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On the degree of irreducible morphisms

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Abstract

We study the degree of irreducible morphisms in generalized standard convex components of the Auslander–Reiten quiver of an artin algebra with the property that paths with the same origin and end vertices have equal length. We call the components with this last property *components with length*. In particular, we give two criteria to determine wether the degree of such an irreducible morphism f is finite or infinite. One of them is given in terms of the compositions of f with non-zero maps between modules in the component. The other states that the left degree of an irreducible map f is finite if and only if $\operatorname{Ker} f$ belongs to the component. We apply our results to irreducible morphisms over artin algebras of finite representation type and over tame hereditary algebras.

Introduction

The notion of irreducible morphism, introduced by Auslander and Reiten, has played an important role in the study of the category $\operatorname{mod} A$ of finitely generated modules over an artin algebra A. The connection with the radical $\Re$ of this category is well known, and

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is given by the fact that a morphism between indecomposable modules is irreducible if and only if it lies in $\Re \setminus \Re^2$. It is then important to further study this radical, in order to get a better understanding of mod A. In particular, it is natural to look at the composition of irreducible morphisms. The composition of n irreducible morphisms belongs to $\Re^n$. An interesting question is when such a composition falls into $\Re^{n+1}$. A partial solution to this problem was given by Igusa and Todorov, who showed that the composition of n irreducible morphisms on a sectional path does not belong to $\Re^{n+1}$.

In order to answer this question Liu [13,14,16] introduced the notion of degree of an irreducible morphism, as follows.

Let A be an artin algebra and $f: X \to Y$ an irreducible morphism in mod A, with X or Y indecomposable. The *left degree* $d_l(f)$ of f is infinite, if for each integer $n \geqslant 0$, each module $Z \in \text{mod}\, A$ and each morphism $g \in \Re^n(Z, X) \setminus \Re^{n+1}(Z, X)$ we have that $fg \notin \Re^{n+2}(Z, Y)$. Otherwise the left degree of f is the least natural m such that there is an A-module Z and a morphism $g \in \Re^m(Z, X) \setminus \Re^{m+1}(Z, X)$ such that $fg \in \Re^{m+2}(Z, Y)$.

The *right degree* $d_r(f)$ of an irreducible morphism f is dually defined.

This notion has been very useful in the study of the components of the Auslander–Reiten quiver Γ_A of an artin algebra A.

We study the degree of irreducible morphisms in generalized standard and convex components Γ of Γ_A having the property that two paths in Γ having the same starting point x and ending point y have equal length, called length from x to y. We call the components with this last property *components with length*. Bongartz and Gabriel proved in [7] that the Auslander–Reiten quiver of a simply connected algebra of finite representation type is a component with length. We show that the convex directed components of a strongly simply connected algebra (not necessarily of finite representation type) are components with length.

We prove that the composition f of n irreducible morphisms in a generalized standard convex component with length Γ of Γ_A belongs to $\Re^{n+1}$ if and only if $f = 0$. We give two different characterizations of the irreducible morphisms in generalized standard convex components with length having finite left (right) degree. The first is given in terms of their compositions with non-zero maps between modules in Γ. Actually, we prove the following theorem.

Theorem A. *Let A be an artin algebra and Γ a generalized standard convex component of Γ_A with length. Let $f: X \to Y$ be an irreducible morphism with $X, Y \in \Gamma$. Then $d_l(f) = \infty$ if and only if $fg \neq 0$ for each non-zero morphism $g: M \to X$ with $M \in \Gamma$.*

Our second characterization allows us to know if the left degree of an irreducible morphism f in a component is finite or infinite, depending on whether Ker f belongs to the component. More precisely we prove the following result.

Theorem B. *Let A be an artin algebra, Γ a generalized standard convex component of Γ_A with length and $f: M \to N$ an irreducible epimorphism with $M, N \in \Gamma$. Then $d_l(f) = \infty$ if and only if* Ker $f \notin \Gamma$.

When we only assume that the component Γ is generalized standard one of the above implications is still true. More precisely, we have in this case that $d_l(f) < \infty$, provided $\operatorname{Ker} f \in \Gamma$.

We apply the above results to algebras of finite representation type proving the following theorem.

Theorem C. *Let A be an artin algebra of finite representation type and $f : M \to N$ an irreducible morphism between indecomposable A-modules. Then*:

If $f : M \to N$ is an epimorphism then $d_l(f) < \infty$.

Moreover, if Γ_A is a component with length, then:

If $f : M \to N$ is an epimorphism then $d_r(f) = \infty$.

The above results refer to the left degree of irreducible morphisms. Dual statements hold for their right degree (see Theorem 3.14).

Finally we give some applications and examples. First we use our results to determine almost all irreducible morphisms with finite degree in the directed components of hereditary algebras of type $\widetilde{E}_p$ and $\widetilde{D}_n$.

Then we use Liu's results in [13], to study the finiteness of the left degree of irreducible morphisms in the directed components of hereditary algebras of type $\widetilde{A}_{pq}$. The same results allow us to compute the degree of any irreducible morphism in the regular components of a tame hereditary algebra.

The paper is organized in the following way.

In Section 1 we give some preliminaries results and recall some definitions. In Section 2 we introduce the notion of component with length and study the relation with the convex directed components of simply connected algebras and of hereditary algebras. In Section 3 we prove Theorems A, B and C. Finally, in Section 4 we apply our results to tame hereditary algebras.

1. Preliminaries

Throughout this paper A will denote an artin algebra, $\operatorname{mod} A$ the category of finitely generated left A-modules and $\Re$ the Jacobson radical of $\operatorname{mod} A$ and k an algebraically closed field.

We denote by Γ_A the Auslander–Reiten quiver of A and by τ and τ^- the Auslander–Reiten translations DTr and TrD, respectively. We are not going to distinguish between an indecomposable module X in $\operatorname{mod} A$ and the corresponding vertex $[X]$ in Γ_A. By $\varepsilon(X)$ we denote the almost split sequence ending at the non-projective indecomposable module X and by $\alpha(X)$ the number of indecomposable summands of the middle term of $\varepsilon(X)$. We denote by $\varepsilon'(X)$ and $\alpha'(X)$ the dual notions, respectively. This is, $\varepsilon'(X)$ is the almost split sequence starting at the non-injective indecomposable module X and $\alpha'(X)$ is the number of indecomposable summands of the middle term of $\varepsilon'(X)$.

Now we recall some definitions and results from [13]. Let A be an artin algebra and $f : X \to Y$ an irreducible morphism in $\operatorname{mod} A$, with X or Y indecomposable. The *left degree* $d_l(f)$ of f is infinite, if for each integer $n \geqslant 0$, each module $Z \in \operatorname{mod} A$ and each morphism $g \in \Re^n(Z, X) \setminus \Re^{n+1}(Z, X)$ we have that $fg \notin \Re^{n+2}(Z, Y)$. Otherwise the left degree of f is the least natural m such that there is an A-module Z and a morphism $g \in \Re^m(Z, X) \setminus \Re^{m+1}(Z, X)$ such that $fg \in \Re^{m+2}(Z, Y)$.

The *right degree* $d_r(f)$ of an irreducible morphism f is dually defined.

If $f : X \to Y$ is an irreducible morphism between indecomposable modules, then it is enough to consider only indecomposable modules Z in the definition of left degree to prove that $d_l(f) = \infty$ (see [9, 2.1]).

A path $Y_n \to Y_{n-1} \to \cdots \to Y_1 \to Y_0 = Y$ in Γ_A is said to be *presectional* if for each i, $1 \leqslant i \leqslant n-1$, $Y_{i-1} = \tau Y_{i+1}$ implies that $Y_{i-1} \oplus \tau Y_{i+1}$ is a summand of the domain of the right almost split morphism for Y_i, or equivalently, $\tau^- Y_{i-1} = Y_{i+1}$ implies that $\tau^- Y_{i-1} \oplus Y_{i+1}$ is a summand of the codomain of the left almost split morphism for Y_i.

Next we recall some known definitions needed throughout the paper.

Let Γ be a component of Γ_A. Then Γ is *generalized standard* if $\Re^\infty(X, Y) = 0$ for all $X, Y \in \Gamma$, and Γ is *convex* if for every chain $X_0 \to X_1 \to \cdots \to X_{n-1} \to X_n$ of non-zero non-isomorphisms between indecomposable modules with $X_0, X_n \in \Gamma$, each X_i belongs to Γ for $i = 1, \ldots, n-1$. Finally, Γ is called *directed* if there is no sequence $M_0 \to M_1 \to \cdots \to M_n$ of non-zero non-isomorphisms between indecomposable A-modules with $M_0 = M_n$.

Given a directed component Γ of Γ_A, its *orbit graph* $O(\Gamma)$ has as points the τ-orbits $O(M)$ of the modules M in Γ. There exists an edge between $O(M)$ and $O(N)$ in $O(\Gamma)$ if there are $m, n \in Z$ and an irreducible morphism $\tau^m M \to \tau^n N$ or $\tau^n N \to \tau^m M$. The number of such edges equals $\dim_k \operatorname{Irr}(\tau^m M, \tau^n N)$ or $\dim_k \operatorname{Irr}(\tau^n N, \tau^m M)$, respectively, where $\operatorname{Irr}(X, Y) = \Re(X, Y)/\Re^2(X, Y)$ and $k = \operatorname{End}(X)/\Re(X, Y)$. A component Γ of Γ_A is of *tree type* if its orbit graph $O(\Gamma)$ is a tree.

Let A be a basic finite dimensional associative algebra (with unit) over the algebraically closed field k. Then $A \simeq kQ/I$ for some finite quiver Q and some admissible ideal I of the path algebra kQ, and the pair (Q, I) is called a presentation for A.

Let now (Q, I) be a connected bound quiver. A relation $\rho = \sum_{i=1}^m \lambda_i w_i \in I(x, y)$ is minimal if $m \geqslant 2$ and, for any non-empty proper subset $J \subset \{1, 2, \ldots, m\}$, we have $\sum_{j \in J} \lambda_i w_i \notin I(x, y)$. A walk in Q from x to y is a path of the quiver formed by Q and the formal inverses α^{-1} of the arrows $\alpha \in Q$. That is, it is a composition $\alpha_1^{\varepsilon_1} \alpha_2^{\varepsilon_2} \cdots \alpha_t^{\varepsilon_t}$ where α_i are arrows in Q and $\varepsilon_i \in \{1, -1\}$ for all i, with source x and target y. We denote by e_x the trivial path at x. Let $\sim$ be the least equivalence relation on the set of all walks in Q such that:

(a) If $\alpha : X \to Y$ is an arrow, then $\alpha^{-1}\alpha \sim e_x$ or $\alpha\alpha^{-1} \sim e_y$.
(b) If $\rho = \sum_{i=1}^m \lambda_i w_i$ is a minimal relation, then $w_i \sim w_j$ for all i, j.
(c) If $u \sim v$, then $wuw' \sim wvw'$ whenever these compositions make sense.

Let $x \in Q_0$ be arbitrary. The set $\pi_1(Q, I, x)$ of equivalence classes $\bar{u}$ of closed paths u starting and ending at x has a group structure defined by the operation $\bar{u}.\bar{v} = \overline{u.v}$. Since Q

is connected then this group does not depend on the choice of x. We denote it $\pi_1(Q, I)$ and call it the *fundamental group* of (Q, I).

A triangular algebra A is simply connected if, for any presentation $\pi_1(Q, I)$ of A, the fundamental group $\pi_1(Q, I)$ is trivial.

An algebra B is a convex subcategory of A if there is a full and convex subquiver Q' of Q such that $B = kQ'/(I \cap kQ')$. The algebra A is said to be *strongly simply connected* if any full convex subcategory of A is simply connected. (See [17].)

2. Components with length

In this section we introduce the concept of component with length. The notion of length of a walk appeared in the work of Bongartz and Gabriel [7] in a different context. Let w be a walk, $w = \alpha_1^{\varepsilon_1}\alpha_2^{\varepsilon_2}\cdots\alpha_t^{\varepsilon_t}$ where $\alpha_i \in Q_1$ and $\varepsilon_i \in \{1, -1\}$ for all i. Then we set $\ell(w) = \sum_{1=1}^{t} \ell(\alpha_i^{\varepsilon_i})$ where $\ell(\alpha_i) = 1$ for all i, while $\ell(\alpha_i^{-1}) = -1$ (see [7]).

Let us recall that paths in Γ_A having the same starting vertex and the same ending vertex are called *parallel paths*.

Definition 2.1. Let Γ be a component of Γ_A. We say that Γ is a *component with length* when parallel paths in Γ have the same length. Otherwise, we say that Γ is a component without length.

Observe that a component of Γ_A with length has no oriented cycles.

Definition 2.2. Let Γ be a component of Γ_A with length and $X, Y \in \Gamma$. We say that the length $\ell(X, Y)$ between X and Y is n if there is a path from X to Y in Γ of length n.

There are many algebras having components with length. In [7], K. Bongartz and P. Gabriel considered the homotopy given by the mesh relations and defined simply connected quivers. The notion of component with length can be extended to translation quivers, and we can state the following result, which has been implicitly proven in [7], in the proof of Proposition 1.6.

Theorem [7]. *Let Γ be a component of a simply connected translation quiver. Then Γ is a component with length.*

Proof. Let $X \in \Gamma$. By the definition of homotopy, the length function ℓ above defined on the set of walks is constant on each homotopy class. Now, since Γ is simply connected, the walks from X to any given $Y \in \Gamma$ are homotopic to each other. □

On the other hand, by [11], we know that if A is a strongly simply connected algebra then any convex directed component of Γ_A is of tree type. Some classes of strongly simply connected algebras have been completely described. In particular, in case that the algebra A is iterated tilted of euclidean type, derived tubular, tame tilted or tame quasi-tilted, it was shown that A is strongly simply connected if and only if the first Hochschild cohomology group $H^1(A)$ vanishes and A is strongly $\widetilde{A}$ free, that is to say, contains no full convex subcategory which is hereditary of type $\widetilde{A}_n$ (see for instance [1–4]). We state the following proposition.

Proposition 2.3. *Let A be a strongly simply connected finite dimensional k-algebra and Γ a convex directed component of Γ_A. Then Γ is a component with length.*

Proof. In [7, Section 4.3], it has been proved that the orbit graph $O(\Gamma)$ is a deformation retract of the component Γ. By [11], $O(\Gamma)$ is a tree. Then Γ is a simply connected translation quiver. Thus, by the theorem stated above [7], it follows that Γ is a component with length. □

The converse of this proposition is not true, since there are components with length whose orbit graph is not a tree. For example, the directed components of a hereditary algebra over a field k of type $\widetilde{A}_{pp}$ are components with length, as the following results will prove.

The remainder of this section is devoted to prove that the directed components of the Auslander–Reiten quiver of a hereditary algebra A over an algebraically closed field k are components with length if and only if the ordinary quiver of A does not contain a subquiver (not necessarily full or convex) of type $\widetilde{A}_{pq}$, with $p \neq q$.

In all that follows k denotes an algebraically closed field.

We start by proving the following lemma:

Lemma 2.4. *Let A be an artin algebra and Γ a semiregular directed component of Γ_A without length. Let $m > 0$ be the least integer such that there are modules $X, Y \in \Gamma$ and paths from X to Y of different length, one of them of length m. Then all paths from X to Y are sectional.*

Proof. Let $C_n = \{\gamma\colon\ X \rightsquigarrow Y / \gamma$ is a path of length $n\}$. We will prove that all paths in C_n are sectional. For a path γ in C_n,

$$\gamma : X = X_0 \to X_1 \to \cdots \to X_n = Y$$

we denote by i_γ the largest integer such that $X = X_0 \to X_1 \to \cdots \to X_{i_\gamma}$ is a sectional path. Now, we assume that C_n is not empty and we choose a path γ_0 in C_n such that

$$i_{\gamma_0} = \min\{i_\gamma / \gamma \in C_n\}.$$

It is enough to prove that $i_{\gamma_0} = n$.

By hypothesis there is a path

$$\mu : X = Y_0 \to Y_1 \to \cdots \to Y_r = Y$$

of length $r \neq n$ and such that either γ_0 or μ has length m. First we assume that the semiregular component Γ has no injective modules. Since any path of length one is sectional, we also assume $n > 1$.

Suppose that $i_{\gamma_0} < n$. If $i_{\gamma_0} = 1$ then $X_2 = \tau^{-1}X_0$. Since there is an irreducible morphism $X_0 \to Y_1$ then there is an irreducible morphism $Y_1 \to \tau^{-1}X_0 = X_2$ and there exist paths

$$Y_1 \to X_2 \to \cdots \to X_n = Y \quad \text{and} \quad Y_1 \to Y_2 \to \cdots \to Y_r = Y$$

of length $n-1$ and $r-1$, respectively, contradicting the minimality of m. Thus $i_{\gamma_0} > 1$. So $X_{i_{\gamma_0}+1} \simeq \tau^{-1}X_{i_{\gamma_0}-1}$. Since there is an irreducible morphism $X_{i_{\gamma_0}-2} \to X_{i_{\gamma_0}-1}$ then we have also an irreducible morphism

$$\tau^{-1}X_{i_{\gamma_0}-2} \to \tau^{-1}X_{i_{\gamma_0}-1}.$$

We can replace in γ_0 the path

$$X_{i_{\gamma_0}-2} \to X_{i_{\gamma_0}-1} \to X_{i_{\gamma_0}} \to X_{i_{\gamma_0}+1}$$

by the path

$$X_{i_{\gamma_0}-2} \to X_{i_{\gamma_0}-1} \to \tau^{-1}X_{i_{\gamma_0}-2} \to \tau^{-1}X_{i_{\gamma_0}-1}$$

obtaining a path $\gamma' : X \to Y$ of length n such that $i_{\gamma'} = i_{\gamma_0} - 1$. This contradicts the minimality of i_{γ_0}, proving that $i_{\gamma_0} = n$.

In case Γ has no projective modules the result follows by duality. □

Lemma 2.5. *Let A be a hereditary k-algebra, Γ a directed component of Γ_A and $X \to Y$ an arrow. If there exists a path from X to Y in Γ of length larger than* 1, *then Q_A contains a subquiver of type $\widetilde{A}_{p1}$ with $p > 1$.*

Proof. First assume that Γ is the preprojective component and let $Y_0 \to Y_1 \to \cdots \to Y_n$ be a path with $n > 1$. Since Γ is directed then the modules Y_i are pairwise different. Let k be the least positive integer such that $\{\tau^k Y_i\}_{i=0,\ldots,n}$ contains a projective module. Then there are irreducible morphisms $\tau^k Y_i \to \tau^k Y_{i+1}$ for $0 \leqslant i \leqslant n-1$.

On the other hand we know by hypothesis that there is an arrow from Y_0 to Y_n, which induces an irreducible morphism $\tau^k Y_0 \to \tau^k Y_n$.

Case 1. Assume $\tau^k Y_n$ is projective. Since there exists an irreducible morphism $\tau^k Y_{n-1} \to \tau^k Y_n$ and A is hereditary, then $\tau^k Y_{n-1}$ is projective. Iterating this argument, we conclude that $\tau^k Y_i$ is projective, for $i = 0, \ldots, n-1$. Thus, Γ contains the subquiver

$$\begin{array}{ccccc} & \tau^k Y_1 & \longrightarrow \cdots \longrightarrow & \tau^k Y_{n-1} & \\ \nearrow & & & & \searrow \\ \tau^k Y_0 & & \longrightarrow & & \tau^k Y_n \end{array}$$

where all $\tau^k Y_i$ are different and projective, proving that Q_A has a subquiver of type $\widetilde{A}_{n1}$ with $n \neq 1$.

Case 2. $\tau^k Y_n$ is not projective. Let t be the largest integer $0 \leqslant t \leqslant n-1$, such that $\tau^k Y_t$ is projective. Thus $\tau^{k+1} Y_i$ is defined for $i > t$ and we get the following path of irreducible morphisms

$$\tau^{k+1} Y_{t+1} \to \tau^{k+1} Y_{t+2} \to \cdots \to \tau^{k+1} Y_n.$$

Since there are irreducible morphisms $\tau^k Y_t \to \tau^k Y_{t+1}$ and $\tau^k Y_0 \to \tau^k Y_n$ then we also have irreducible morphisms $\tau^{k+1} Y_{t+1} \to \tau^k Y_t$ and $\tau^{k+1} Y_n \to \tau^k Y_0$.

Since Γ is directed there are no oriented cycles in Γ. Using this and the fact that $\tau^k Y_t$ is projective we get that all modules in the subquiver

$$\begin{array}{ccc} & \cdots \longrightarrow \cdots \longrightarrow \tau^{k+1} Y_n \longrightarrow \tau^k Y_0 \longrightarrow \cdots \longrightarrow \tau^k Y_{t-1} & \\ \nearrow & & \searrow \\ \tau^{k+1} Y_{t+1} & \xrightarrow{\hspace{6cm}} & \tau^k Y_t \end{array}$$

of Γ_A are different and projective. So Q_A has a subquiver of type $\widetilde{A}_{n1}$ with $n \neq 1$, proving the lemma in this case.

The result for the preinjective component follows by duality. $\square$

Proposition 2.6. *Let A be a hereditary k-algebra. Then the directed components of Γ_A are components with length if and only if Q_A does not contain a subquiver of type $\widetilde{A}_{pq}$, with $p \neq q$.*

Proof. If Q_A has a subquiver of type $\widetilde{A}_{pq}$ with $p \neq q$ then the directed components of Γ_A contain Q_A as a convex subquiver. Thus, they are components without length.

To prove the converse, we first assume that the preprojective component $\mathcal{P}$ of Γ_A is without length. We will prove that there exist $p \neq q$ such that Q_A contains a subquiver of type $\widetilde{A}_{pq}$. First we will prove this in case there exists a particular type of cyclic walk, and then we will show that such a walk always exists. So we start by assuming that there is a cyclic walk in $\mathcal{P}$ of the form

$$P_0 - P_1 - \cdots - P_d = P_0$$

where each edge stands for an arrow $\to$ or $\leftarrow$, such that:

(i) P_i is projective for all $i = 0, 1, \ldots, d$.
(ii) There exists n, with $0 < n < d$, such that the modules $P_0, \ldots, P_n$ are pairwise non-isomorphic, and also $P_n, P_{n+1}, \ldots, P_d$ are pairwise non-isomorphic.
(iii) The number r of arrows in one direction is different from the number s of arrows in the opposite direction.

If the modules $P_0, \ldots, P_d$ are pairwise non-isomorphic then Q_A would contain a subquiver of type $\widetilde{A}_{rs}$. If not, there exist $i < j$ such that $P_i \simeq P_j$. Then we obtain two cyclic walks of shorter length satisfying (i) and (ii), and such that one of them, say C, satisfies also (iii). If all modules in C are non-isomorphic we are done. Otherwise we iterate the procedure until we finally reach a cyclic walk with non-isomorphic modules satisfying (i), (ii) and (iii), having a subquiver of type $\widetilde{A}_{pq}$ with $p \neq q$, as desired.

Let m be the least integer such that there exist indecomposable modules X and Y in $\mathcal{P}$ and paths from X to Y of different length, one of them of length m. We assume that $m \geqslant 2$, since otherwise the result holds by the previous lemma, and we will prove that there exists a cyclic walk between projective modules satisfying (i), (ii) and (iii). Let then

$$\gamma : X = X_0 \to X_1 \to \cdots \to X_n = Y$$

and

$$\mu : X = Y_0 \to Y_1 \to \cdots \to Y_m = Y$$

be paths with $n \neq m$. Since Γ_A has no oriented cycles and by the minimality of m, the modules $X_0, X_1, \ldots, X_n, Y_1, Y_2, \ldots, Y_m$ are pairwise different.

Then in $\mathcal{P}$ we have the following subquiver

$$\begin{array}{ccccc} & \nearrow & X_1 \to X_2 \to \cdots \to X_{n-2} \to X_{n-1} & \searrow & \\ Y_0 = X_0 & & & & X_n = Y_m \\ & \searrow & Y_1 \to Y_2 \to \cdots \to Y_{m-2} \to Y_{m-1} & \nearrow & \end{array}$$

By applying τ^k to this subquiver if necessary, we may assume that X_0 is projective. Let $k_1 \in \{1, \ldots, n-1, n\}$ be the largest integer such that X_{k_1} is projective. If $k_1 < n$ then τX_{k_1+1} is projective, because there is an irreducible morphism from τX_{k_1+1} to the projective module X_{k_1}. Let $k_2 \in \{k_1+1, k_1+2, \ldots, n\}$ be largest so that τX_{k_2} is projective.

Iterating this process we find $k_1 < k_2 < \cdots < k_r$ such that the modules

$$X_1, X_2, \ldots, X_{k_1},$$

$$\tau X_{k_1+1}, \tau X_{k_1+2}, \ldots, \tau X_{k_2}$$

and

$$\tau^r X_{k_r+1}, \tau^r X_{k_r+2}, \ldots, \tau^r X_n$$

are projective. We construct a walk of irreducible morphisms between projective modules

$$X_0 \to \cdots \to X_{k_1} \leftarrow \tau X_{k_1+1} \to \cdots \to \tau X_{k_2} \leftarrow \tau^2 X_{k_2+1} \to \cdots \leftarrow \cdots \to \tau^r X_n$$

with r arrows in one direction and $n - r$ arrows in the opposite direction. Moreover, there is always a non-sectional path from X_i to $\tau^{-k} X_i$ if $k > 0$. So $X_j \neq \tau^{-k} X_i$ for $i > j$ and

$k > 0$ because all paths from X to Y are sectional, according Lemma 2.4. Thus all the projective modules in the walk are pairwise different.

In a similar way we find in $\mathcal{P}$ a walk of irreducible morphisms between different projective modules

$$Y_0 \to \cdots \to Y_{t_1} \leftarrow \tau Y_{t_1+1} \to \cdots \to \tau Y_{t_2} \leftarrow \tau^2 Y_{t_2+1} \to \cdots \leftarrow \cdots \to \tau^s Y_m$$

with $m - s$ arrows in the same direction. Since $X_n = Y_m$ and $\tau^r X_n$, $\tau^s Y_m$ are projective it follows that $r = s$.

Since $X_0 = Y_0$ and $\tau^r X_n = \tau^s Y_m$, combining the two walks we get a cyclic walk with exactly n arrows in one direction and m in the opposite direction, satisfying (i), (ii) and (iii), as we wish. Then we conclude that there exist $p \neq q$ such that Q_A contains a subquiver of type $\widetilde{A}_{pq}$.

The result for the preinjective component follows by duality. Then we conclude that the directed components of Γ_A are components with length. □

Finally, we remark that if a hereditary algebra contains a full convex subquiver of type $\widetilde{A}_{pp}$ then it is not strongly simply connected. However, the directed components of Γ_A may be components with length.

3. On the degree of irreducible morphisms in components with length

In this section we are mainly interested in giving two different characterizations of the irreducible morphisms in generalized standard convex components with length having finite left (right) degree.

In these components it is possible to find a handy criterion to determine if the degree of an irreducible morphism is finite, depending on whether Ker f (Coker f) is in Γ.

Before we state our first characterization we prove a result useful for our purposes.

Proposition 3.1. *Let Γ in Γ_A be a generalized standard convex component with length. Let $X, Y \in \Gamma$ such that $\ell(X, Y) = n$. Then*:

(a) $\Re^{n+1}(X, Y) = 0$.
(b) *If $g : X \to Y$ is a non-zero morphism then $g \in \Re^n(X, Y) \setminus \Re^{n+1}(X, Y)$.*
(c) *$\Re^j(X, Y) = \Re^n(X, Y)$, for each $j = 1, \dots, n$.*

Proof. (a) Assume that there is a morphism $g \neq 0$ such that $g \in \Re^{n+1}(X, Y)$, with $X, Y \in \Gamma$. Then there exist an integer $s \geqslant 1$, indecomposable modules $B_1, B_2, \dots, B_s$, morphisms $f_i \in \Re(X, B_i)$ and $g_i : B_i \to Y$ with each g_i a sum of compositions of n irreducible morphism between indecomposable modules such that $g = \sum_{i=1}^{s} g_i f_i$ with $g_i f_i \neq 0$ [6]. Since Γ is a convex component, the modules $B_i \in \Gamma$, for $i = 1, \dots, s$. Moreover, as Γ is a generalized standard component, by [6, 7.5], each $f_i : X \to B_i$ can be written as $f_i = \sum_{k=1}^{r} \mu_{ik}$, where μ_{ik} is composition of irreducible morphisms, for $k = 1, \dots, r$.

So the paths $g_i\mu_{ik}: X \to Y$ have length greater than n, contradicting that Γ is a component with length. Thus $\Re^{n+1}(X, Y) = 0$.

(b) Since Γ is generalized standard and $g: X \to Y$ is not zero then $g = \sum_{i=1}^{s} g_i$, where each g_i is a path from X to Y. Since $\ell(X, Y) = n$, the length of each g_i is n. Hence $g \in \Re^n(X, Y)$. By (a) we know that $\Re^{n+1}(X, Y) = 0$. So $g \in \Re^n(X, Y) \setminus \Re^{n+1}(X, Y)$.

(c) Follows immediately from the fact that Γ is a component with length and $\ell(X, Y) = n$. □

The following corollaries are direct consequences of the proposition.

Corollary 3.2. *Let X and Y be modules in a generalized standard convex component with length. If* $\mathrm{Hom}(X, Y) \neq 0$ *then there is a unique k such that $\Re^k(X, Y) \setminus \Re^{k+1}(X, Y)$ is non-empty and such k coincides with $\ell(X, Y)$.*

Corollary 3.3. *Let Γ in Γ_A be a generalized standard convex component with length. Then the composition f of n irreducible morphisms with modules in Γ is in $\Re^{n+1}$ if and only if $f = 0$.*

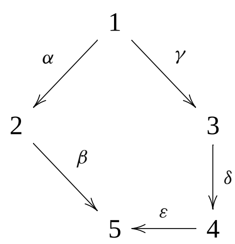
When Γ is a generalized standard convex component with length, given an irreducible map and a morphism in $\Re^n \setminus \Re^{n+1}$, their composition is in $\Re^{n+2}$ only if it is zero. As a consequence we can state the following result:

Corollary 3.4. *Let Γ be a generalized standard convex component of Γ_A with length. Let $f: X \to Y$ be an irreducible morphism and let $\varphi \in \Re^n(Y, Z) \setminus \Re^{n+1}(Y, Z)$, with $X, Y, Z \in \Gamma$. Then $\varphi f \in \Re^{n+2}(X, Z)$ if and only if $\varphi f = 0$.*

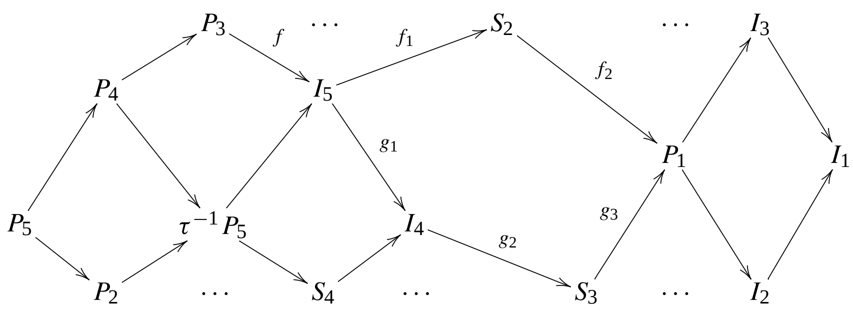
Proof. Let $\varphi \in \Re^n(Y, Z) \setminus \Re^{n+1}(Y, Z)$ be such that $\varphi f \in \Re^{n+2}(X, Z)$. By Corollary 3.2 we know that $\ell(Y, Z) = n$, since $\Re^n(Y, Z) \setminus \Re^{n+1}(Y, Z)$ is non-empty. Then $\ell(X, Z) = n + 1$, because $f: X \to Y$ is irreducible. Then we obtain from Proposition 3.1(a) that $\Re^{n+2}(X, Z) = 0$, and consequently $\varphi f = 0$. □

In the preceding corollary the hypothesis that Γ has length can not be omitted, as the example below shows.

Example 3.5. Let A be the algebra given by the quiver:



with relations $\beta\alpha = 0$ and $\delta\gamma = 0$. Then A is a triangular algebra of finite representation type, tilted of type $\widetilde{A}_n$, and the Auslander–Reiten quiver Γ_A of A is the following:



So the unique component of Γ_A is a component without length. The irreducible morphism f is left almost split, so $d_r(f) = 1$.

Let $\gamma = g_3g_2g_1$ and $\mu = f_2f_1$. Consider $\varphi = \gamma + \mu$. This morphism belongs to $\Re^2(I_5, P_1)\backslash\Re^3(I_5, P_1)$, since μ is a sectional path of length 2. We claim that $\varphi f \in \Re^4(P_3, P_1)$ and $\varphi f \neq 0$. In fact, since $\mu f = 0$ and φf is a sectional path of length 4, we have that $\varphi f \in \Re^4(P_3, P_1)\backslash\Re^5(P_3, P_1)$.

In the next proposition we consider a similar situation in a semiregular directed component of Γ_A, having sectional parallel paths of different length.

Proposition 3.6. *Let A be an artin algebra and Γ a semiregular directed component of Γ_A without length. Let $m > 0$ be the least integer such that there are modules $X, Y \in \Gamma$ and paths from X to Y of different length, one of them of length m. Let γ, μ be such paths, with $\ell(\gamma) = n$, $\ell(\mu) = m$. Assume that there is an irreducible map $f : Z \to X$ in Γ such that $\mu f = 0$. Then $\varphi = \gamma + \mu$ is in $\Re^m(X, Y)\backslash\Re^{m+1}(X, Y)$ and $\varphi f \in \Re^{n+1}(Z, Y)\backslash\Re^{n+2}(Z, Y)$. In particular, $d_r(f) \leqslant m$.*

Proof. By Lemma 2.4 we know that both paths γ and μ are sectional. It follows from the minimality of m that γf is also sectional, since $\mu f = 0$. Since γf is a composition of $n+1$ irreducible morphisms we know by [12] that it belongs to $\Re^{n+1}(Z, Y)\backslash\Re^{n+2}(Z, Y)$. Thus $\varphi f = \gamma f + \mu f = \gamma f \in \Re^{n+1}(Z, Y)\backslash\Re^{n+2}(Z, Y)$. Since $n > m$ we obtain that $\varphi = \gamma + \mu$ is in $\Re^m(X, Y)\backslash\Re^{m+1}(X, Y)$ and therefore $d_r(f) \leqslant m$. □

Now we state the first of our main theorems.

Theorem 3.7. *Let A be an artin algebra and Γ be a generalized standard convex component of Γ_A with length. Let $f : X \to Y$ be an irreducible morphism with $X, Y \in \Gamma$. Then*:

(a) *$d_r(f) = \infty$ if and only if $gf \neq 0$ for each non-zero morphism $g : Y \to M$ with $M \in \Gamma$.*
(b) *$d_l(f) = \infty$ if and only if $fg \neq 0$ for each non-zero morphism $g : M \to X$ with $M \in \Gamma$.*

Proof. (a) First assume that for each non-zero morphism $g : Y \to M$, with $M \in \Gamma$, we have that $gf \neq 0$.

Consider $g \in \Re^n(Y, M) \setminus \Re^{n+1}(Y, M)$. Thus $g \neq 0$. By Corollary 3.3 we have that $\ell(Y, M) = n$. Since $f : X \to Y$ is irreducible we get that $\ell(X, M) = n + 1$, and Proposition 3.1(a) implies that $\Re^{n+2}(X, M) = 0$. We assumed that $gf \neq 0$, so $gf \notin \Re^{n+2}(X, M)$ and we conclude that $d_r(f) = \infty$.

The converse is an immediate consequence of the definition of infinite right degree.

The result stated in (b) follows by duality. □

From the above theorem we deduce the following useful result.

Corollary 3.8. *Let A be an artin algebra. If Γ is a generalized standard convex component of Γ_A with length then*:

(a) *If $f : X \to Y$ is an irreducible monomorphism with $X, Y \in \Gamma$ then $d_l f = \infty$.*
(b) *If $f : X \to Y$ is an irreducible epimorphism with $X, Y \in \Gamma$ then $d_r f = \infty$.*

Let A be an artin algebra and Γ a generalized standard convex component of Γ_A with length. To determine if the degree of an irreducible morphism between indecomposable modules $f : X \to Y$ is infinite we only need to prove that the composition of f with non-zero maps is non-zero. It would be interesting to know if it is enough to consider only compositions of f with paths in Γ_A. Though we do not know the answer in general, we can prove that this is the case when $\alpha(\Gamma) \leqslant 2$. That is, $\alpha(X) \leqslant 2$ for every X in Γ. In fact, we prove the following result.

Proposition 3.9. *Let A be an artin algebra and Γ a generalized standard convex component of Γ_A with length such that $\alpha(\Gamma) \leqslant 2$. Let $f : X \to Y$ be an irreducible morphism with $X, Y \in \Gamma$. Then the following conditions are equivalent*:

(a) $d_r(f) = \infty$.
(b) $\gamma f \neq 0$ *for each non-zero path $\gamma : Y \to M$ in Γ.*

Proof. (a) implies (b) is an immediate consequence of Theorem 3.7.

Now, assume that (b) holds. By Theorem 3.7 it is enough to prove that $gf \neq 0$ for each non-zero morphism $g : Y \to M$ with $M \in \Gamma$. Since Γ is a generalized standard convex component then $g \in \Re^n(Y, M) \setminus \Re^{n+1}(Y, M)$ for some $n \geqslant 0$.

We prove that $gf \neq 0$ by induction on n. Assume that $n = 1$, that is, $g \in \Re(Y, M) \setminus \Re^2(Y, M)$. Since Y, M are indecomposable then g is an irreducible morphism. Therefore g is a non-zero path and it follows by hypothesis that $gf \neq 0$.

Assume now that the composition of an irreducible morphism h in Γ with a non-zero morphism in $\Re^{n-1} \setminus \Re^n$ is non-zero, whenever the composition of h with non-zero paths does not vanish.

Suppose that there exists a morphism $g \in \Re^n(Y, M) \setminus \Re^{n+1}(Y, M)$ with $M \in \Gamma$ such that $gf = 0$. Thus $\ell(Y, M) = n$. By [13, Lemma 1.3] the irreducible morphism $f : X \to Y$ is such that X is not injective. Since we are assuming (b) it follows that $\alpha'(X) \neq 1$. Otherwise,

f is minimal left almost split and there is a non-zero path $g : Y \to \tau^{-1}X$ such that $gf = 0$. So $\alpha'(X) = 2$. Consider the almost split sequence

$$0 \to X \xrightarrow{(f' f)^T} Y' \oplus Y \xrightarrow{(t' t)} \tau^{-1}X \to 0.$$

It is enough to prove that there exists a non-zero path $\gamma : \tau^{-1}(X) \to M$ such that $\gamma t' = 0$. In fact, if this is the case, then $\gamma t \neq 0$ because (t, t') is surjective. Thus the path γt satisfies $\gamma t f = \gamma t' f' = 0$, contradicting (b).

We prove next the existence of such γ using the induction hypothesis. Since $gf = 0 \in \Re^{n+2}$ [13, Lemma 1.3], states that there exists a morphism $q : \tau^{-1}X \to M$ in $\operatorname{mod} A$, $q \notin \Re^n(\tau^{-1}X, M)$ such that $g + qt \in \Re^{n+1}(Y, M)$ and $qt' \in \Re^{n+1}(Y', M)$. Since Γ is a component with length and $\ell(Y, M) = n$, we get that $\Re^{n+1}(Y, M) = 0$ and therefore $-qt = g$. It follows from Corollary 3.2 that the morphism q belongs to $\Re^{n-1}(\tau^{-1}X, M)$ and not to $\Re^n(\tau^{-1}X, M)$.

On the other hand, $qt' = 0$ since $qt' \in \Re^{n+1}(Y', M)$ and $\ell(Y', M) = n$. Thus, since $q \neq 0$ we know by the induction hypothesis that there exists a path γ such that $\gamma t' = 0$, proving the desired result and ending the proof of the proposition. □

One can easily find components Γ satisfying the hypothesis of the proposition. An example is given by the convex directed components of strongly simply connected string algebras. In fact, such components Γ satisfy $\alpha(\Gamma) \leqslant 2$ [8, p. 175], and by Proposition 2.3 we know that convex directed components are components with length.

Next we give another characterization of the degree of an irreducible morphism between indecomposable modules, which allows us to determine if the right (left) degree of an irreducible morphism is finite depending on whether $\operatorname{Ker} f$ ($\operatorname{Coker} f$) belongs to the component.

Observe that if $f : X \to Y$ is an irreducible epimorphism (monomorphism) then $\operatorname{Ker} f$ ($\operatorname{Coker} f$) is an indecomposable module [5].

The following lemma will be very helpful in the sequel. For $X, Y \in \Gamma_A$, we recall that X is a *predecessor of* Y or that Y is a *successor of* X if there exists a sequence $X = X_0 \to X_1 \to \cdots \to X_t = Y$ of non-zero non-isomorphisms between indecomposable modules. When X and Y are in a generalized standard convex component Γ the non-zero non-isomorphisms in the definition may be replaced by irreducible morphisms.

Lemma 3.10. *Let A be an artin algebra and Γ a generalized standard convex component of Γ_A.*

(a) *Let $g : X \to M$ be a non-zero morphism with $X, M \in \Gamma$ and let $f : M \to N$ be an irreducible epimorphism such that $fg = 0$. Then* $\operatorname{Ker} f$ *belongs to Γ and is a successor of X.*
(b) *Let $g : N \to X$ be a non-zero morphism with $N, X \in \Gamma$ and $f : M \to N$ an irreducible monomorphism such that $gf = 0$. Then* $\operatorname{Coker} f$ *belongs to Γ and is a predecessor of X.*

Proof. We prove the first statement, the second follows by duality. Let f and g be as in (a). Then g factors through $\operatorname{Ker} f$, because $fg = 0$. Hence there is a morphism $g' : X \to \operatorname{Ker} f$ such that $g = jg'$, where $j : \operatorname{Ker} f \to M$ is the inclusion map and $g' \neq 0$. Since $X, M \in \Gamma$ and Γ is convex then $\operatorname{Ker} f \in \Gamma$ and is a successor of X. □

We are now in a position to prove the following criterion to determine if an irreducible morphism has infinite left degree.

Theorem 3.11. *Let Γ be a generalized standard component of Γ_A and $f : M \to N$ be an irreducible morphism with $M, N \in \Gamma$.*

(a) *If* $\operatorname{Ker} f \in \Gamma$ *then* $d_l(f) < \infty$.
(b) *If* $\operatorname{Coker} f \in \Gamma$ *then* $d_r(f) < \infty$.

If Γ is, moreover, convex and with length, then

(a′) *If $f : M \to N$ is an epimorphism then $d_l(f) = \infty$ if and only if* $\operatorname{Ker} f \notin \Gamma$.
(b′) *If $f : M \to N$ is a monomorphism then $d_r(f) = \infty$ if and only if* $\operatorname{Coker} f \notin \Gamma$.

Proof. (a) Let $f : M \to N$ be an irreducible epimorphism with $M, N \in \Gamma$ and consider the inclusion $j : \operatorname{Ker} f \to M$. Since Γ is a generalized standard component then $\Re^\infty(X, Y) = 0$ for each $X, Y \in \Gamma$. Then $j \notin \Re^\infty(\operatorname{Ker} f, M)$. So there is an integer $k > 1$ such that $j \in \Re^k(\operatorname{Ker} f, M) \setminus \Re^{k+1}(\operatorname{Ker} f, M)$. Since $fj = 0$ it follows from the definition of left degree that $d_l(f) < \infty$.

The statement (b) follows by duality.

(a′) First suppose $d_l(f)$ is finite. This means that there is a non-zero morphism $g : X \to M$ with $X \in \Gamma$, such that $fg = 0$, by Theorem 3.7(b). Then the above lemma implies that $\operatorname{Ker} f$ belongs to Γ.

Assume now that $\operatorname{Ker} f \in \Gamma$. Then the inclusion $j : \operatorname{Ker} f \to M$ is a non-zero morphism and $fj = 0$. Thus $d_l(f)$ is finite, by Theorem 3.7(b).

The last statement follows by duality. □

As an application we have the following corollary.

Corollary 3.12. *Let $0 \to \tau Z \xrightarrow{(g_1 g_2)^t} Y_1 \sqcup Y_2 \xrightarrow{(f_1 f_2)} Z \to 0$ be an almost split sequence with Y_1 and Y_2 indecomposable. Then* $\operatorname{Ker}(g_1) \simeq \operatorname{Ker}(f_2)$. *If, moreover, Z belongs to a generalized standard convex component of Γ_A with length, then $d_l(g_1) < \infty$ if and only if $d_l(f_2) < \infty$.*

Proof. If g_1 is a monomorphism then so is f_2, and $d_l(g_1) = d_l(f_2) = \infty$ by Corollary 3.8. So we may assume that g_1 is an epimorphism. Since $(g_1 g_2)^t$ is injective it follows that $g_2|_{\operatorname{Ker} g_1}$ is injective.

On the other hand, $f_2 g_2 = -f_1 g_1$, so $g_2|_{\operatorname{Ker} g_1} : \operatorname{Ker} g_1 \to \operatorname{Ker} f_2$. Using a standard argument we get that $l(\operatorname{Ker} g_1) = l(\operatorname{Ker} f_2)$, thus the monomorphism $g_2|_{\operatorname{Ker} g_1} : \operatorname{Ker} g_1 \to \operatorname{Ker} f_2$

is an isomorphism, proving the first statement. The second statement follows from the first and the preceding theorem. □

Remark 3.13. S. Liu proved in [13, Corollary 1.2] that if $d_l(f_2)$ is finite then $d_l(g_1) < d_l(f_2)$.

Another interesting application of the above theorem is the following result.

Theorem 3.14. *Let A be a finite dimensional k-algebra of finite representation type, such that Γ_A is a component with length. Then*:

(a) *If $f : M \to N$ is an irreducible epimorphism then $d_l(f) < \infty$ and $d_r(f) = \infty$.*
(b) *If $f : M \to N$ is an irreducible monomorphism then $d_r(f) < \infty$ and $d_l(f) = \infty$.*

Proof. (a) It is known that Γ_A is connected and generalized standard. The result is a direct application of Theorems 3.11 and 3.7. □

We remark that if A is a finite dimensional algebra over an algebraically closed field and Γ is a standard component of Γ_A then Γ is generalized standard [15]. Moreover, if A is of finite representation type and Γ_A is a component with length then Γ_A is standard [7].

4. On the degree of irreducible morphisms over a tame hereditary algebra

We start this section by applying our results to determine the degree of almost all irreducible morphisms lying in the directed components of tame hereditary algebras of type $\widetilde{E}_p$ or $\widetilde{D}_n$.

Note that such components are convex, generalized standard and with length. Since we are dealing with tame hereditary algebras, we can use the defect associated to an indecomposable module to decide if the module is in the preinjective component $\mathcal{I}$. Thus, for a particular irreducible morphism f in $\mathcal{I}$ we can determine whether $\operatorname{Ker} f$ belongs to $\mathcal{I}$ in this way (see [10, Proposition 1.9]). Then, using the results of the preceding section one can decide if the left degree of f is finite. However, we are going to give a direct description of the irreducible morphisms of finite degree, up to a finite number of them.

Then we turn our attention the hereditary algebras of type $\widetilde{A}_n$, and determine the left degree of any irreducible morphism in the directed components of these algebras. We know that these components are not always with length, so the results of the previous section do not apply to them. We use results due to S. Liu [13, Proposition 1.6 and Corollary 1.6] to determine the degree of such irreducible morphisms.

On the other hand, we are going to consider the regular components of a tame hereditary algebra. It is well known that these components are without length. Using again Liu's results we determine the degree of the irreducible morphisms between indecomposable regular modules.

For the convenience of the reader we state now the above mentioned results of Liu.

Proposition [13, Proposition 1.6]. *Let $f : X \to Y$ be an irreducible morphism of finite left degree in* mod A *with Y indecomposable. Assume that*

$$Y_n \to Y_{n-1} \to \cdots \to Y_1 \to Y_0 = Y$$

is a presectional path in Γ_A with $n \geqslant 1$. If $X \oplus Y_1$ is a summand of the middle term of $\varepsilon(Y)$ then $d_l(f) > n$.

Corollary [13, Corollary 1.6]. *Let $f : X \to Y$ be an irreducible morphism in* mod A *with Y indecomposable. Assume that there is an infinite presectional path*

$$\cdots Y_n \to Y_{n-1} \to Y_{n-2} \to \cdots \to Y_1 \to Y_0 = Y$$

in Γ_A, such that $Y_1 \oplus X$ is a summand of the domain of the left almost split morphism ending at Y. Then $d_l f = \infty$.

Dual results hold for the right degree.

4.1. Hereditary algebras of type $\widetilde{E}_p$ for $p = 6, 7$ and 8

Let A be a hereditary algebra of type $\widetilde{E}_p$ for $p = 6, 7$ and 8 and let $\mathcal{I}$ be the preinjective component of Γ_A. By Corollary 3.8, we know that the irreducible monomorphisms in $\mathcal{I}$ have infinite left degree. On the other hand, it follows from Theorem 3.11 that the right degree of such morphisms is finite, since their cokernel is in $\mathcal{I}$.

We also know, by Corollary 3.8, that the right degree of any irreducible epimorphism is infinite.

Using the shape of the Auslander–Reiten quiver and arguments on the length of the modules (similar to those in [6, VIII, 277–289]) we can determine which irreducible morphisms in $\mathcal{I}$ are monomorphisms, up to a finite number (see [9, 4.4]). On the other hand, using the results of Liu stated above, one can determine the right degree of almost all irreducible epimorphism between indecomposable preinjective modules.

We recall the following notation. Let $S(\to M)$ be the full subquiver of Γ_A given by all modules $X \in \Gamma_A$ such that there is a path $X \rightsquigarrow M$ and any such path is sectional.

Now, consider $\mathcal{S} = \bigsqcup_{j \in Q_0} S(\to I_j)$ where I_j is the indecomposable injective corresponding to the vertex j of the ordinary quiver Q of A.

Let r_j be the maximum length of any sectional path ending at I_j and $r = \max_{j \in Q_0}\{r_j\}$.

Let $\mathcal{C} = \{X \in \mathcal{I} / X$ is a predecessor of a module in $\tau^r \mathcal{S}\}$. Then $\mathcal{C}$ is cofinite in ind A and the just mentioned results hold for irreducible morphisms with codomain in $\mathcal{C}$. More precisely:

Proposition 4.1. *Let A be a hereditary algebra of type $\widetilde{E}_p$, with $p = 6, 7, 8$ and let*

$$0 \to \tau N \xrightarrow{(f_1 f_2 f_3)^t} B_1 \sqcup B_2 \sqcup B_3 \xrightarrow{(g_1 g_2 g_3)} N \to 0$$

be an almost split sequence, with N in $\mathcal{C}$. Then, for $i = 1, 2, 3$,

(a) *f_i is an epimorphism, $d_l(f_i) < \infty$ and $d_r(f_i) = \infty$.*
(b) *g_i is a monomorphism, $d_r(g_i) < \infty$ and $d_l(g_i) = \infty$.*

Proposition 4.2. *Let A be a hereditary k-algebra of type $\widetilde{E}_p$, for $p = 6, 7$ and 8. Let*

$$0 \to \tau N \xrightarrow{(f_1 f_2)^t} N_1 \sqcup B_2 \xrightarrow{(g_1 g_2)} N \to 0$$

be an almost split sequence with N in $\mathcal{C}$. Then either f_1 or f_2 is a monomorphism. If f_1 is a monomorphism then

(a) *g_2 is a monomorphism, $d_l(f_1) = \infty$, $d_l(g_2) = \infty$ and $d_r(f_1) < \infty$, $d_r(g_2) < \infty$.*
(b) *f_2 and g_1 are epimorphisms, $d_l(f_2) < \infty$, $d_l(g_1) < \infty$ and $d_r(f_2) = \infty$, $d_r(g_1) = \infty$.*

We do not include the proof of these results, which are done following the ideas explained at the beginning of 4.1. The details can be found in [9].

4.2. Hereditary algebras of type $\widetilde{D}_n$ with $n \geqslant 5$

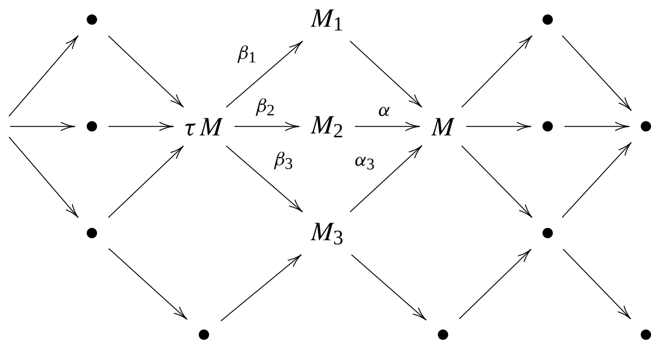
Using the shape of the Auslander–Reiten quiver and arguments similar to those used in [6, VIII, 277–289], we can prove that the only irreducible monomorphisms in $\mathcal{I}$ are the left almost split morphisms between indecomposable modules (see [9, 4.4]), which have left degree one. On the other hand, the left degree of any irreducible monomorphism is infinite, by Corollary 3.8, and so is the right degree of irreducible epimorphisms.

So we know the right and left degree of all irreducible morphisms in $\mathcal{I}$, except the degree of the irreducible epimorphisms which are not minimal right almost split. We will prove that the left degree of almost all such epimorphisms is infinite. We start with some technical results.

Lemma 4.3. *Let $\mathcal{I}$ be the preinjective component of Γ_A, where A is a hereditary k-algebra of type $\widetilde{D}_n$, with $n \geqslant 5$. Let $X, M \in \mathcal{I}$ such that $\alpha'(X) \neq 1$ and $\mathrm{Hom}(X, M) \neq 0$. Then there exists a path from X to M that is a composition of irreducible epimorphisms.*

Proof. We prove the result by induction on the length d of the paths $X \rightsquigarrow M$ in Γ_A. If $d = 1$ then the irreducible morphisms starting at X are epimorphisms, because $\alpha'(X) \neq 1$.

Let $\ell(X \rightsquigarrow M) = d > 1$. Suppose that the result is true for paths of length smaller than d. We write the path $X \rightsquigarrow M$ as the composition $X \rightsquigarrow M' \xrightarrow{\alpha} M$, with α an irreducible morphism. Since $\ell(X \rightsquigarrow M') < d$, by the inductive hypothesis there exists a path of irreducible epimorphisms $\gamma : X \rightsquigarrow M'$. If $M' \xrightarrow{\alpha} M$ is an epimorphism there is nothing to prove. Otherwise $M' \xrightarrow{\alpha} M$ is a monomorphism. Then α is left almost split, since the unique irreducible monomorphisms are the left almost split morphisms. Thus, we have in $\mathcal{I}$ the subquiver



where without loss of generality we assume that $M' = M_2$.

Then $\gamma : X \rightsquigarrow M'$ is of the form $X \rightsquigarrow \tau M \overset{\beta_2}{\to} M_2$, with $X \rightsquigarrow \tau M$ a path of epimorphisms or $X = \tau M$. In either case we can build a path of epimorphisms from X to M, through M_3. $\square$

Corollary 4.4. *Let A be a hereditary k-algebra of type $\widetilde{D}_n$ with $n \geqslant 5$. Let $X \in \mathcal{I}$ be such that $\alpha'(X) \neq 1$. Then there is no monomorphism $\gamma : X \to M$, with M indecomposable not isomorphic to X.*

Proof. Suppose there is a monomorphism $\gamma : X \to M$ which is not an isomorphism. Then $l(X) < l(M)$. This is a contradiction because by the above lemma there exists a path of epimorphisms from $X \to M$, so $l(X) \geqslant l(M)$. $\square$

Corollary 4.5. *Let A be a hereditary k-algebra of type $\widetilde{D}_n$ with $n \geqslant 5$. Let $f : M \to N$ be an irreducible epimorphism with $M \in \mathcal{I}$. If $\alpha'(\operatorname{Ker} f) \neq 1$ then $\operatorname{Ker} f \notin I$.*

Proof. Assume that $\operatorname{Ker} f \in \mathcal{I}$. The inclusion of $\operatorname{Ker} f$ in M is a proper monomorphism, thus by Corollary 4.4, $\alpha'(\operatorname{Ker} f) = 1$. $\square$

In the study of the remaining cases the following result will be useful.

Lemma 4.6. *Let A be a hereditary k-algebra of infinite representation type, $\mu : X \to M$ a monomorphism, with $X, M \in \mathcal{I}$ and $t \geqslant 1$. Then μ does not factor through $\tau^{-t} X$.*

Proof. Let $\mu = \mu_2 \mu_1$ with $\mu_1 : X \to \tau^{-t} X$ and $\mu_2 : \tau^{-t} X \to M$. If μ is a monomorphism, then μ_1 is also a monomorphism. Using that $\mathcal{I}$ contains no projective modules and τ preserves monomorphisms, we get a chain of irreducible monomorphisms

$$\cdots \to \tau^{(n+1)t} X \to \tau^{nt} X \to \cdots \to \tau^t X \to X$$

of arbitrary length, contradicting that $l(X)$ is finite. $\square$

We consider again the set $\mathcal{S} = \bigsqcup_{j\in Q_0} S(\to I_j)$. Note that only finitely many irreducible morphisms in $\mathcal{I}$ end at a module in $\mathcal{S}$. The following result holds for the remaining ones.

Proposition 4.7. *Let A be a hereditary k-algebra of type $\widetilde{D}_n$, with $n \geqslant 5$. Let $M \in \mathcal{I}$ be not injective and $f : M \to N$ an irreducible epimorphism which is not an almost split morphism and such that $N \notin \mathcal{S}$. Then* $\operatorname{Ker} f \notin \mathcal{I}$ *and $d_l(f) = \infty$.*

Proof. Using Corollary 3.12 and the shape of the Auslander–Reiten quiver we obtain that it is enough to prove the result when $\alpha(N) = 3$.

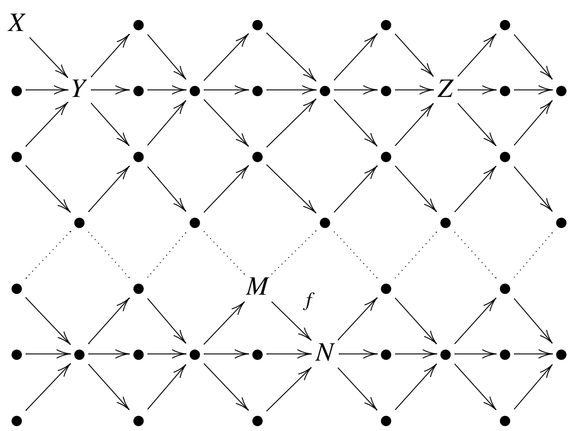
Let $X = \operatorname{Ker} f$ and let $j : X \to M$ be the inclusion. Suppose $X \in \mathcal{I}$. By Corollary 4.4 we have that $\alpha'(X) = 1$, so $\varepsilon'(X)$ has indecomposable middle term Y with $\alpha(Y) = 3$. First we analyze the case when $N = \tau^{-k}Y$, with $k \geqslant 1$. We have the almost split sequence

$$0 \to \tau N \xrightarrow{(\beta_1\beta_2\beta_3)^t} \bigsqcup_{i=1}^{3} M_i \xrightarrow{(\alpha_1\alpha_2\alpha_3)} N \to 0$$

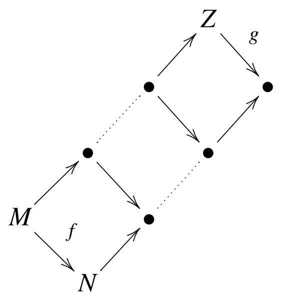
with $M = M_3$ and $f = \alpha_3$. Then $\tau^{-k}X \simeq M_1$ or $\tau^{-k}X \simeq M_2$. Assume that $\tau^{-k}X \simeq M_2$.

We have that $\operatorname{Im} j = \operatorname{Ker} f$. Moreover, $(0, 0, \operatorname{Ker} f)^t \subset \operatorname{Ker}(\alpha_1, \alpha_2, f) = \operatorname{Im}(\beta_1\beta_2\beta_3)^t$. Then there exists a submodule $U \subset \tau N$ such that $(\beta_1\beta_2\beta_3)^t(U) = (00 \operatorname{Ker} f)^t$. Therefore $\beta_1(U) = \beta_2(U) = 0$ and $\beta_3|_U : U \to \operatorname{Ker} f$ is an isomorphism. Now, $U \subset \operatorname{Ker} \beta_2 = \tau M_2$, since $\varepsilon(M_2)$ is a sequence with indecomposable middle term. Then there exists a monomorphism from X to τM_2. Since $\tau M_2 \simeq \tau^{-k+1}X$, with $k > 1$, there exists a monomorphism from X to $\tau^{-k+1}X$, contradicting the above lemma, and proving the result if N and Y belong to the same τ-orbit.

Now assume N and Y are in different τ-orbits. We illustrate the situation in the following picture:



Consider a sectional path from $M \rightsquigarrow Z$ with Z, Y in the same τ-orbit. From the commutative diagram of irreducible morphisms



we get that $\operatorname{Ker} g \simeq \operatorname{Ker} f$, by successive application of Corollary 3.12. Thus there is a monomorphism $\varphi : \operatorname{Ker} f = X \to Z$ such that $\operatorname{Im}\varphi = \operatorname{Ker} g$. Let

$$0 \to Z \xrightarrow{(g_1 g_2 g_3)^t} \coprod_{i=1}^{3} X_i \xrightarrow{(h_1 h_2 h_3)} \tau^{-1} Z \to 0$$

be almost split, with $X_1 = \tau^{-k} X$ and $g_3 = g$. Since h_2 is a monomorphism, we get that $(g_1 g)^t$ is injective. Since $\operatorname{Im}\varphi = \operatorname{Ker} f$ and φ is injective, we get that $(g_1 g)^t \varphi$ is also injective, and so is $g_1\varphi : X \to X_1 = \tau^{-k} X$, contradicting that there are no monomorphisms from X to $\tau^{-k} X$. Therefore $\operatorname{Ker} f \notin \mathcal{I}$ and by Theorem 3.11 $d_l(f) = \infty$. $\square$

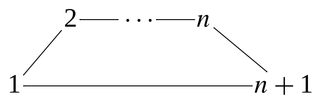
4.3. Hereditary algebras of type $\widetilde{A}_n$

We know that any irreducible morphism between preinjective indecomposable modules over a hereditary k-algebra of type $\widetilde{A}_n$ has infinite left degree. This follows from [13, Theorem 2.3], because all almost split sequences in the directed components of Γ_A have exactly two indecomposable summands in the middle term. This result can also be proven using the existence of certain presectional paths. More precisely, we prove that given any irreducible morphism f between indecomposable modules over a hereditary algebra of type $\widetilde{A}_n$, we can find an infinite presectional path in the preinjective component $\mathcal{I}$ ending at f. Moreover, when $n \geqslant 2$ such a path is sectional. Since the latter fact is interesting by itself, we include its proof here. We start by proving the following lemma.

Lemma 4.8. *Let A be a hereditary artin algebra of type $\widetilde{A}_n$ with $n \geqslant 2$ and let $m \leqslant n$. Then there exists a sectional path*

$$\tau^{k_{m+1}} I_{m+1} \to \tau^{k_m} I_m \to \cdots \to \tau^{k_3} I_3 \to \tau^{k_2} I_2 \to I_1, \tag{4.1}$$

where I_k is the indecomposable injective corresponding to the vertex k of Q_A,



is the underlying graph of Q_A, and $0 \leqslant k_2 \leqslant k_3 \leqslant \cdots \leqslant k_{m+1}$.

Proof. We prove the result by induction on m. Let $m = 1$. If $1 \to 2$ is an arrow in Q_A there exists an irreducible morphism $I_2 \to I_1$. Otherwise $2 \to 1$ is an arrow in Q_A and there is an irreducible morphism $\tau I_2 \to I_1$.

Now, let $m > 1$ and assume that the result holds for $m - 1$. Then there exists a sectional path of the form

$$\tau^{r_m} I_m \to \tau^{r_{m-1}} I_{m-1} \to \cdots \to \tau^{r_2} I_2 \to I_1 \tag{4.2}$$

with $0 \leqslant r_2 \leqslant \cdots \leqslant r_m$.

There is either an arrow $m + 1 \to m$ or an arrow $m \to m + 1$ in Q_A.

In the first case there are irreducible morphisms $I_m \to I_{m+1}$ and $\tau I_{m+1} \to I_m$. Thus we also have an irreducible morphism $\tau^{r_m+1} I_{m+1} \to \tau^{r_m} I_m$, and by composing it with the above path we get the path

$$\tau^{r_m+1} I_{m+1} \to \tau^{r_m} I_m \to \cdots \to \tau^{r_2} I_2 \to I_1$$

which is clearly sectional. In a similar way, if there is an arrow $m \to m + 1$ in Q_A then there is an irreducible morphism $I_{m+1} \to I_m$ and the required sectional path is

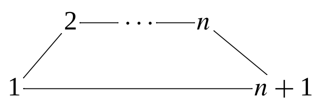
$$\tau^{r_m} I_{m+1} \to \tau^{r_m} I_m \to \cdots \to \tau^{r_2} I_2 \to I_1. \qquad \square$$

Proposition 4.9. *Let A be a hereditary artin algebra of type $\widetilde{A}_n$ with $n \geqslant 1$. Given an irreducible morphism $f : X_1 \to X_0$ in $\mathcal{I}$ there exists an infinite presectional path*

$$\cdots \to X_{n+1} \to X_n \to \cdots \to X_1 \xrightarrow{f} X_0.$$

Moreover, if $n \geqslant 2$ such a path is sectional.

Proof. Let



be the underlying graph of Q_A, and let $f : X_1 \to X_0$ be an irreducible morphism in $\mathcal{I}$.

First consider $n = 1$, so A is the Kronecker algebra and Q_A is $1 \rightrightarrows 2$.

We may assume that either $X_1 = \tau^{k_2} I_2$, $X_0 = \tau^{k_1} I_1$, or $X_1 = \tau^{k_1} I_1$, $X_0 = \tau^{k_2} I_2$. In the first case $k_1 = k_2$ and in the second $k_2 = k_1 - 1$.

In case $X_1 = \tau^{k_1} I_2$, $X_0 = \tau^{k_1} I_1$ we get a presectional path

$$\cdots \to \tau^{k_1+t} I_2 \to \tau^{k_1+t} I_1 \to \cdots \to \tau^{k_1+1} I_2 \to \tau^{k_1+1} I_1 \to \tau^{k_1} I_2 \xrightarrow{f} \tau^{k_1} I_1. \quad (4.3)$$

On the other hand, if $X_1 = \tau^{k_1} I_1$ and $X_0 = \tau^{k_1-1} I_2$, we build a presectional path

$$\tau^{k_1+t} I_1 \to \tau^{k_1+(t-1)} I_2 \to \cdots \to \tau^{k_1+1} I_1 \to \tau^{k_1} I_2 \to \tau^{k_1} I_1 \xrightarrow{f} \tau^{k_1-1} I_2.$$

Thus we get the desired result in case $n = 1$.

Now, suppose $n \geqslant 2$. By the above proposition there exists a sectional path

$$\tau^{k_{n+1}} I_{n+1} \to \tau^{k_n} I_n \to \cdots \to \tau^{k_2} I_2 \to I_1 \quad (4.4)$$

with $0 \leqslant k_2 \leqslant k_3 \leqslant \cdots \leqslant k_{n+1}$.

If there is an arrow $1 \to n+1$ in Q_A then there is an irreducible morphism $I_{n+1} \to I_1$ and thus another $\tau I_1 \to I_{n+1}$. By applying the functor τ to (4.4) and composing with the morphism τI_1 to I_{n+1} we obtain the sectional path

$$\tau^{k_{n+1}+1} I_{n+1} \to \tau^{k_n+1} I_n \to \cdots \to \tau^{k_3+1} I_3 \to \tau^{k_2+1} I_2 \to \tau I_1 \to I_{n+1}.$$

Now we may apply $\tau^{i(k_{n+1}+1)}$ obtaining a path γ_i, for each $i \geqslant 0$. The composition $\cdots \gamma_i \cdots \gamma_1 \gamma_0$ is an infinite sectional path, ending at the irreducible morphism $\tau I_1 \to I_{n+1}$.

In case there is an arrow $n+1 \to 1$ we obtain, in analogous way, an infinite sectional path ending at the irreducible morphism $I_1 \to I_{n+1}$.

Then we proved that there is either an infinite sectional path ending at $I_1 \to I_{n+1}$ or at $\tau I_1 \to I_{n+1}$.

Now consider the given irreducible morphism $f : X_1 \to X_0$. Let I and I' be injective modules such that $X_1 = \tau^r I$ and $X_2 = \tau^s I'$ with $r, s \geqslant 0$. Since A is hereditary, we have $s = r$ or $s = r - 1$. So there is an arrow $I \to I'$ or $\tau I \to I'$. We label the vertices of Q_A so that $I_1 = I$, $I_{n+1} = I'$. Then there is an arrow $n+1 \to 1$ when $s = r$, and an arrow in the opposite direction otherwise. In any case, we apply τ^r to the corresponding infinite sectional path above constructed. We obtain an infinite sectional path ending at $\tau^r I_1 \to \tau^s I_{n+1}$. This is, at $X_1 \to X_0$, as desired. □

Corollary 4.10. *Let A be a hereditary artin algebra of type $\widetilde{A}_n$, with $n \geqslant 1$ and $f : X \to Y$ be an irreducible morphism, with $X \in \mathcal{I}$. Then $d_l(f) = \infty$.*

Proof. Let $\varepsilon(Y)$ be the almost split sequence ending at Y

$$0 \to \tau Y \to X \oplus Z \to Y \to 0.$$

By the above proposition there exists an infinite presectional path ending at the irreducible morphism $g : Z \to Y$, such that $Z \oplus X$ is a summand of the domain of the right almost split morphism for Y. Then from the result due to S. Liu stated at the beginning of this section [16, Proposition 1.6] we conclude that $d_l f = \infty$. □

4.4. The regular components

In this section we calculate the right and left degree of the irreducible morphisms between indecomposable regular modules in stable tubes or in components of type ZA_∞ of the Auslander–Reiten quiver of an artin algebra.

Let $\mathcal{T}$ be such a component. We recall that a co-ray of $\mathcal{T}$ is an infinite sectional path $\cdots \to X_n \to X_{n-1} \to \cdots \to X_2 \to X_1$ with $X_1 \in \mathcal{T}$ and $\alpha(X_1) = 1$. Dually, a ray of $\mathcal{T}$ is an infinite sectional path $X_1 \to X_2 \to \cdots \to X_n \to X_{n-1} \to \cdots$ with $X_1 \in \mathcal{T}$ and $\alpha(X_1) = 1$.

An irreducible morphism in a stable tube belongs either to a ray or to a co-ray, and we determine its degree in the following proposition.

Proposition 4.11. *Let A be an artin algebra and assume that the component $\mathcal{T}$ of Γ_A is either a tube or of type ZA_∞. Let f in Γ_A be an irreducible morphism. Then*

(a) *If f belongs to a co-ray of $\mathcal{T}$, then $d_r(f) = \infty$, and $d_l(f)$ is finite and coincides with the length of the longest sectional path starting at f.*
(b) *If f belongs to a ray of $\mathcal{T}$, then $d_l(f) = \infty$, and $d_r(f)$ is finite and coincides with the length of the longest sectional path ending at f.*

Proof. Let $f : X \to Y$ be an irreducible morphism of a co-ray of $\mathcal{T}$. Then $\epsilon'(X)$ is of the form $0 \to X \to Y \oplus X_1 \to \tau^{-1}X \to 0$ with X_1 indecomposable, and there exists an infinite sectional path in $\mathcal{T}$ of the form

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \to X_n \xrightarrow{f_n} \cdots .$$

Then, by the dual of Corollary 1.6 [13], due to S. Liu, we conclude that $d_r(f) = \infty$.

Let now

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \to X_n \xrightarrow{f_n} \cdots$$

be a ray of $\mathcal{T}$. We are going to prove that f_k has finite right degree $k+1$. The ray induces a sectional path

$$X_{k+1} \xrightarrow{g_{k+1}} \tau^{-1}X_k \xrightarrow{g_k} \tau^{-2}X_{k-1} \to \cdots \xrightarrow{g_0} \tau^{-k}X_0$$

in $\mathcal{T}$, and $(g_0 \cdots g_k g_{k+1})f = 0$, due to the mesh relations. This proves that $d_r(f_k) \leqslant k+1$. To prove that the converse inequality holds, we use that there is also a sectional path

$$X_k \xrightarrow{g_k} \tau^{-1}X_{k-1} \xrightarrow{g_{k-1}} \tau^{-2}X_{k-2} \to \cdots \xrightarrow{g_0} \tau^{-k+1}X_0$$

and that $X_{k+1} \oplus \tau^{-1}X_{k-1}$ is a summand of $\varepsilon'(X_k)$. Using again Proposition 1.6 [13], we conclude that $d_r(f) = k+1$. Thus we proved the statements concerning the right degree of f. Those concerning the left degree follow by duality. □

Remark 4.12. Note that all irreducible morphism in a component of type ZA_∞^∞ have infinite right and left degree by [13, Theorem 2.3].

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