tolerant metainferences. In C. Başkent & T. M. Ferguson (Eds.), *Graham Priest on dialetheism and paraconsistency*. Dordrecht: Springer. Forthcoming.

French, R. (2016). Structural reflexivity and the paradoxes of self-reference. *Ergo*, *3*(5), 113–131.

Girard, J.-Y. (1987). *Proof theory and logical complexity*. Napoli: Bibliopolis.

Humberstone, L. (1996). Valuational semantics of rule derivability. *Journal of Philosophical Logic*, *25*(5), 451–461.

Humberstone, L. (2000). Contra-classical logics. *Australasian Journal of Philosophy*, *78*(4), 438–474.

Keefe, R. (2014). What Logical Pluralism cannot be. *Synthese*, *191*(7), 1375–1390.

McCall, S. (1966). Connexive implication. *Journal of Symbolic Logic*, *31*(3), 415–433.

McCall, S. (2012). A history of connexivity. In D. Gabbay, F. Pelletier, & J. Woods (Eds.), Logic: A history of its central concepts, volume 11 of handbook of the history of logic (pp. 415–449). Amsterdam: North-Holland.

Meyer, R. K., & Slaney, J. K. (1989). Abelian logic from A to Z. In G. Priest, R. Routley, & J. Norman (Eds.), *Paraconsistent logic: Essays on the inconsistent*. München: Philosophia Verlag.

Paoli, F., Spinks, M., & Veroff, R. (2008). Abelian logic and the logics of pointed lattice-ordered varieties. *Logica Universalis*, *2*(2), 209–233.

Priest, G. (1979). The logic of paradox. *Journal of Philosophical logic*, *8*(1), 219–241.

Priest, G. (1999). Negation as cancellation, and connexive logic. *Topoi*, *18*(2), 141–148.

Priest, G. (2006). Logic: One or many? In J. Woods, & B. Brown (Eds.), Logical consequence: Rival approaches. Proceedings of the 1999 conference of the society of exact philosophy. Hermes, Stanmore.

Quine, W. V. (1986). *Philosophy of logic* (2nd ed.). Cambridge: Harvard University Press.
Read, S. (2006). Monism: The one true logic. In D. de Vidi & T. Kenyon (Eds.), *A logical approach to philosophy: Essays in memory of Graham Solomon*. Dordrecht: Springer.
Ripley, D. (2012). Conservatively extending classical logic with transparent truth. *Review of Symbolic Logic*, *5*(02), 354–378.
Russell, G. (2017). Logic isn't normative. *Inquiry*, 1–18. https://doi.org/10.1080/0020174X.2017.1372305.
Stei, E. (2017). Rivalry, normativity, and the collapse of Logical Pluralism. *Inquiry*, 1–22. https://doi.org/10.1080/0020174X.2017.1327370.
Wansing, H. (2016). Connexive logic. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy*. Stanford: Stanford University. Spring 2016 edition.
Williamson, T. (1987). Equivocation and existence. *Proceedings of the Aristotelian Society*, *88*, 109–127.
Woods, J. (2018). Intertranslatability, theoretical equivalence, and perversion. *Thought*, *7*(1), 58–68.