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Configurations of trivial extensions of Dynkin type $\mathbf{A}_n$ ☆

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We dedicate this paper to Claus Michael Ringel on his sixtieth birthday

Abstract

Let Λ be a trivial extension of Cartan class $\mathbf{A}_n$. We find a combinatorial algorithm giving the configurations of $\mathbb{Z}\mathbf{A}_n$ associated to Λ.

Introduction

In this paper we will consider basic finite dimensional algebras over a fixed algebraically closed field k. It is well known that an algebra A of this type is isomorphic to kQ_A/I, where Q_A is the ordinary quiver associated to A and I is an admissible ideal of the path algebra kQ_A. That is, we have a presentation (Q_A, I) for the algebra A. For a quiver Q we denote by Q_0 the set of vertices and by Q_1 the set of arrows.

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Given a k-algebra A and a vertex j of Q_A we will denote by S_j the simple A-module corresponding to j. So P_j will denote the projective cover of S_j, and I_j the injective envelope of S_j.

Let A be an iterated tilted algebra of Dynkin type Δ, see [2], and let $T(A) = A \ltimes D_A(A)$ be the trivial extension of A by its minimal injective cogenerator $D_A(A) = \mathrm{Hom}_k(A, k)$. The set of vertices $(\Gamma_A)_0$ of the Auslander–Reiten quiver Γ_A of A can be embedded in the stable part ${}_S\Gamma_{T(A)}$ of the Auslander–Reiten quiver $\Gamma_{T(A)}$ of $T(A)$. Moreover, since $\mathbb{Z}\Delta \to {}_S\Gamma_{T(A)}$ is the universal covering of ${}_S\Gamma_{T(A)}$, we get that the vertices of Γ_A can be embedded in $\mathbb{Z}\Delta$, and in such a way that knowing the vertices of $\mathbb{Z}\Delta$ corresponding to A-modules we can obtain the arrows of Γ_A, see [10]. So, Γ_A is embedded in $\mathbb{Z}\Delta$ and we want to describe this embedding explicitly. In [10] we divided this problem in two parts as follows.

Let T be a trivial extension of finite representation type and Cartan class Δ.

(1) Assume that we know the vertices of $\mathbb{Z}\Delta$ corresponding to the radicals of the indecomposable projective T-modules. Determine the embedding of Γ_A in $\mathbb{Z}\Delta$ for any algebra A such that $T(A) \simeq T$.
(2) Describe an algorithm to determine which subsets of vertices in $\mathbb{Z}\Delta$ represent the radicals of the indecomposable projective modules over the trivial extension T.

The first problem was solved in [10] (see also [9]). The subsets of vertices of $\mathbb{Z}\Delta$ of the second part have been considered by Chr. Riedtmann, who called them configurations, in a more general setting [5,12–14]. The configurations of selfinjective algebras of finite type were computed in these works. One could use the results for selfinjective algebras and then decide which configurations correspond to trivial extension. With a different approach, we present here a new algorithm giving directly the configuration of a given trivial extension of Dynkin type $\mathbf{A}_n$. The case $\mathbf{D}_n$ will be considered in a forthcoming paper. Both cases have been studied in the first author PhD thesis [9].

Let Λ be a trivial extension of Cartan class $\mathbf{A}_n$. The ordinary quiver Q_Λ of Λ is a union of oriented cycles. We fix an appropriate oriented cycle $\mathcal{C}$ of Q_Λ, and associated to $\mathcal{C}$ we define the height function $h_\Lambda : (Q_\Lambda)_0 \to \mathbb{N}$ and the border function $\partial_\Lambda : (Q_\Lambda)_0 \to \{-, +\}$, as follows. For a vertex i, the quiver Q_Λ can be written in a unique way as the union of two connected subquivers $Q_\Lambda^{-,i}$ and $Q_\Lambda^{+,i}$ meeting at the vertex i, such that $Q_\Lambda^{-,i}$ is a union of cycles and contains $\mathcal{C}$. Then $h_\Lambda(i)$ is the number of vertices of $Q_\Lambda^{+,i}$. On the other hand, ∂_Λ takes the value $+$ in $\mathcal{C}$, and is defined inductively on the cycles in such a way that if $\mathcal{C}'$ and $\mathcal{C}''$ are minimal oriented cycles meeting at the vertex t and ∂_Λ is defined on $\mathcal{C}'$, then we define $\partial_\Lambda(x) = -\partial_\Lambda(t)$ for the vertices x of $\mathcal{C}''$ different from t. We may assume that $(Q_\Lambda)_0 = \{1, 2, \dots, n\}$ and that the vertex 1 belongs only to the cycle $\mathcal{C}$.

Now we outline the algorithm. Let $\{x_1, x_2, \dots, x_n\}$ be vertices in $\mathbb{Z}\mathbf{A}_n$ defined inductively by the following rules:

(1) x_1 is an arbitrary vertex in the top border of $\mathbb{Z}\mathbf{A}_n$,
(2) if $i \to j$ is an arrow of Q_Λ, $x_j = (a, b)$ and x_i has not been defined, we set $x_i = (a + h_\Lambda(i), n - h_\Lambda(i) + 1)$ if $\partial_\Lambda(i) = +$, and $x_i = (a + b, h_\Lambda(i))$ otherwise.

Then $x_1, x_2, \ldots, x_n$ define a lifting of the radicals $rP_1, rP_2, \ldots, rP_n$ of the indecomposable projective Λ-modules.

Though the algorithm is stated in a simple way and has an easy geometric interpretation, proving that it works is technically complicated. We prove it by induction on the number of minimal oriented cycles of Q_Λ. Let Γ be a trivial extension of Cartan class $\mathbf{A}_k$ obtained from Λ by eliminating an oriented cycle of Q_Λ. The inductive hypothesis applies to Γ and we need to compare the universal coverings $\mathbb{Z}A_k \to {}_S\Gamma_{T(\Gamma)}$ and $\mathbb{Z}A_n \to {}_S\Gamma_{T(\Lambda)}$. To do this we find an appropriate embedding $\imath : \underline{\mathrm{mod}}\,\Gamma \to \underline{\mathrm{mod}}\,\Lambda$ of stable module categories, and an embedding $\Phi : k(\mathbb{Z}\mathbf{A}_k) \to k(\mathbb{Z}\mathbf{A}_n)$ lifting $\imath : \underline{\mathrm{ind}}\,\Gamma \to \underline{\mathrm{ind}}\,\Lambda$ through the corresponding universal coverings.

We observe first that $\Gamma = \mathrm{End}_\Lambda(P)^{\mathrm{op}}$ for some projective Λ-module P. There are several well-known embeddings of $\mathrm{mod}\,\Gamma$ in $\mathrm{mod}\,\Lambda$ given by M. Auslander. More precisely, he described full subcategories of $\mathrm{mod}\,\Lambda$ which are equivalent to $\mathrm{mod}\,\Gamma$ via the restriction of the evaluation functor $\mathrm{Hom}_\Lambda(P, -) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$ to them. The one suited for our purpose is the full subcategory $\mathcal{C}_P$ consisting of the Λ-modules whose projective cover and injective envelope have, respectively, their top and socle in $\mathrm{add}\,P/rP$. As usual, for a module M, $\mathrm{add}\,M$ denotes the full subcategory of $\mathrm{mod}\,\Lambda$ whose objects are isomorphic to sums of direct summands of M. Let $\underline{\mathcal{C}_P}$ be the full subcategory of $\underline{\mathrm{mod}}\,\Lambda$ induced by the objects of $\mathcal{C}_P$. Then the equivalence $\mathrm{mod}\,\Gamma \to \mathcal{C}_{\mathcal{P}}$ induces an equivalence $\underline{\mathrm{mod}}\,\Gamma \xrightarrow{\sim} \underline{\mathcal{C}_P}$ between the corresponding stable categories. By composing this equivalence with the inclusion $\underline{\mathcal{C}_P} \subseteq \underline{\mathrm{mod}}\,\Lambda$, we obtain the desired embedding $\imath : \underline{\mathrm{mod}}\,\Gamma \to \underline{\mathrm{mod}}\,\Lambda$.

We need to compare the maps: ∂_Γ and ∂_Λ, h_Γ and h_Λ. The restriction of ∂_Λ to $(Q_\Gamma)_0$ is ∂_Γ. However the relationship between h_Γ and h_Λ is more complicated and is one of the important technical difficulties in our proof.

1. Preliminaries

Let Q be a quiver. Given an arrow $\alpha \in Q_1$, we say it starts at $o(\alpha)$ and ends at $e(\alpha)$. A path in Q is either an oriented sequence of arrows $p = \alpha_n \cdots \alpha_1$ with $e(\alpha_t) = o(\alpha_{t+1})$ for $1 \leqslant t < n$, or the symbol e_i for $i \in Q_0$. For any path $p = \alpha_n \cdots \alpha_1$ we define $o(p) = o(\alpha_1)$ and $e(p) = e(\alpha_n)$. If δ is a path in Q, we denote by $\underline{\delta}$ the *support* of δ in Q. Thus, $\underline{\delta}$ is a subquiver of Q having as vertices and arrows those belonging to δ. A nontrivial path p in Q is said to be an *oriented cycle* if $o(p) = e(p)$. Let $\mathcal{C} = \alpha_n\alpha_{n-1} \cdots \alpha_2\alpha_1$ be an oriented cycle in Q. We call $\mathcal{C}$ *minimal oriented cycle* if all the vertices $o(\alpha_1), o(\alpha_2), \ldots, o(\alpha_n)$ are pairwise different. We recall that Q' is a *full subquiver* of Q, if it is a subquiver of Q and for all vertices $i, j \in Q'$ we have that each arrow $i \xrightarrow{\alpha} j$ of Q is also an arrow of Q'. A full subquiver Q' of Q is called *convex*, if for any path $a_0 \to a_1 \to \cdots \to a_t$ in Q, with $a_0, a_t \in Q'_0$ we have $a_i \in Q'_0$ for all i.

The description of the quiver and relations of trivial extensions of Cartan class A_n will be needed throughout the paper. For this reason we state the following known result.

Proposition 1.1 [6]. *Let $\Lambda = kQ_\Lambda/I$ be a trivial extension of Cartan class $\mathbf{A}_n$, with $n > 1$. Then*:

(a) (i) *Q_Λ has n vertices,*
(ii) *Q_Λ is the union of oriented cycles and there are no loops in Q_Λ,*
(iii) *any two minimal oriented cycles of Q_Λ meet in at most one vertex,*
(iv) *every vertex $i \in Q_\Lambda$ belongs to at most two minimal oriented cycles,*
(v) *if $C_1, C_2, \ldots, C_m$ are minimal oriented cycles in Q_Λ such that*

$$\underline{C_1} \cap \underline{C_2} \neq \emptyset, \quad \underline{C_2} \cap \underline{C_3} \neq \emptyset, \quad \ldots, \quad \underline{C_{m-1}} \cap \underline{C_m} \neq \emptyset,$$

then $\underline{C_1} \cap \underline{C_m} = \emptyset$.

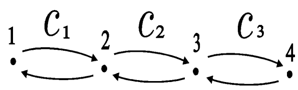
(b) *The admissible ideal I can be chosen such that it is generated by*:
(i) *the paths consisting of $t+1$ arrows in an oriented cycle of length t,*
(ii) *the paths whose arrows do not belong to a single minimal oriented cycle,*
(iii) *the difference $q - q'$, where q and q' are paths starting and ending at the same vertices and such that there exists a path v with vq and vq' minimal oriented cycles.*

Definition 1.2. Let Λ be a trivial extension of Cartan class $\mathbf{A}_n$ (respectively $\mathbf{D}_n$), and let Γ be a trivial extension of Cartan class $\mathbf{A}_k$ (respectively $\mathbf{D}_k$). If C is a (nonzero) minimal oriented cycle of Q_Λ and Q_Γ is the union of the remaining cycles of Q_Λ, we say that C is an *elimination cycle* of Q_Λ and that Γ is obtained from Λ by *eliminating the cycle C*. Then $C \cap Q_\Gamma$ is a single vertex z, and we also say that Λ is obtained from Γ by *inserting the cycle C* at z. Vertices x of Q_Γ where a cycle C can be inserted in order to obtain a trivial extension Λ of Cartan class $\mathbf{A}_n$ (respectively $\mathbf{D}_n$) with $n > k$, are called *insertion vertices*.

Remark 1.3.

(1) Suppose that Λ is obtained from Γ by inserting the cycle C at the vertex z. Then $\Gamma \simeq \mathrm{End}_\Lambda({}_\Lambda P)^{\mathrm{op}}$ where ${}_\Lambda P = \coprod_{i \in (Q_\Gamma)_0} {}_\Lambda P_i$.
(2) Let Λ be a trivial extension of Cartan class $\mathbf{A}_n$. Then a vertex x of Q_Λ is an insertion vertex if and only if it belongs to a single minimal oriented cycle.

Example. Let Λ be the trivial extension of Cartan class $\mathbf{A}_4$ given by the quiver:



where $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ denote cycles in the quiver. The elimination cycles are $\mathcal{C}_1$ and $\mathcal{C}_3$, and the insertion vertices are 1 and 4.

We will freely use properties of the module category $\mathrm{mod}\,\Lambda$ of finitely generated left Λ-modules, the stable category $\underline{\mathrm{mod}}\,\Lambda$ modulo projectives, the Auslander–Reiten quiver Γ_Λ and the Auslander–Reiten translations $\tau = \mathrm{DTr}$ and $\tau^{-1} = \mathrm{TrD}$, as can be found in [3]. We denote by $\mathrm{ind}\,\Lambda$ (respectively by $\underline{\mathrm{ind}}\,\Lambda$) the full subcategory of $\mathrm{mod}\,\Lambda$ (respectively,

$\underline{\mathrm{mod}}\,\Lambda$) formed by chosen representatives of the isomorphism classes of indecomposable modules. Let X be an object of $\mathrm{mod}\,\Lambda$, then $P_0(X)$ and $I_0(X)$ denote respectively the projective cover and the injective envelope of X.

Moreover, we will freely use the notions of locally finite k-category, translation quiver, covering functor, well behaved functor and related notions. We refer the reader to [3,4,7, 11,12] for their definitions and basic properties.

Let Λ be a trivial extension of Cartan class Δ with Δ a Dynkin quiver, and $\pi : \mathbb{Z}\Delta \to {}_S\Gamma_\Lambda$ the universal covering of ${}_S\Gamma_\Lambda$. Let M be an object of $\underline{\mathrm{ind}}\,\Lambda$. In [10, 3.5] we introduced the notion of *lifting of* ${}_S\Gamma_\Lambda$ *to* $\mathbb{Z}\Delta$ *at the vertex* M. We recall that this procedure starts by fixing an element $M[0]$ of the fibre $\pi^{-1}(M)$, afterwards we consider a slice of ${}_S\Gamma_\Lambda$ starting at M and lift it through the universal covering $\pi : \mathbb{Z}\Delta \to {}_S\Gamma_\Lambda$ to the unique slice of $\mathbb{Z}\Delta$ starting at $M[0]$. We iterate this procedure for $\tau^{-1}(M), \tau^{-2}(M), \ldots,$ until all the vertices of ${}_S\Gamma_\Lambda$ have been lifted. The minimal connected subquiver of $\mathbb{Z}\Delta$ which contains all the lifted slices is denoted by ${}_S\Gamma_\Lambda[0]$ and is called *the lifting of* ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\Delta$ at M. Then $\pi|_{{}_S\Gamma_\Lambda[0]} : {}_S\Gamma_\Lambda[0] \to {}_S\Gamma_\Lambda$ is a quiver morphism, which is a bijection on the vertices of ${}_S\Gamma_\Lambda[0]$. The inverse $\varphi_M : ({}_S\Gamma_\Lambda)_0 \to (\mathbb{Z}\Delta)_0$ of this bijection defines an embedding of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\Delta$. For $X \in \underline{\mathrm{ind}}\,\Lambda$ we denote by $X[i]$ the vertex $\tau^{-im_\Delta}X[0]$ of $\mathbb{Z}\Delta$, where $X[0] = \varphi_M(X)$ (see [10]).

2. The category $\underline{\mathrm{mod}}\,\mathrm{End}_\Lambda(P)^{\mathrm{op}}$ as a subcategory of $\underline{\mathrm{mod}}\,\Lambda$

Given an algebra Λ and a projective Λ-module P we consider the endomorphism algebra $\Gamma = \mathrm{End}_\Lambda(P)^{\mathrm{op}}$. We will study the relationship between the stable module categories $\underline{\mathrm{mod}}\,\Gamma$ and $\underline{\mathrm{mod}}\,\Lambda$ when Λ is weakly-symmetric. Let us start by comparing the module categories $\mathrm{mod}\,\Gamma$ and $\mathrm{mod}\,\Lambda$. To do that, it is convenient to view $\mathrm{mod}\,\Gamma$ as an appropriate full subcategory of $\mathrm{mod}\,\Lambda$. Maurice Auslander showed several ways to do this. The most convenient one for our problem is the following. Let Λ be an artin algebra and P be a finitely generated projective Λ-module. We denote by $\mathcal{C}_P$ the full subcategory of $\mathrm{mod}\,\Lambda$ whose objects are the modules X such that $P_0(X) \in \mathrm{add}\,P$ and $I_0(X) \in \mathrm{add}\,I_0(P/rP)$. In the next proposition we collect results on the equivalence between $\mathcal{C}_P$ and $\mathrm{mod}\,\Gamma$, which will be used throughout the paper.

Proposition 2.1. *Let Λ be an artin algebra, P a finitely generated projective Λ-module, $\Gamma = \mathrm{End}_\Lambda(P)^{\mathrm{op}}$ and $\mathcal{C}_P$ be the category defined above. Then*:

(a) [3] *The evaluation functor $e_P = \mathrm{Hom}_\Lambda(P, -) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$ induces by restriction equivalences of categories $\mathrm{add}\,P \xrightarrow{\sim} \mathcal{P}_\Gamma$ and $\mathcal{C}_P \xrightarrow{\sim} \mathrm{mod}\,\Gamma$, where $\mathcal{P}_\Gamma$ is the full subcategory of $\mathrm{mod}\,\Gamma$ whose objects are the projective Γ-modules.*
(b) [3] *Let M be in $\mathcal{C}_P$. Then M is a simple Λ-module if and only if $e_P(M)$ is a simple Γ-module. Moreover, $e_P(M/rM) \simeq e_P(M)/r_\Gamma e_P(M)$.*
(c) *Let $Q \in \mathcal{C}_P$ be an indecomposable Λ-module and let $X \in \mathcal{C}_P$ be such that $r_\Gamma e_P(Q) = e_P(X)$. Then $X \simeq r_\Lambda Q$ if and only if $r_\Lambda Q \in \mathcal{C}_P$.*

Assume moreover than Λ is weakly-symmetric. Then:

(d) $X \in \mathcal{C}_P$ *if and only if* $P_0(X)$ *and* $I_0(X)$ *are in* add P. *Therefore* add $P \subseteq \mathcal{C}_P$.
(e) $e_P(\mathbf{P}_\Lambda(X,Y)) = \mathbf{P}_\Gamma(e_P(X), e_P(Y))$ *for* $X, Y \in \mathcal{C}_P$, *where* $\mathbf{P}_\Lambda(X,Y)$ *denotes the set of* Λ*-morphisms* $f: X \to Y$ *which factor through a projective* Λ*-module.*

Let Λ be a weakly-symmetric artin algebra, P a finitely generated projective Λ-module and $\Gamma = \mathrm{End}_\Lambda(P)^{\mathrm{op}}$. We denote by $\underline{\mathcal{C}_P}$ the full subcategory of $\underline{\mathrm{mod}}\,\Lambda$ whose objects are the objects of $\mathcal{C}_P$. Since $e_P(\mathbf{P}_\Lambda(X,Y)) = \mathbf{P}_\Gamma(e_P(X), e_P(Y))$ we have that the functor $\underline{e}_P : \underline{\mathcal{C}_P} \to \underline{\mathrm{mod}}\,\Gamma$ defined by $\underline{e}_P(f + \mathbf{P}_\Lambda(X,Y)) = e_P(f) + \mathbf{P}_\Gamma(e_P(X), e_P(Y))$ is well defined. Moreover, since e_P is a full and dense functor we get that the functor $\underline{e}_P$ inherits these properties obtaining the following result.

Proposition 2.2. *Let* Λ *be a weakly-symmetric artin algebra,* P *a finitely generated projective* Λ*-module and* $\Gamma = \mathrm{End}_\Lambda(P)^{\mathrm{op}}$. *Then the functor* $\underline{e}_P : \underline{\mathcal{C}_P} \to \underline{\mathrm{mod}}\,\Gamma$ *induced by* $e_P : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$ *is an equivalence of categories.*

Throughout the paper we identify $\mathrm{mod}\,\Gamma$ with $\mathcal{C}_P$, and $\underline{\mathrm{mod}}\,\Gamma$ with $\underline{\mathcal{C}_P}$ if Λ is weakly-symmetric. The next proposition will be useful to know when an object of $\underline{\mathrm{mod}}\,\Lambda$ belongs to $\underline{\mathcal{C}_P}$.

Proposition 2.3. *Let* Λ *be a selfinjective artin algebra,* P *an indecomposable projective* Λ*-module and* $X \in \mathrm{mod}\,\Lambda$. *If* X *has no nonzero projective summands then*:

(a) $\underline{\mathrm{Hom}}_\Lambda(P/\mathrm{soc}\,P, X) \neq 0$ *if and only if* P *is a direct summand of* $P_0(X)$,
(b) $\underline{\mathrm{Hom}}_\Lambda(X, rP) \neq 0$ *if and only if* P *is a direct summand of* $I_0(X)$.

Proof. This proposition can be proven using standard arguments. □

In the next theorem we describe the objects of $\underline{\mathrm{ind}}\,\Lambda$ which are not in ind $\underline{\mathcal{C}_P}$.

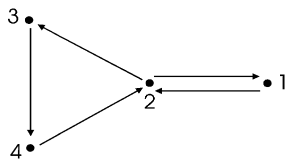
Let $\mathcal{C}$ be a k-category and let $F : \mathcal{C} \to \mathrm{mod}\,k$ be a functor. Then Supp F denotes the support of the induced functor $F : \mathrm{ind}\,\mathcal{C} \to \mathrm{mod}\,k$, that is, the set of indecomposable objects $X \in \mathcal{C}$ such that $F(X) \neq 0$.

Theorem 2.4. *Let* Λ *be a weakly-symmetric basic artin algebra. If* $\Lambda = P \amalg Q$ *then*:

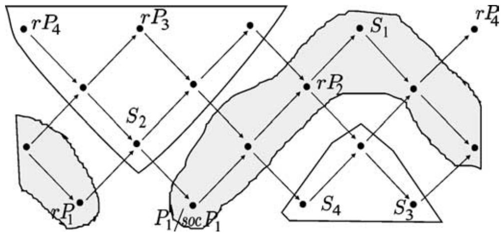
$$\underline{\mathrm{ind}}\,\Lambda \setminus \mathrm{ind}\,\underline{\mathcal{C}_P} = \mathrm{Supp}\ \underline{\mathrm{Hom}}_\Lambda(Q/\mathrm{soc}\,Q, -) \cup \mathrm{Supp}\ \underline{\mathrm{Hom}}_\Lambda(-, rQ).$$

Proof. Follows from 2.1(d) and 2.3. □

Example. Let Λ be the trivial extension of Cartan class $\mathbf{A}_4$ given by the quiver:



and the relations of 1.1(b). Let $P = P_2 \amalg P_3 \amalg P_4$ and $Q = P_1$. The shaded regions of the picture below show which vertices of ${}_S\Gamma_\Lambda$ are in $\operatorname{Supp}\underline{\operatorname{Hom}}_\Lambda(Q/\operatorname{soc} Q, -) \cup \operatorname{Supp}\underline{\operatorname{Hom}}_\Lambda(-, rQ)$. The remaining vertices correspond to the objects of $\operatorname{ind}\underline{\mathcal{C}_P}$.



Let Λ be a trivial extension of finite representation type and let $i \neq j$ be vertices of Q_Λ. The following fact will be useful later: there exists an arrow $i \to j$ in Q_Λ if and only if $\underline{\operatorname{Hom}}_\Lambda(\tau^{-1}rP_j, rP_i) \neq 0$. We will prove this result in the more general context of quasi-schurian algebras. We recall from [8] that an algebra Λ is *quasi-schurian* if it satisfies:

(a) $\dim_k \operatorname{Hom}_\Lambda(P, Q) \leqslant 1$ if P and Q are nonisomorphic indecomposable projective Λ-modules and
(b) $\dim_k \operatorname{End}_\Lambda(P) = 2$ for any indecomposable projective Λ-module P.

Proposition 2.5. *Let $\Lambda = kQ_\Lambda/I$ be a quasi-schurian selfinjective k-algebra, with I an admissible ideal. If $i \neq j$ are vertices of Q_Λ the following conditions are equivalent*:

(a) *There exists an arrow $i \xrightarrow{\alpha} j$ in Q_Λ.*
(b) $\underline{\operatorname{Hom}}_\Lambda(P_j/\operatorname{soc} P_j, rP_i) \neq 0$.

Moreover, if one of the preceding conditions holds then the canonical epimorphism $\operatorname{Hom}_\Lambda(P_j/\operatorname{soc} P_j, rP_i) \to \underline{\operatorname{Hom}}_\Lambda(P_j/\operatorname{soc} P_j, rP_i)$ *is a k-linear isomorphism.*

To prove this proposition we use the following two lemmas.

Lemma 2.6. *Let $\Lambda = kQ_\Lambda/I$ be a quasi-schurian k-algebra, with I an admissible ideal. Then for any vertices $i, j \in (Q_\Lambda)_0$ the following conditions are equivalent*:

(a) *There exists an arrow $i \xrightarrow{\alpha} j$ in Q_Λ.*
(b) $\operatorname{Hom}_\Lambda(P_j, P_i) \neq 0$, *and for any $f : P_j \to P_t$ and $g : P_t \to P_i$ with $t \in (Q_\Lambda)_0$, $gf \neq 0$ implies that either f or g is an isomorphism.*

Proof. Let δ be a path of Q_Λ. We denote by ρ_δ the morphism $\rho_\delta : P_{e(\delta)} \to P_{o(\delta)}$ given by $\rho_\delta(x) = x\bar{\delta}$.

(a) $\Rightarrow$ (b). Follows from the fact that $\dim_k \operatorname{Hom}_\Lambda(P_t, P_r) \leqslant 1$ for $t \neq r$, and $\dim_k \operatorname{rad}\operatorname{End}_\Lambda(P_t) = 1$ because Λ is quasi-schurian.

(b) $\Rightarrow$ (a). $\operatorname{Hom}_\Lambda(P_j, P_i) \neq 0$ implies that there exists a nontrivial path γ from i to j which is nonzero in Λ. Let $\gamma = \delta\alpha$, where α is an arrow. Then $\rho_\gamma = \rho_\alpha\rho_\delta$ and consequently

ρ_γ factors through the projective P_t for $t = e(\alpha)$. By hypothesis we get that $t = i$ or $t = j$. From Lemma 2 in [8] we know that being Λ quasi-schurian, the left or right composition of an arrow and an oriented cycle is zero in Λ. Therefore δ is a trivial path, and consequently $\gamma = \alpha$ is an arrow from i to j. □

Lemma 2.7. *Let Λ be a selfinjective k-algebra, P be an indecomposable projective Λ-module and let $\pi : P \to P/\operatorname{soc} P$ be the canonical epimorphism. Let Q be an indecomposable projective Λ-module not isomorphic to P, and let $\nu : rQ \to Q$ be the inclusion map. Then the map $\Phi : \operatorname{Hom}_\Lambda(P/\operatorname{soc} P, rQ) \to \operatorname{Hom}_\Lambda(P, Q)$ defined by $\Phi(g) = \nu g \pi$ is a k-linear isomorphism.*

Proof of Proposition 2.5. Let $i \neq j$ be vertices of Q_Λ.

(a) ⇒ (b). Let $i \xrightarrow{\alpha} j$ be an arrow in Q_Λ. Then there is a nonzero morphism $f : P_j \to P_i$, and by 2.7 we get that $\operatorname{Hom}_\Lambda(P_j/\operatorname{soc} P_j, rP_i) \neq 0$. Thus, it is enough to prove that the canonical epimorphism $\operatorname{Hom}_\Lambda(P_j/\operatorname{soc} P_j, rP_i) \to \underline{\operatorname{Hom}}_\Lambda(P_j/\operatorname{soc} P_j, rP_i)$ is injective. Let $f : P_j/\operatorname{soc} P_j \to rP_i$ be nonzero in $\operatorname{mod}\Lambda$. Then the composition

$$P_j \xrightarrow{\pi} P_j/\operatorname{soc} P_j \xrightarrow{f} rP_i \xrightarrow{\nu} P_i$$

is nonzero, where π is the canonical epimorphism and ν is the inclusion map. Suppose that f factors through a projective P. Then there exists $t \in (Q_\Lambda)_0$ and maps $h : P_j/\operatorname{soc} P_j \to P_t$, $g : P_t \to rP_i$ such that $gh \neq 0$. Thus, $\nu g h \pi \neq 0$ and from 2.6 we obtain that either $\nu g : P_t \to P_i$ or $h\pi : P_j \to P_i$ is an isomorphism, and this is a contradiction.

(b) ⇒ (a). Assume $\underline{\operatorname{Hom}}_\Lambda(P_j/\operatorname{soc} P_j, rP_i) \neq 0$. Since $i \neq j$ we conclude from 2.7 that $\operatorname{Hom}_\Lambda(P_j/\operatorname{soc} P_j, rP_i) \simeq \operatorname{Hom}_\Lambda(P_j, P_i)$, and since Λ is quasi-schurian we obtain that $\dim_k \operatorname{Hom}_\Lambda(P_j/\operatorname{soc} P_j, rP_i) = 1$. Thus

$$\underline{\operatorname{Hom}}_\Lambda(P_j/\operatorname{soc} P_j, rP_i) = \operatorname{Hom}_\Lambda(P_j/\operatorname{soc} P_j, rP_i).$$

Let now $g : P_j \to P_t$ and $h : P_t \to P_i$ be nonisomorphisms. According to 2.6, to conclude that there exists an arrow $i \to j$ we only need to prove that $hg = 0$. Since h and g are not isomorphisms we can write $g = g'\pi$, $h = \nu h'$, with $g' : P_j/\operatorname{soc} P_j \to P_t$, $h' : P_t \to rP_i$, and π, ν as above. Since $h'g'$ factors through a projective module and $\underline{\operatorname{Hom}}_\Lambda(P_j/\operatorname{soc} P_j, rP_i) = \operatorname{Hom}_\Lambda(P_j/\operatorname{soc} P_j, rP_i)$, we conclude that $h'g' = 0$. Thus $hg = \nu h' g' \pi = 0$, proving (a). □

Let Γ be a trivial extension of Cartan class $\mathbf{A}_n$ or $\mathbf{D}_n$. Let z be an insertion vertex of Q_Γ and let Λ be the trivial extension obtained from Γ by inserting a cycle C at z (see 1.2). Then $\Gamma \simeq \operatorname{End}_\Lambda({}_\Lambda P)^{\mathrm{op}}$, where ${}_\Lambda P$ is the projective Λ-module $\coprod_{i \in (Q_\Gamma)_0} {}_\Lambda P_i$. We saw in 2.2 that the evaluation functor at P induces an equivalence of stable categories $\underline{e}_P : \underline{\mathcal{C}_P} \to \underline{\operatorname{mod}}\,\Gamma$. Given a vertex $i \in (Q_\Gamma)_0$ it is important to know when the Λ-modules ${}_\Lambda S_i$ and $r_\Lambda P_i$ belong to $\underline{\mathcal{C}_P}$. The following result gives the answer to this question.

Theorem 2.8. *Let Γ be a trivial extension of Cartan class $\mathbf{A}_n$ or $\mathbf{D}_n$, and let z be an insertion vertex of Q_Γ. Let Λ be the trivial extension obtained from Γ by inserting the* *cycle $C = z \leftarrow z_1 \leftarrow z_2 \leftarrow \cdots \leftarrow z_{m-1} \leftarrow z$ at z. Then the following conditions hold for the projective Λ-module ${}_\Lambda P = \coprod_{i\in(Q_\Gamma)_0} {}_\Lambda P_i$:*

(a) *${}_\Lambda S_i \in \underline{\mathcal{C}_P}$ and $\underline{e}_P({}_\Lambda S_i) \simeq {}_\Gamma S_i$ for any vertex i of Q_Γ.*
(b) *$r_\Lambda P_i \in \underline{\mathcal{C}_P}$ and $\underline{e}_P(r_\Lambda P_i) \simeq r_\Gamma P_i$ for any vertex i of Q_Γ, $i \neq z$.*

Proof. By 2.1(b) we get that $e_P({}_\Lambda S_i) \simeq {}_\Gamma S_i$ and $e_P(r_\Lambda P_i) \simeq r_\Gamma P_i$ for any $i \in (Q_\Gamma)_0$. Then to obtain the result it is enough to prove that ${}_\Lambda S_i \in \underline{\mathcal{C}_P}$ for any $i \in (Q_\Gamma)_0$ and $r_\Lambda P_i \in \underline{\mathcal{C}_P}$ for any $i \in (Q_\Gamma)_0$ not equal to z.

Let $X \in \operatorname{mod} \Lambda$ be such that X has no nonzero projective summands. By 2.4 we have that $X \in \underline{\mathcal{C}_P}$ if and only if $\underline{\operatorname{Hom}}_\Lambda({}_\Lambda P_j/\operatorname{soc}_\Lambda P_j, X) = 0 = \underline{\operatorname{Hom}}_\Lambda(X, r_\Lambda P_j)$, for $j = z_1, z_2, \ldots, z_{m-1}$. Being Λ weakly symmetric, these equalities hold for $X = S_i$, if $i \neq z_1, \ldots, z_{m-1}$. So we only need to prove that they hold for $X = r_\Lambda P_i$ for $i \neq z, z_1, \ldots, z_{m-1}$.

Since the syzygy functor $\Omega : \underline{\operatorname{mod}}\, \Lambda \to \underline{\operatorname{mod}}\, \Lambda$ is an equivalence of categories, and $\Omega({}_\Lambda S_i) \simeq r_\Lambda P_i$, we get that $\underline{\operatorname{Hom}}_\Lambda(r_\Lambda P_i, r_\Lambda P_j) \simeq \underline{\operatorname{Hom}}_\Lambda({}_\Lambda S_i, {}_\Lambda S_j) = 0$ because $i \neq j$. On the other hand, there is no arrow starting at $i \in (Q_\Gamma)_0 \setminus \{z\}$ and ending at $j \in \{z_1, z_2, \ldots, z_{m-1}\}$. By 2.5 this implies that $\underline{\operatorname{Hom}}_\Lambda({}_\Lambda P_j/\operatorname{soc}_\Lambda P_j, r_\Lambda P_i) = 0$, proving (b). □

In the next proposition we collect results on the irreducible morphisms of $\underline{\operatorname{mod}}\, \Gamma$ and $\underline{\operatorname{mod}}\, \Lambda$, which will be useful in Section 4.

Proposition 2.9. *Let Λ be a trivial extension of Cartan class Δ with Δ a Dynkin diagram. Let P be a projective Λ-module, $\Gamma = \operatorname{End}_\Lambda(P)^{\mathrm{op}}$ and let $\underline{e}_P : \underline{\mathcal{C}_P} \xrightarrow{\sim} \underline{\operatorname{mod}}\, \Gamma$ be the equivalence of categories induced by the evaluation functor at P. Then for any $X, Y \in \operatorname{ind} \underline{\mathcal{C}_P}$ we have*:

(a) *If $f : X \to Y$ is irreducible in $\underline{\operatorname{mod}}\, \Lambda$, then $\underline{e}_P(f) : \underline{e}_P(X) \to \underline{e}_P(Y)$ is irreducible in $\underline{\operatorname{mod}}\, \Gamma$.*
(b) *Let $X \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_r} M_r \xrightarrow{f_{r+1}} Y$ be a sectional path in ${}_S\Gamma_\Lambda$ and $f = f_{r+1} f_r \cdots f_1$. If $M_i \notin \underline{\mathcal{C}_P}$ for all $i = 1, 2, \ldots, r$, then $\underline{e}_P(f) : \underline{e}_P(X) \to \underline{e}_P(Y)$ is irreducible in $\underline{\operatorname{mod}}\, \Gamma$.*
(c) *If $f : X \to Y$ in $\underline{\operatorname{mod}}\, \Lambda$ is not irreducible and $\underline{e}_P(f) : \underline{e}_P(X) \to \underline{e}_P(Y)$ is irreducible in $\underline{\operatorname{mod}}\, \Gamma$, then for each chain of irreducible morphisms in $\underline{\operatorname{ind}}\, \Lambda$*

$$X = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \to \cdots \to M_{r-1} \xrightarrow{f_r} M_r = Y$$

with nonzero composition we have that $M_i \notin \underline{\mathcal{C}_P}$, for all $i = 1, 2, \ldots, r-1$.

Proof. The proof is straightforward and follows from the following lemma. □

Lemma 2.10. *Let Λ be a trivial extension of Cartan class Δ with Δ a Dynkin diagram. If $X \to M_1 \to \cdots \to M_r \to Y$ is a sectional path in ${}_S\Gamma_\Lambda$ then*:

$$\operatorname{Supp} \underline{\operatorname{Hom}}_\Lambda(X, -) \cap \operatorname{Supp} \underline{\operatorname{Hom}}_\Lambda(-, Y) = \{X, M_1, \ldots, M_r, Y\}.$$

Proof. Let $\pi : \mathbb{Z}\Delta \to {}_S\Gamma_\Lambda$ be the universal covering of ${}_S\Gamma_\Lambda$, and let ${}_S\Gamma_\Lambda[0]$ be a lifting of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\Delta$ at X [10, 3.5]. Then the sectional path $X \to M_1 \to \cdots \to M_r \to Y$ lifts to a sectional path $X[0] \to M_1[0] \to \cdots \to M_r[0] \to Y[0]$ in $\mathbb{Z}\Delta$. By the isomorphisms $\operatorname{Supp} k(\mathbb{Z}\Delta)(x,-) \xrightarrow{\sim} \operatorname{Supp} \underline{\operatorname{Hom}}_\Lambda(\pi(x),-)$ and $\operatorname{Supp} k(\mathbb{Z}\Delta)(-,x) \xrightarrow{\sim} \operatorname{Supp} \underline{\operatorname{Hom}}_\Lambda(-,\pi(x))$ induced by the universal covering $\pi : \mathbb{Z}\Delta \to {}_S\Gamma_\Lambda$ [10, 3.3] it is enough to prove that

$$\operatorname{Supp} k(\mathbb{Z}\Delta)\big(X[0],-\big) \cap \operatorname{Supp} k(\mathbb{Z}\Delta)\big(-,Y[0]\big) = \big\{X[0], M_1[0], \ldots, M_r[0], Y[0]\big\}.$$

This equality is a consequence of the fact that $X[0] \to M_1[0] \to \cdots \to M_r[0] \to Y[0]$ is a sectional path in $\mathbb{Z}\Delta$ and of the shape of the supports of the functors $k(\mathbb{Z}\Delta)(x,-)$ and $k(\mathbb{Z}\Delta)(-,y)$. □

3. Configurations arising from trivial extensions

Let Λ be a trivial extension of Cartan class $\mathbf{A}_n$, C an elimination cycle of Q_Λ, and Γ a trivial extension obtained from Λ by eliminating C. The main result of this section gives a lifting to $\mathbb{Z}\mathbf{A}_n$ of the radical rP_t for any vertex t of the cycle C. This is an important step in the proof of our main theorem, since, being $(Q_\Lambda)_0 = (Q_\Gamma)_0 \cup (\underline{C})_0$, it will allow us to use inductive arguments on the number of cycles of the quiver.

We recall (see [12]) that if Γ is a stable translation quiver and $k(\Gamma)$ the mesh-category associated to Γ, *a configuration* $\mathcal{C}$ of Γ is a set of vertices of Γ satisfying:

(a) for any vertex $x \in \Gamma_0$ there exists a vertex $y \in \mathcal{C}$ such that $k(\Gamma)(x,y) \neq 0$,
(b) $k(\Gamma)(x,y) = 0$ if x and y are different elements of $\mathcal{C}$,
(c) $k(\Gamma)(x,x) = k$ for all $x \in \mathcal{C}$.

Remark 3.1. Let Δ be a Dynkin diagram, Λ be a selfinjective algebra of Cartan class Δ, $\pi : \mathbb{Z}\Delta \to {}_S\Gamma_\Lambda$ be the universal covering of translation quivers, $\mathcal{C}_\Lambda = \{rP_i \colon i \in (Q_\Lambda)_0\}$ and $\widetilde{\mathcal{C}_\Lambda} = \pi^{-1}(\mathcal{C}_\Lambda)$. From [12] we know that $\widetilde{\mathcal{C}_\Lambda}$ is a configuration of $\mathbb{Z}\Delta$ and $\mathcal{C}_\Lambda$ is a configuration of ${}_S\Gamma_\Lambda$. We recall that the Nakayama permutation $\nu_\Delta : (\mathbb{Z}\Delta)_0 \to (\mathbb{Z}\Delta)_0$ and the Loewy length m_Δ of $k(\mathbb{Z}\Delta)$ satisfy the equality

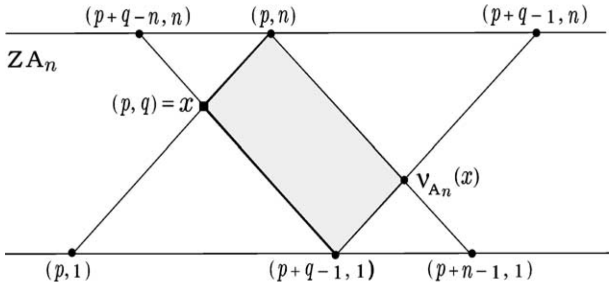
$$\tau^{-m_\Delta} = \nu_\Delta^2 \tau^{-1}.$$

Moreover, if Λ is a trivial extension then the fundamental group $\Pi({}_S\Gamma_\Lambda, x)$ associated with $\pi : \mathbb{Z}\Delta \to {}_S\Gamma_\Lambda$ is generated by τ^{m_Δ}, see [1,5].

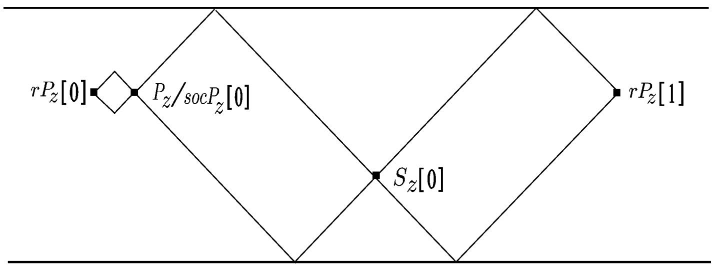
The points in the shaded area in the following picture, are those of

$$\operatorname{Supp} k(\mathbb{Z}\mathbf{A}_n)(x,-) = \operatorname{Supp} k(\mathbb{Z}\mathbf{A}_n)\big(-,\nu_{\mathbf{A}_n}(x)\big),$$

where $x = (p,q)$.



Remark 3.2. The picture illustrates the next proposition in the case $\Delta = \mathbf{A}_n$.



Proposition 3.3. *Let Λ be a trivial extension of Cartan class Δ with Δ a Dynkin diagram, and let z be a vertex of Q_Λ. Then for a lifting ${}_S\Gamma_\Lambda[0]$ of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\Delta$ at rP_z we have that* $\operatorname{Supp} k(\mathbb{Z}\Delta)(\tau^{-1}rP_z[0], -) \cap \operatorname{Supp} k(\mathbb{Z}\Delta)(-, rP_z[1]) = \{S_z[0]\}$ *and* $\nu_\Delta(\tau^{-1}rP_z[0]) = S_z[0] = \nu_\Delta^{-1}(rP_z[1])$.

Proof. By definition we have that $rP_z[1] = \tau^{-m_\Delta} rP_z[0]$. We proved in [10, 3.1] that $\operatorname{Supp} k(\mathbb{Z}\Delta)(x, -) \cap \operatorname{Supp} k(\mathbb{Z}\Delta)(-, \nu_\Delta^2(x)) = \{\nu_\Delta(x)\}$. Using that $\tau^{-m_\Delta} = \nu_\Delta^2 \tau^{-1}$ we obtain that $\operatorname{Supp} k(\mathbb{Z}\Delta)(\tau^{-1}rP_z[0], -) \cap \operatorname{Supp} k(\mathbb{Z}\Delta)(-, rP_z[1]) = \{\nu_\Delta \tau^{-1} rP_z[0]\}$. On the other hand, $\underline{\operatorname{Hom}}_\Lambda(P_z/\operatorname{soc} P_z, S_z) \neq 0$ and $\underline{\operatorname{Hom}}_\Lambda(S_z, rP_z) \neq 0$. So $S_z[0] \in \operatorname{Supp}(\tau^{-1}rP_z[0], -) \cap \operatorname{Supp}(-, rP_z[1])$ and therefore $\nu_\Delta \tau^{-1} rP_z[0] = S_z[0]$, proving the result. □

Proposition 3.4. *Let Λ be a trivial extension of Cartan class Δ with Δ a Dynkin diagram. The following conditions are equivalent for vertices $i \neq j$ of Q_Λ:*

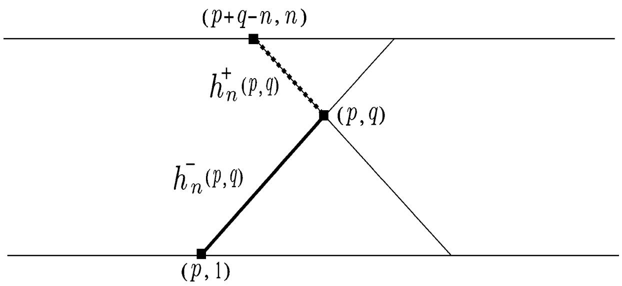
(a) *There exists an arrow $i \xrightarrow{\alpha} j$ in Q_Λ.*
(b) *For any lifting ${}_S\Gamma_\Lambda[0]$ of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\Delta$ we have that either $k(\mathbb{Z}\Delta)(\tau^{-1}rP_j[0], rP_i[0]) \neq 0$ or $k(\mathbb{Z}\Delta)(\tau^{-1}rP_j[0], rP_i[1]) \neq 0$.*

Proof. By 2.6 we have that there exists an arrow $i \xrightarrow{\alpha} j$ in Q_Λ if and only if $\underline{\operatorname{Hom}}_\Lambda(P_j/\operatorname{soc} P_j, rP_i) \neq 0$. Then the proposition is now an easy consequence of Remark 3.6 in [10]. □

We introduce now the notions of height functions and borders in $\mathbb{Z}\mathbf{A}_n$. To do that, we label the vertices of $\mathbf{A}_n$ as follows:

Definition 3.5. The *positive height* in $\mathbb{Z}\mathbf{A}_n$ is the function $h_n^+ : (\mathbb{Z}\mathbf{A}_n)_0 \to \{1, 2, \ldots, n\}$ defined by $h_n^+(p, q) = n - q + 1$, and the *top border* is the set $\{(p, n)\colon\ p \in \mathbb{Z}\}$ of vertices of $\mathbb{Z}\mathbf{A}_n$. Likewise, the *negative height* in $\mathbb{Z}\mathbf{A}_n$ is the function $h_n^- : (\mathbb{Z}\mathbf{A}_n)_0 \to \{1, 2, \ldots, n\}$ defined by $h_n^-(p, q) = q$, and the *bottom border* is the set $\{(p, 1)\colon\ p \in \mathbb{Z}\}$ of vertices of $\mathbb{Z}\mathbf{A}_n$.

Remark 3.6. For any vertex $(p, q) \in \mathbb{Z}\mathbf{A}_n$ we have that $h_n^-(p, q)$ is the "distance" from the bottom border of $\mathbb{Z}\mathbf{A}_n$ to (p, q), and $h_n^+(p, q)$ is the "distance" from the top border of $\mathbb{Z}\mathbf{A}_n$ to the vertex (p, q).



Proposition 3.7. *Let Λ be a trivial extension of Cartan class $\mathbf{A}_n$, j a vertex of Q_Λ, and let ${}_S\Gamma_\Lambda[0]$ be a lifting of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{A}_n$. Then*:

(a) *$r P_j / \operatorname{soc} P_j$ is indecomposable if and only if $r P_j[0]$ belongs to a border of $\mathbb{Z}\mathbf{A}_n$,*
(b) *if there is an arrow $i \xrightarrow{\alpha} j$ in Q_Λ and $r P_i \simeq P_j / \operatorname{soc} P_j$, then $r P_j / \operatorname{soc} P_j$ is indecomposable.*

Proof. The proof of (a) is straightforward and (b) follows from the description of the presentation for Λ given in 1.1 $\square$

Proposition 3.8. *Let Λ be a trivial extension of Cartan class $\mathbf{A}_n$, with $n > 1$. For any vertex z of Q_Λ the following conditions are equivalent*:

(i) *z is an insertion vertex of Q_Λ.*
(ii) *The projective P_z associated to z is uniserial.*
(iii) *$r P_z[0]$ belongs to a border of $\mathbb{Z}\mathbf{A}_n$.*
(iv) *$S_z[0] = P_z / r P_z[0]$ belongs to a border of $\mathbb{Z}\mathbf{A}_n$.*

In particular, the number of vertices $z \in Q_\Lambda$ such that $r P_z[0]$ belongs to a border of $\mathbb{Z}\mathbf{A}_n$ is larger than 1, and coincides with the number of insertion vertices of Q_Λ.

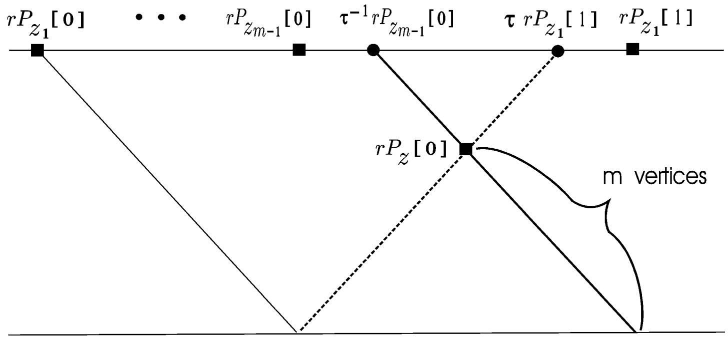
Proof. The result follows from the description of Λ given in 1.1, the fact that the cosyzygy functor $\Omega^{-1}: \underline{\mathrm{mod}}\,\Lambda \to \underline{\mathrm{mod}}\,\Lambda$ is an equivalence of categories, and the equality $S_z = \Omega^{-1}(rP_z)$. □

The following proposition is the main result of this section and will be useful throughout the paper.

Proposition 3.9. *Let Λ be a trivial extension of Cartan class $\mathbf{A}_n$, and let $C = .^z \leftarrow .^{z_1} \leftarrow \cdots \leftarrow .^{z_{m-1}} \leftarrow .^z$ be an oriented cycle of Q_Λ such that $z_1, \dots, z_{m-1}$ are insertion vertices of Q_Λ. Then for any lifting ${}_S\Gamma_\Lambda[0]$ of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{A}_n$ at rP_{z_1} we have*:

(a) *$rP_{z_1}[0]$ belongs to a border of $\mathbb{Z}\mathbf{A}_n$ and $\tau^{-t} rP_{z_1}[0] = rP_{z_{t+1}}[0]$ for $1 \leqslant t < m-1$,*
(b) *$\{rP_z[0]\} = \mathrm{Supp}\, k(\mathbb{Z}\mathbf{A}_n)(\tau^{-1} rP_{z_{m-1}}[0], -) \cap \mathrm{Supp}\, k(\mathbb{Z}\mathbf{A}_n)(-, \tau rP_{z_1}[1])$,*
(c) *$h_n^\varepsilon(rP_z[0]) = n - m + 1$, where $\varepsilon = +$ if $rP_{z_1}[0]$ belongs to the top border of $\mathbb{Z}\mathbf{A}_n$, and $\varepsilon = -$ otherwise.*

The next picture illustrates the situation when rP_{z_1} is in the top border of $\mathbb{Z}\mathbf{A}_n$.



Proof. Since $C = z \leftarrow z_1 \leftarrow \cdots \leftarrow z_{m-1} \leftarrow z$ is a minimal oriented cycle and $z_1, \dots, z_{m-1}$ are insertion vertices of Q_Λ we get by 3.8 that the projective P_{z_i} is uniserial for $i = 1, 2, \dots, m-1$, and therefore

$$P_{z_1}/\mathrm{soc}\, P_{z_1} \simeq rP_{z_2}, \quad \dots, \quad P_{z_{m-2}}/\mathrm{soc}\, P_{z_{m-2}} \simeq rP_{z_{m-1}}.$$

Then by 3.7 we obtain (a). Since $z_{m-1} \leftarrow z$ and $z \leftarrow z_1$ are arrows of Q_Λ we deduce from 3.4 that

$$rP_z[0] \in \mathrm{Supp}\, k(\mathbb{Z}\mathbf{A}_n)\big(\tau^{-1} rP_{z_{m-1}}[0], -\big) \quad \text{and} \quad \tau^{-1} rP_z[0] \in \mathrm{Supp}\, k(\mathbb{Z}\mathbf{A}_n)\big(-, rP_{z_1}[1]\big).$$

Hence

$$rP_z[0] \in \mathrm{Supp}\, k(\mathbb{Z}\mathbf{A}_n)\big(\tau^{-1} rP_{z_{m-1}}[0], -\big) \cap \mathrm{Supp}\, k(\mathbb{Z}\mathbf{A}_n)\big(-, \tau rP_{z_1}[1]\big).$$

Since $rP_{z_{m-1}}[0]$ and $rP_{z_1}[1]$ are in the same border of $\mathbb{Z}\mathbf{A}_n$ we get that this intersection of supports contains an unique vertex. Thus $\mathrm{Supp}\, k(\mathbb{Z}\mathbf{A}_n)(\tau^{-1} rP_{z_{m-1}}[0], -) \cap$

$\operatorname{Supp} k(\mathbb{Z}\mathbf{A}_n)(-, \tau r P_{z_1}[1]) = \{r P_z[0]\}$. This intersection determines the height of the vertex $r P_z[0]$. $\square$

Let Δ be a Dynkin diagram, Λ a trivial extension of Cartan class Δ, and $\pi : \mathbb{Z}\Delta \to {}_S\Gamma_\Lambda$ the universal covering of ${}_S\Gamma_\Lambda$. We recall that the Nakayama permutation ν_Δ has the following property: for each vertex x of $\mathbb{Z}\Delta$ there exists a path $w : x \to \nu_\Delta(x)$ whose image $\overline{w}$ in the mesh-category $k(\mathbb{Z}\Delta)$ is not zero, and w has longest length among all nonzero paths starting at x. Furthermore, it can be proven that ν_Δ commutes with the translation τ of $\mathbb{Z}\Delta$. So, ν_Δ induces a permutation ν_Λ on $({}_S\Gamma_\Lambda)_0$, since the fundamental group $\Pi({}_S\Gamma_\Lambda, x)$ is generated by τ^{m_Δ}. That is, the following diagram is commutative

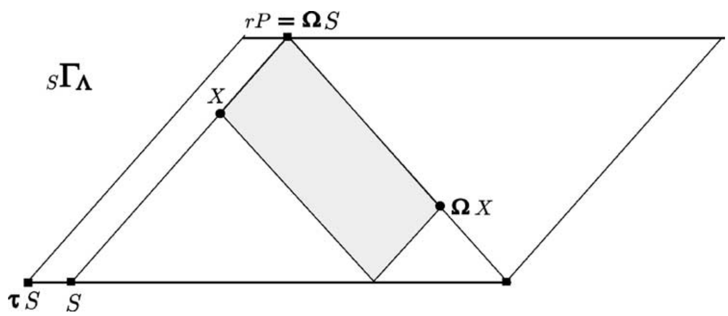
$$\begin{array}{ccc} \mathbb{Z}\Delta & \xrightarrow{\nu_\Delta} & \mathbb{Z}\Delta \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ {}_S\Gamma_\Lambda & \xrightarrow{\nu_\Lambda} & {}_S\Gamma_\Lambda \end{array}$$

In the following proposition we prove that ν_Λ is the syzygy functor when $\Delta = \mathbf{A}_n$.

Proposition 3.10. *Let Λ be a trivial extension of Cartan class $\mathbf{A}_n$, and let $\Omega : \underline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{mod}}\,\Lambda$ be the syzygy functor. Then for any $X \in \underline{\operatorname{ind}}\,\Lambda$ we have that*

$$\Omega(X) = \nu_\Lambda(X) \quad \textit{and} \quad \Omega^{-1}(X) = \tau^{-1}\nu_\Lambda(X).$$

Proof. We know that Ω commutes with the translation $\tau = DTr$ of ${}_S\Gamma_\Lambda$ and preserves sectional paths (see Chapter X in [3]). Then to prove that $\Omega = \nu_\Lambda$ on ${}_S\Gamma_\Lambda$ it is enough to see that $\Omega = \nu_\Lambda$ on a section of ${}_S\Gamma_\Lambda$. From 3.8 we have that there exists a simple Λ-module S in a border of ${}_S\Gamma_\Lambda$. Let $S \to \cdots \to X$ be a sectional path of length r in ${}_S\Gamma_\Lambda$. Then $rP = \Omega(S) \to \cdots \to \Omega(X)$ is also a sectional path of length r, where P is the projective cover of S and $\nu_\Lambda(S) = rP$ (see 3.2).



So $\Omega(X) = \nu_\Lambda(X)$ (see 3.1) and we get that Ω and ν_Λ coincide on the section starting at S, proving that $\Omega = \nu_\Lambda$. Thus $\Omega^{-1} = \tau^{-1}\nu_\Lambda$, since $\Omega^{-2} = \tau^{-1}$. $\square$

Corollary 3.11. *Let Λ be a trivial extension of Cartan class $\mathbf{A}_n$, let $\Omega : \underline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{mod}}\,\Lambda$ be the syzygy functor, and let $X \in \underline{\operatorname{ind}}\,\Lambda$. Then*:

(a) $\operatorname{Supp} \underline{\operatorname{Hom}}_\Lambda(X,-) = \operatorname{Supp} \underline{\operatorname{Hom}}_\Lambda(-,\Omega(X))$,
(b) $\operatorname{Supp} \underline{\operatorname{Hom}}_\Lambda(-,X) = \operatorname{Supp} \underline{\operatorname{Hom}}_\Lambda(\tau^{-1}\Omega(X),-)$.

Proof. Using 3.1 and 3.3 from [10] we have that $\operatorname{Supp} \underline{\operatorname{Hom}}_\Lambda(X,-) = \operatorname{Supp} \underline{\operatorname{Hom}}_\Lambda(-, \nu_\Lambda(X))$. Thus, the corollary follows from 3.10. □

4. The embedding of ${}_S\Gamma_{\operatorname{End}_\Lambda(P)^{\mathrm{op}}}$ in ${}_S\Gamma_\Lambda$

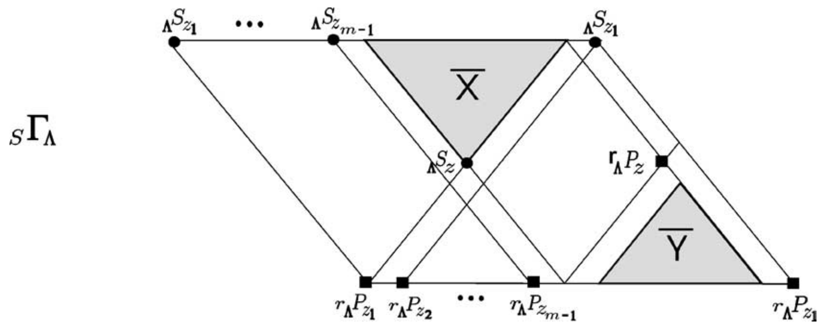
Let Γ be a trivial extension of Cartan class $\mathbf{A}_n$ and let z be an insertion vertex of Q_Γ. Consider the trivial extension Λ of Cartan class $\mathbf{A}_{n+m-1}$ obtained from Γ by inserting a cycle C at z. We recall that $\Gamma \simeq \operatorname{End}_\Lambda(P)^{\mathrm{op}}$ where P is the projective Λ-module $\coprod_{i\in(Q_\Gamma)_0} {}_\Lambda P_i$. In Section 2 we saw that the evaluation functor $e_P : \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$ allows us to identify $\operatorname{mod}\Gamma$ with the full subcategory $\mathcal{C}_P$ of $\operatorname{mod}\Lambda$. Moreover the functor e_P induces the equivalence of stable categories $\underline{e}_P : \underline{\mathcal{C}_P} \to \underline{\operatorname{mod}}\,\Gamma$. Let $\iota : \underline{\operatorname{mod}}\,\Gamma \to \underline{\operatorname{mod}}\,\Lambda$ be the full and faithful functor obtained by composing the inverse equivalence of $\underline{e}_P : \underline{\mathcal{C}_P} \to \underline{\operatorname{mod}}\,\Gamma$ and the inclusion $\underline{\mathcal{C}_P} \subseteq \underline{\operatorname{mod}}\,\Lambda$. In this section we will study the behavior of the irreducible morphisms of $\underline{\operatorname{mod}}\,\Gamma$ through the embedding $\iota : \underline{\operatorname{mod}}\,\Gamma \to \underline{\operatorname{mod}}\,\Lambda$. We start with some preliminaries.

Let $\mathcal{A}$ be a full subcategory of $\underline{\operatorname{ind}}\,\Lambda$. We denote by $\overline{\mathcal{A}}$ the full subquiver of ${}_S\Gamma_\Lambda$ whose vertices correspond to the objects of $\mathcal{A}$.

Let $\mathcal{C}$ be a k-linear category and let $\mathcal{A}$ be a class of objects in $\mathcal{C}$. We denote by ${}^\perp\mathcal{A} = \{X \in \mathcal{C}\colon \mathcal{C}(X,-)|_\mathcal{A} = 0\}$ the left orthogonal category of $\mathcal{A}$, and by $\mathcal{A}^\perp = \{X \in \mathcal{C}\colon \mathcal{C}(-,X)|_\mathcal{A} = 0\}$ the right orthogonal category of $\mathcal{A}$.

Proposition 4.1. *Let Γ be a trivial extension of Cartan class $\mathbf{A}_n$, and z be an insertion vertex of Q_Γ. Let Λ be the trivial extension obtained from Γ by inserting the cycle $C = z \leftarrow z_1 \leftarrow z_2 \leftarrow \cdots \leftarrow z_{m-1} \leftarrow z$ at z. Let $P = \coprod_{i\in(Q_\Gamma)_0} {}_\Lambda P_i$, $Q = \coprod_{i=1}^{m-1} {}_\Lambda P_{z_i}$, $\mathcal{B} = \operatorname{Supp} \underline{\operatorname{Hom}}_\Lambda(-, r_\Lambda Q)$, $\mathcal{B}' = \operatorname{Supp} \underline{\operatorname{Hom}}_\Lambda(Q/\operatorname{soc} Q, -)$, $\mathsf{X} = {}^\perp\mathcal{B}$ and $\mathsf{Y} = \mathcal{B}^\perp \cap \operatorname{ind}\underline{\mathcal{C}_P}$. Then*:

(a) *$\overline{\mathsf{X}}$ and $\overline{\mathsf{Y}}$ are the connected components of $\overline{\operatorname{ind}\underline{\mathcal{C}_P}}$ in ${}_S\Gamma_\Lambda$. Moreover, we have the following picture.*

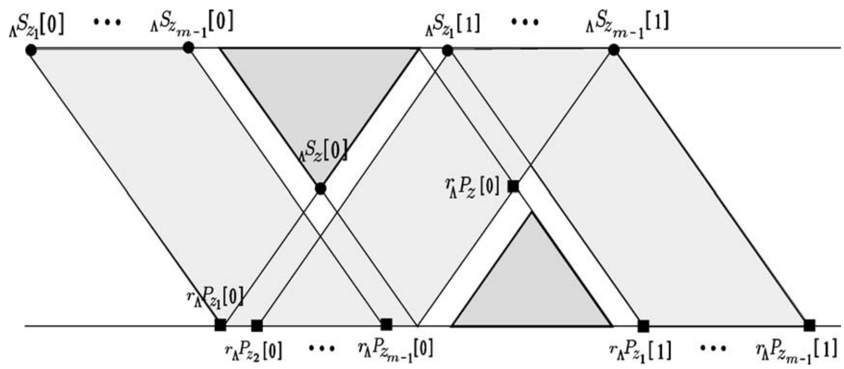


(b) *$\Omega^{-1}\mathcal{B}=\mathcal{B}'$, where $\Omega^{-1}:\underline{\mathrm{mod}}\,\Lambda\to\underline{\mathrm{mod}}\,\Lambda$ is the cosyzygy functor. Therefore $\underline{\mathrm{ind}}\,\Lambda\setminus\mathrm{ind}\,\underline{\mathcal{C}_P}=\mathcal{B}\cup\Omega^{-1}\mathcal{B}$.*
(c) *$\mathsf{X}=(\mathcal{B}')^{\perp}$ and $\mathsf{Y}={}^{\perp}(\mathcal{B}')\cap\mathrm{ind}\,\underline{\mathcal{C}_P}$.*
(d) *For any morphism $f:M\to N$ in $\mathrm{ind}\,\underline{\mathcal{C}_P}$, the following conditions are equivalent:*
  (d_1) *$\underline{e}_P(f):\underline{e}_P(M)\to\underline{e}_P(N)$ is irreducible in $\underline{\mathrm{mod}}\,\Gamma$,*
  (d_2) *there is a sectional path $M=M_0\to M_1\to M_2\to\cdots\to M_{r-1}\to M_r=N$ in ${}_S\Gamma_\Lambda$ with $r\geqslant 1$ and such that $M_j\notin\mathrm{ind}\,\underline{\mathcal{C}_P}$ for $j=1,2,\ldots,r-1$.*
(e) *For any $M,N\in\mathrm{ind}\,\underline{\mathcal{C}_P}$ the following conditions are equivalent:*
  (e_1) *$f:M\to N$ is irreducible in $\underline{\mathrm{mod}}\,\Lambda$.*
  (e_2) *$M,N\in\mathsf{X}$ or $M,N\in\mathsf{Y}$, and $\underline{e}_P(f):\underline{e}_P(M)\to\underline{e}_P(N)$ is irreducible in $\underline{\mathrm{mod}}\,\Gamma$.*
(f) *The functor $\underline{e}_P:\mathrm{ind}\,\underline{\mathcal{C}_P}\to\underline{\mathrm{mod}}\,\Gamma$ induces by restriction isomorphisms of quivers*

$$\overline{\mathsf{X}}\xrightarrow{\sim}\overline{\underline{e}_P(\mathsf{X})}\quad\text{and}\quad\overline{\mathsf{Y}}\xrightarrow{\sim}\overline{\underline{e}_P(\mathsf{Y})}.$$

(g) *$\underline{e}_P(\tau_\Lambda S_{z_1})\simeq r_\Gamma P_z$.*

Proof. (a) We know by 2.4 that $\underline{\mathrm{ind}}\,\Lambda\setminus\mathrm{ind}\,\underline{\mathcal{C}_P}=\mathcal{B}\cup\mathcal{B}'$. Let $\pi:\mathbb{Z}\mathbf{A}_{m+n-1}\to{}_S\Gamma_\Lambda$ be the universal covering of ${}_S\Gamma_\Lambda$. Since $\mathbb{Z}\mathbf{A}_{m+n-1}$ has no oriented cycles, it will be easier to prove (a) if we lift ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{A}_{m+n-1}$. Since z_1 is an insertion vertex of Q_Λ we have by 3.8 that the simple ${}_\Lambda S_{z_1}$ lifts to some border of $\mathbb{Z}\mathbf{A}_{m+n-1}$, which we may assume is the top border. Let ${}_S\Gamma_\Lambda[0]$ be a lifting of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{A}_{m+n-1}$ at ${}_\Lambda S_{z_1}$. Using 3.2 we determine the position of ${}_\Lambda S_{z_1}[0], r_\Lambda P_{z_1}[0]$ and ${}_\Lambda S_{z_1}[1]$ in $\mathbb{Z}\mathbf{A}_{n+m-1}$. On the other hand, 3.9 gives the position of $r_\Lambda P_{z_1}[0], r_\Lambda P_{z_2}[0],\ldots,r_\Lambda P_{z_{m-1}}[0]$. Using 3.2 again we can complete the following picture.



It follows from the definitions of $\mathcal{B}$ and $\mathcal{B}'$ that the lighter shaded area of this picture corresponds to $\pi^{-1}(\mathcal{B}\cup\mathcal{B}')$ (in fact, in this picture we just sketched $\mathcal{B}[0]\cup\mathcal{B}'[0]\cup\mathcal{B}[1]$). By [10, 3.3] we know that the covering π induces bijections $\mathrm{Supp}\,k(\mathbb{Z}\mathbf{A}_{m+n-1})(x,-)\xrightarrow{\sim}\mathrm{Supp}\,\underline{\mathrm{Hom}}_\Lambda(\pi(x),-)$ and $\mathrm{Supp}\,k(\mathbb{Z}\mathbf{A}_{m+n-1})(-,x)\xrightarrow{\sim}\mathrm{Supp}\,\underline{\mathrm{Hom}}_\Lambda(-,\pi(x))$ for any vertex x of $\mathbb{Z}\mathbf{A}_{m+n-1}$. Now (a) follows by looking at the supports of the functors $k(\mathbb{Z}\mathbf{A}_{m+n-1})(x,-)$ and $k(\mathbb{Z}\mathbf{A}_{m+n-1})(-,x)$.

(b) We know that $\Omega^{-1}(r_\Lambda Q)=Q/r_\Lambda Q$. Therefore $\Omega^{-1}\mathcal{B}=\mathrm{Supp}\,\underline{\mathrm{Hom}}_\Lambda(-,Q/r_\Lambda Q)$ $=\mathrm{Supp}\,\underline{\mathrm{Hom}}_\Lambda(Q/\mathrm{soc}\,Q,-)$, as follows from 3.11 or just from the above picture.

(c) Follows from (b) and the above picture.

(d) Let $f: M \to N$ in $\operatorname{ind}\underline{\mathcal{C}_P}$.

$(d_1) \Rightarrow (d_2)$. If $f: M \to N$ is irreducible in $\underline{\operatorname{mod}}\,\Lambda$ then $M \to N$ is sectional and (d_2) holds with $r = 1$.

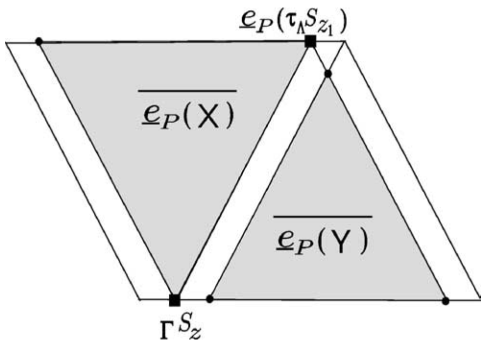
Suppose that $f: M \to N$ is not irreducible in $\underline{\operatorname{mod}}\,\Lambda$. Since $f \neq 0$ and Λ is of finite representation type we obtain from 2.9(c) a nonzero path $\mu: M = M'_0 \to M'_1 \to \cdots \to M'_{t-1} \to M'_t = N$ in $k({}_S\Gamma_\Lambda)$ such that $M'_j \notin \underline{\mathcal{C}_P}$ for $j = 1, 2, \ldots, t-1$. Let $\mathcal{S}$ be the set of nonzero paths $M = M_0 \to M_1 \to \cdots \to M_{r-1} \to M_r$ in $k({}_S\Gamma_\Lambda)$ such that $M_r \in \underline{\mathcal{C}_P}$ and $M_j \notin \underline{\mathcal{C}_P}$ for $j = 1, 2, \ldots, r-1$. It follows from (a) that $\mathcal{S}$ contains a unique sectional path $\gamma: M = M_0 \to M_1 \to \cdots \to M_{r-1} \to M_r$ in ${}_S\Gamma_\Lambda$. Moreover, any path of $\mathcal{S}$ factors through γ in $k({}_S\Gamma_\Lambda)$ and so does μ. Since $\underline{e}_P(f): \underline{e}_P(M) \to \underline{e}_P(N)$ is irreducible in $\underline{\operatorname{mod}}\,\Gamma$ we get that $M_r = N$, proving that (d_1) implies (d_2).

$(d_2) \Rightarrow (d_1)$. Follows from 2.9(a), (b).

(e) Follows from (a) and (d).

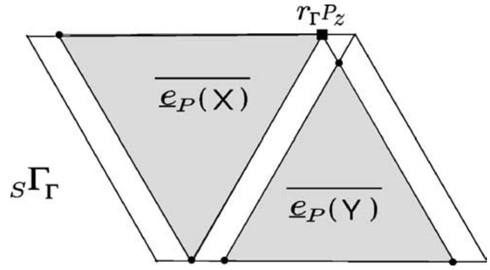
(f) Follows from (a) and (e).

(g) We observe that $\tau_\Lambda S_{z_1}$ is on a border of ${}_S\Gamma_\Lambda$ and belongs to the section $\vec{\mathcal{S}}_{{}_\Lambda S_z}$ starting at $\underline{{}_\Lambda S_z}$. Since $\underline{e}_P({}_\Lambda S_z) \simeq {}_\Gamma S_z$ (by 2.8) and $\underline{e}_P$ induces an isomorphism of quivers $\overline{\mathsf{X}} \xrightarrow{\sim} \overline{\underline{e}_P(\mathsf{X})}$ we get the following picture.



Therefore by 3.10 we obtain that $\underline{e}_P(\tau_\Lambda S_{z_1}) = \nu_\Gamma({}_\Gamma S_z) = r_\Gamma P_z$. $\square$

The partition $\{\mathsf{X}, \mathsf{Y}\}$ of $\operatorname{ind}\underline{\mathcal{C}_P}$ induces through the equivalence $\underline{e}_P: \underline{\mathcal{C}_P} \to \underline{\operatorname{mod}}\,\Gamma$ the partition $\{\underline{e}_P(\mathsf{X}), \underline{e}_P(\mathsf{Y})\}$ in $\underline{\operatorname{ind}}\,\Gamma$. Moreover, we proved that $\underline{e}_P(\tau_\Lambda S_{z_1}) \simeq r_\Gamma P_z$.



The embedding $\iota : \underline{\mathrm{mod}}\,\Gamma \to \underline{\mathrm{mod}}\,\Lambda$ induces a map $\iota : {}_S\Gamma_\Gamma \to {}_S\Gamma_\Lambda$ defined as follows. Let $\alpha : M \to N$ be an arrow in ${}_S\Gamma_\Gamma$. Then by 4.1(d) we know that there is only one sectional path in ${}_S\Gamma_\Lambda$ starting at $\iota(M)$ and ending at $\iota(N)$. We define $\iota(\alpha)$ to be such sectional path. It is not difficult to see that if ρ is a mesh relation in ${}_S\Gamma_\Gamma$ then $\iota(\rho)$ is zero in $k({}_S\Gamma_\Lambda)$. Therefore the map ι induces a functor, denoted also by $\iota : k({}_S\Gamma_\Gamma) \to k({}_S\Gamma_\Lambda)$.

Corollary 4.2. *The functor $\iota : k({}_S\Gamma_\Gamma) \to k({}_S\Gamma_\Lambda)$ above defined is full and faithful. Moreover, an arrow α belongs to one of the quivers $\overline{\underline{e}_P(\mathsf{X})}$, $\overline{\underline{e}_P(\mathsf{Y})}$ if and only if $\iota(\alpha)$ is an arrow in ${}_S\Gamma_\Lambda$.*

5. The embedding of $k(\mathbb{Z}\mathbf{A}_n)$ in $k(\mathbb{Z}\mathbf{A}_{n+m-1})$

Throughout this section Γ is a trivial extension of Cartan class $\mathbf{A}_n$, z is an insertion vertex of Q_Γ, and Λ is the trivial extension of Cartan class $\mathbf{A}_{n+m-1}$ obtained from Γ by inserting a cycle $C_z = z \leftarrow z_1 \leftarrow z_2 \leftarrow \cdots \leftarrow z_{m-1} \leftarrow z$ at z. In Section 4 we studied the embedding $\iota : \underline{\mathrm{mod}}\,\Gamma \to \underline{\mathrm{mod}}\,\Lambda$ and we showed that this functor induces a map $\iota : {}_S\Gamma_\Gamma \to {}_S\Gamma_\Lambda$ and a full and faithful functor $\iota : k({}_S\Gamma_\Gamma) \to k({}_S\Gamma_\Lambda)$. Let $\pi : \mathbb{Z}\mathbf{A}_n \to {}_S\Gamma_\Gamma$ and $\pi' : \mathbb{Z}\mathbf{A}_{n+m-1} \to {}_S\Gamma_\Lambda$ be the universal coverings of ${}_S\Gamma_\Gamma$ and ${}_S\Gamma_\Lambda$, respectively. We will define a functor $\Phi : k(\mathbb{Z}\mathbf{A}_n) \to k(\mathbb{Z}\mathbf{A}_{n+m-1})$ in such way that the following diagram is commutative:

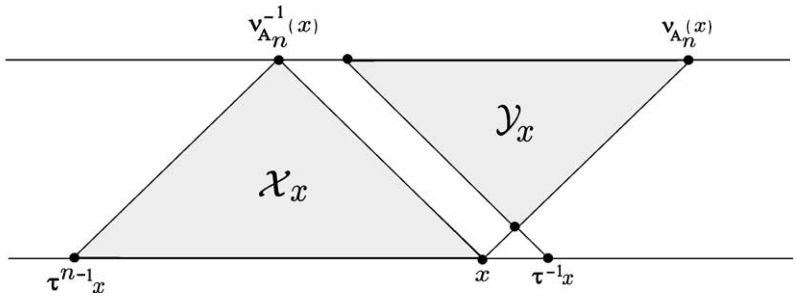
$$\begin{array}{ccc} k(\mathbb{Z}\mathbf{A}_n) & \xrightarrow{\Phi} & k(\mathbb{Z}\mathbf{A}_{n+m-1}) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ \underline{\mathrm{ind}}\,\Gamma & \xrightarrow{\iota} & \underline{\mathrm{ind}}\,\Lambda, \end{array}$$

where F and F' are well-behaved functors induced by the coverings π and π' respectively. In order to describe Φ, we lift the partition $\{\overline{\underline{e}_P(\mathsf{X})}, \overline{\underline{e}_P(\mathsf{Y})}\}$ of ${}_S\Gamma_\Gamma$ (respectively $\{\bar{\mathsf{X}}, \bar{\mathsf{Y}}\}$ of ${}_S\Gamma_\Lambda$) through π (respectively π') in an appropriate way. To do that, we introduce some definitions.

Let Q be a subquiver of $\mathbb{Z}\mathbf{A}_n$. We recall that the convex closure $\mathrm{Conv}(Q)$ in $\mathbb{Z}\mathbf{A}_n$, is the smallest convex subquiver of $\mathbb{Z}\mathbf{A}_n$ containing the set of vertices Q_0 of Q. Let $x \in \mathbb{Z}\mathbf{A}_n$ be a vertex in a border of $\mathbb{Z}\mathbf{A}_n$. We define the following full subquivers of $\mathbb{Z}\mathbf{A}_n$:

$$\mathcal{X}_x = \mathrm{Conv}\big(\big\{x, \nu_{\mathbf{A}_n}^{-2}(x)\big\}\big) \quad \text{and} \quad \mathcal{Y}_x = \mathrm{Conv}\big(\big\{\nu_{\mathbf{A}_n}(x), \tau^{-1}\nu_{\mathbf{A}_n}^{-1}(x)\big\}\big).$$

The picture below shows the shape of $\mathcal{X}_x$ and $\mathcal{Y}_x$ in $\mathbb{Z}\mathbf{A}_n$, if x is a vertex of the bottom border of $\mathbb{Z}\mathbf{A}_n$. We observe that $\nu_{\mathbf{A}_n}^{-2}(x) = \tau^{n-1}x$. Moreover, when $\pi(x) = r_\Gamma P_z$ then $\pi(\mathcal{X}_x) = \bar{\mathsf{X}}$ and $\pi(\mathcal{Y}_x) = \bar{\mathsf{Y}}$.

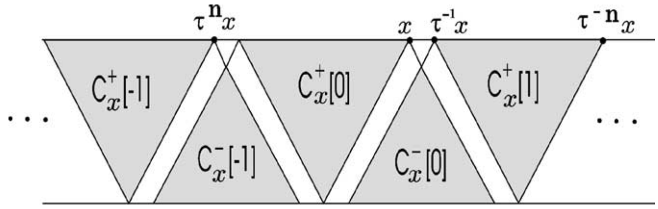


Let $\mathcal{Z}$ be a subquiver of $\mathbb{Z}\mathbf{A}_n$. For any integer i the shifted quiver $\mathcal{Z}[i]$ is $\tau^{-in}\mathcal{Z}$, and $\mathcal{Z}[\mathbb{Z}] = \bigcup_{i\in\mathbb{Z}} \mathcal{Z}[i]$. Moreover, for any vertex x belonging to a border of $\mathbb{Z}\mathbf{A}_n$ we define the partition $\{C_x^+[i], C_x^-[i]\colon i \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{A}_n$, where

$$C_x^+ = \begin{cases} \mathcal{Y}_x & \text{if } x \text{ is in the bottom border of } \mathbb{Z}\mathbf{A}_n, \\ \mathcal{X}_x & \text{if } x \text{ is in the top border of } \mathbb{Z}\mathbf{A}_n, \end{cases}$$

$$C_x^- = \begin{cases} \mathcal{X}_x & \text{if } x \text{ is in the bottom border of } \mathbb{Z}\mathbf{A}_n, \\ \mathcal{Y}_x & \text{if } x \text{ is in the top border of } \mathbb{Z}\mathbf{A}_n. \end{cases}$$

The following picture illustrates the situation when x is a vertex of the top border of $\mathbb{Z}\mathbf{A}_n$.



This partition induces in a natural way the *signature function* $\delta_n = \delta_n^x : (\mathbb{Z}\mathbf{A}_n)_0 \to \{-,+\}$, defined by $\delta_n(y) = -$ if $y \in C_x^-[\mathbb{Z}]$, and $\delta_n(y) = +$ otherwise.

The four pictures given in the preceding section, illustrating how ${}_S\Gamma_\Gamma$ can be considered inside ${}_S\Gamma_\Lambda$ by inserting the "bands" $\mathcal{B}$ and $\mathcal{B}'$, suggest the following definition.

Definition of the functor $\Phi : k(\mathbb{Z}\mathbf{A}_n) \to k(\mathbb{Z}\mathbf{A}_{n+m-1})$

Let x be a vertex in a border of $\mathbb{Z}\mathbf{A}_n$. Then we have the partition $\{C_x^+[i], C_x^-[i]\colon i \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{A}_n$. Using this partition, we will define the functor $\Phi = \Phi_x$.

Definition of Φ *on the vertices of* $\mathbb{Z}\mathbf{A}_n$

- For $(p,q) \in C_x^- \cup C_x^+$

$$\Phi(p,q) = \begin{cases} (p,q) & \text{if } (p,q) \in C_x^- \text{ and } x \text{ is in the bottom border of } \mathbb{Z}\mathbf{A}_n, \\ (p+m-1,q) & \text{if } (p,q) \in C_x^- \text{ and } x \text{ is in the top border of } \mathbb{Z}\mathbf{A}_n, \\ (p,q+m-1) & \text{if } (p,q) \in C_x^+. \end{cases}$$

- $\Phi(y) = \Phi(y[-i])[i]$ if $y \in C_x^{\pm}[i]$ and $i \neq 0$.

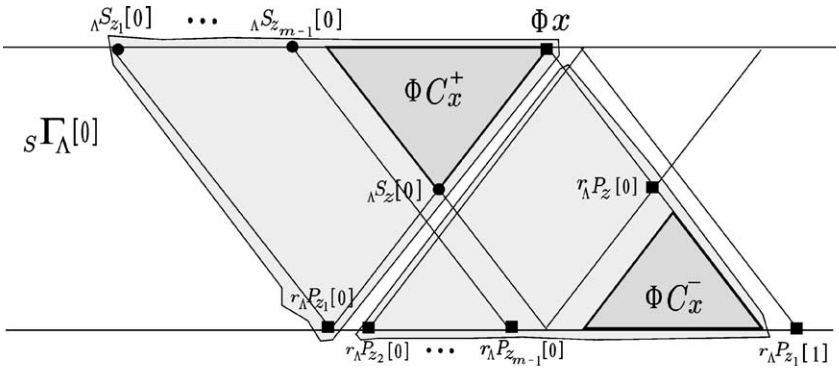
Definition of Φ on the arrows of $\mathbb{Z}\mathbf{A}_n$

Let $y \xrightarrow{\alpha} z$ be an arrow of $\mathbb{Z}\mathbf{A}_n$. By the definition of Φ on the vertices of $\mathbb{Z}\mathbf{A}_n$ we obtain that there is a unique sectional path γ in $\mathbb{Z}\mathbf{A}_{n+m-1}$ starting at $\Phi(y)$ and ending at $\Phi(z)$. Then we define $\Phi(\alpha) = \gamma$. We observe that $\Phi(\alpha)$ is an arrow in $\mathbb{Z}\mathbf{A}_{n+m-1}$ if and only if α is an arrow of $C_x^{\varepsilon}[j]$ for some integer j and some $\varepsilon = -, +$. Moreover, it is not difficult to see that, if ρ is a mesh relation in $\mathbb{Z}\mathbf{A}_n$, then $\Phi(\rho)$ is zero in $k(\mathbb{Z}\mathbf{A}_{n+m-1})$. Therefore the map $\Phi : \mathbb{Z}\mathbf{A}_n \to \mathbb{Z}\mathbf{A}_{n+m-1}$ induces a fully faithful functor $\Phi = \Phi_x : k(\mathbb{Z}\mathbf{A}_n) \to k(\mathbb{Z}\mathbf{A}_{n+m-1})$.

The rest of this section is devoted to study the behavior of the partition and the signature functions of $\mathbb{Z}\mathbf{A}_n$ under the functor Φ, as well as other properties of Φ.

Let $\pi : \mathbb{Z}\mathbf{A}_n \to {}_S\Gamma_\Gamma$ be the universal covering of ${}_S\Gamma_\Gamma$. Since z is an insertion vertex of Q_Γ we have by 3.8 that $\pi^{-1}(r_\Gamma P_z)$ belongs to a border of $\mathbb{Z}\mathbf{A}_n$. Then we obtain a functor $\Phi = \Phi_x : k(\mathbb{Z}\mathbf{A}_n) \to k(\mathbb{Z}\mathbf{A}_{n+m-1})$ for each $x \in \pi^{-1}(r_\Gamma P_z)$.

The next picture illustrates the following lemma if $r_\Gamma P_z$ lifts to the top border of $\mathbb{Z}\mathbf{A}_n$.



Lemma 5.1. *Let z be an insertion vertex of Q_Γ, and $\{\overline{\mathsf{X}}, \overline{\mathsf{Y}}\}$ the induced partition of $\overline{\operatorname{ind}\underline{\mathcal{C}_P}}$ in ${}_S\Gamma_\Lambda$ defined in 4.1. Let $\pi : \mathbb{Z}\mathbf{A}_n \to {}_S\Gamma_\Gamma$ and $\pi' : \mathbb{Z}\mathbf{A}_{n+m-1} \to {}_S\Gamma_\Lambda$ be the universal coverings of ${}_S\Gamma_\Gamma$ and ${}_S\Gamma_\Lambda$, respectively. Let ${}_S\Gamma_\Gamma[0]$ be a lifting of ${}_S\Gamma_\Gamma$ to $\mathbb{Z}\mathbf{A}_n$ at $\tau^{-1}r_\Gamma P_z$, and $x = r_\Gamma P_z[0]$.*

We fix a lifting of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{A}_{n+m-1}$ at ${}_\Lambda S_{z_1}$ by choosing ${}_\Lambda S_{z_1}[0] \in \pi'^{-1}({}_\Lambda S_{z_1})$ such that ${}_\Lambda S_{z_1}[0] = \nu_{\mathbf{A}_{n+m-1}}^{-2}(\Phi(x))$, where $\Phi = \Phi_x : k(\mathbb{Z}\mathbf{A}_n) \to k(\mathbb{Z}\mathbf{A}_{n+m-1})$. Then:

(a) ${}_S\Gamma_\Gamma[0] = \operatorname{Conv}(C_x^+ \cup C_x^-)$ *and* ${}_S\Gamma_\Lambda[0] = \operatorname{Conv}(C_{\Phi(x)}^+ \cup C_{\Phi(x)}^-)$.
(b) *For any vertex y of a border of $\mathbb{Z}\mathbf{A}_n$ we have that*

$$\Phi_x\big(C_y^{\pm}\big) \subseteq C_{\Phi_x(y)}^{\pm} \quad \textit{and} \quad \delta_n^y = \delta_{n+m-1}^{\Phi_x(y)} \circ \Phi_x.$$

(c) $\iota\pi|_{\mathcal{X}_x} : \mathcal{X}_x \to \overline{\mathsf{X}}$ *and* $\iota\pi|_{\mathcal{Y}_x} : \mathcal{Y}_x \to \overline{\mathsf{Y}}$ *are isomorphisms of quivers.*
(d) $\pi'\Phi|_{\mathcal{X}_x} : \mathcal{X}_x \to \overline{\mathsf{X}}$ *and* $\pi'\Phi|_{\mathcal{Y}_x} : \mathcal{Y}_x \to \overline{\mathsf{Y}}$ *are isomorphisms of quivers.*
(e) $\pi'\Phi = \iota\pi$.

Proof. (a), (b) and (d). The proof is straightforward.

(c) Follows from 4.1(f).

(e) Follows from (c), (d) and the fact that the group of automorphisms of $\mathcal{X}_x$ (respectively $\mathcal{Y}_x$) is trivial. □

Theorem 5.2. *Let Γ be a trivial extension of Cartan class $\mathbf{A}_n$, z an insertion vertex of Q_Γ and Λ the trivial extension obtained from Γ by inserting cycle $C_z = z \leftarrow z_1 \leftarrow \cdots \leftarrow z_{m-1} \leftarrow z$ at z. Let $\pi : \mathbb{Z}\mathbf{A}_n \to {}_S\Gamma_\Gamma$ and $\pi' : \mathbb{Z}\mathbf{A}_{n+m-1} \to {}_S\Gamma_\Lambda$ be the universal coverings of ${}_S\Gamma_\Gamma$ and ${}_S\Gamma_\Lambda$, respectively. Let ${}_S\Gamma_\Gamma[0]$ be a lifting of ${}_S\Gamma_\Gamma$ to $\mathbb{Z}\mathbf{A}_n$ at $\tau^{-1} r_\Gamma P_z$, and let $x = r_\Gamma P_z[0]$.*

We fix a lifting of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{A}_{n+m-1}$ at ${}_\Lambda S_{z_1}$ by choosing ${}_\Lambda S_{z_1}[0] \in \pi'^{-1}({}_\Lambda S_{z_1})$ such that ${}_\Lambda S_{z_1}[0] = \nu^{-2}_{\mathbf{A}_{n+m-1}}(\Phi(x))$, where $\Phi = \Phi_x : k(\mathbb{Z}\mathbf{A}_n) \to k(\mathbb{Z}\mathbf{A}_{n+m-1})$. Then:

(a) *For any vertex t of Q_Γ we have that Φ satisfies*:

(a$_1$) $\Phi(r_\Gamma P_t[0]) = \begin{cases} r_\Lambda P_t[0] & \text{if } t \neq z, \\ \tau_\Lambda S_{z_1}[1] & \text{if } t = z. \end{cases}$

(a$_2$) $\Phi({}_\Gamma S_t[0]) = {}_\Lambda S_t[0]$.

(b) *Let $F' : k(\mathbb{Z}\mathbf{A}_{m+n-1}) \to \underline{\mathrm{ind}}\,\Lambda$ be a well-behaved functor induced by $\pi' : \mathbb{Z}\mathbf{A}_{m+n-1} \to {}_S\Gamma_\Lambda$. Then there exists a well-behaved functor $F : k(\mathbb{Z}\mathbf{A}_n) \to \underline{\mathrm{ind}}\,\Gamma$ induced by $\pi : \mathbb{Z}\mathbf{A}_n \to {}_S\Gamma_\Gamma$ such that the following diagram is commutative*:

$$\begin{array}{ccc} k(\mathbb{Z}\mathbf{A}_n) & \xrightarrow{\Phi} & k(\mathbb{Z}\mathbf{A}_{m+n-1}) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ \underline{\mathrm{ind}}\,\Gamma & \xrightarrow{\iota} & \underline{\mathrm{ind}}\,\Lambda, \end{array}$$

Proof. (a) Let $t \in (Q_\Gamma)_0$. We prove (a$_1$) only, since (a$_2$) can be proved analogously. If $t = z$ then $\Phi(r_\Gamma P_t[0]) = \tau({}_\Lambda S_{z_1}[1])$ follows from the definition of Φ. Assume that $t \neq z$. Then by 2.8 we obtain that $r_\Lambda P_t \in \underline{\mathcal{C}_P}$ and $\iota(r_\Gamma P_t) = r_\Lambda P_t$. Thus, by 5.1(e) we get $\pi'\Phi(r_\Gamma P_t[0]) = \iota\pi(r_\Gamma P_t[0]) = \iota(r_\Gamma P_t) = r_\Lambda P_t$, proving (a).

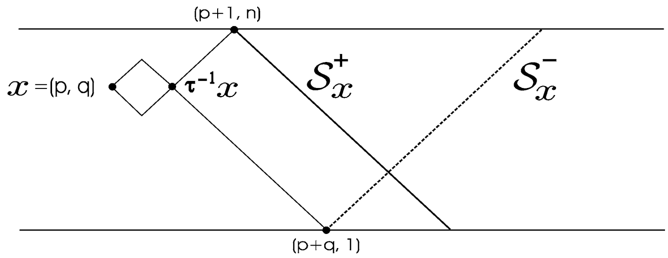
(b) From 5.1(e) we have that $\pi'\Phi = \iota\pi$. Now, we go on to define F on the arrows of $\mathbb{Z}\mathbf{A}_n$. Let $x \xrightarrow{\alpha} y$ be an arrow of $\mathbb{Z}\mathbf{A}_n$, we define $F(\alpha) = \underline{e}_P F'\Phi(\alpha)$ where $\underline{e}_P = \underline{\mathrm{Hom}}_\Lambda(P, -) : \underline{\mathcal{C}_P} \to \underline{\mathrm{mod}}\,\Gamma$ is the equivalence of categories giving in 2.2. Then we have a functor $F : k(\mathbb{Z}\mathbf{A}_n) \to \underline{\mathrm{ind}}\,\Gamma$. Moreover, by 4.1(d) we get that $\underline{e}_P F'\Phi(\alpha)$ is irreducible in $\underline{\mathrm{mod}}\,\Gamma$ for any arrow $x \xrightarrow{\alpha} y$ of $\mathbb{Z}\mathbf{A}_n$. □

6. Construction of the configuration associated to a trivial extension of Cartan class $\mathbf{A}_n$

Let Λ be a trivial extension of Cartan class $\mathbf{A}_n$, and let $\pi : \mathbb{Z}\mathbf{A}_n \to {}_S\Gamma_\Lambda$ be the universal covering of ${}_S\Gamma_\Lambda$. In this section we give an algorithm to determine the configuration $\widetilde{\mathcal{C}_\Lambda}$ of $\mathbb{Z}\mathbf{A}_n$ associated to Λ. We recall that $\widetilde{\mathcal{C}_\Lambda} = \pi^{-1}(\mathcal{C}_\Lambda)$, where $\mathcal{C}_\Lambda$ is the set of vertices of

${}_S\Gamma_\Lambda$ representing the radicals of the indecomposable projective Λ-modules. We define the subset $r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{A}_n)$ of $\mathbb{Z}\mathbf{A}_n$ and prove that $r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{A}_n)[\mathbb{Z}] = \bigcup_{i\in\mathbb{Z}} r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{A}_n)[i]$ is the desired configuration. We start with some useful definitions.

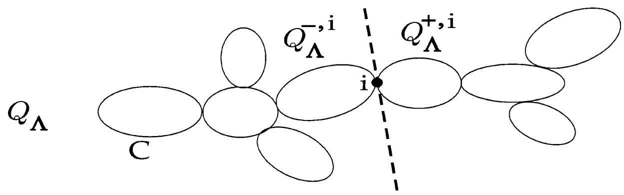
Definition 6.1. Let (p, q) be a vertex of $\mathbb{Z}\mathbf{A}_n$. We associate to this vertex the sections $\mathcal{S}^+_{(p,q)}$ and $\mathcal{S}^-_{(p,q)}$ of $\mathbb{Z}\mathbf{A}_n$ starting at $(p+1, n)$ and $(p+q, 1)$, respectively.



Definition 6.2. Let Λ be a trivial extension of Cartan class $\mathbf{A}_n$ and let C be a minimal oriented cycle of Q_Λ. We call C *cycle of reference*, if C meets at most one of the remainder cycles of Q_Λ.

Definition 6.3. Let C be a cycle of reference in Q_Λ. For each vertex $i \in Q_\Lambda$ we have that C induces a partition $\{Q_\Lambda^{-,i}, Q_\Lambda^{+,i}\}$ in Q_Λ, defined as follows:

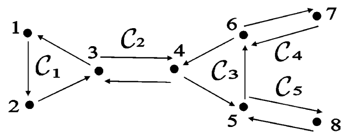
(a) the quivers $Q_\Lambda^{-,i}$ and $Q_\Lambda^{+,i}$ are full connected subquivers of Q_Λ which meet only at the vertex i,
(b) $Q_\Lambda^{-,i}$ is union of minimal oriented cycles and contains the cycle of reference C.



Definition 6.4. Let C be a cycle of reference in Q_Λ. Associated to C we define the *height map* $h_\Lambda = h_{\Lambda,C} : (Q_\Lambda)_0 \to \mathbf{N}$ and the *border map* $\partial_\Lambda = \partial_{\Lambda,C} : (Q_\Lambda)_0 \to \{-,+\}$ by:

- $h_\Lambda(i)$ is the number of vertices of the quiver $Q_\Lambda^{+,i}$.
- $\partial_\Lambda(x) = +$ for any vertex x of the cycle C, and ∂_Λ is defined inductively on the remaining cycles as follows. Let C' and C'' be different minimal oriented cycles which meet at the vertex t and assume that ∂_Λ is defined on C', then we define $\partial_\Lambda(x) = -\partial_\Lambda(t)$ for the vertices $x \in (\underline{C''})_0$, $x \neq t$. The function $-\partial_\Lambda : (Q_\Lambda)_0 \to \{-,+\}$ is obtained from ∂_Λ following the rules $-- = +$ and $-+ = -$.

Example. Let Λ be the trivial extension of Cartan class $\mathbf{A}_8$ given by the quiver:



The cycles of reference in Q_Λ are: $\mathcal{C}_1, \mathcal{C}_4, \mathcal{C}_5$. In the following table we give the height and border maps associated to the reference cycles $\mathcal{C}_1$ and $\mathcal{C}_4$.

$i \in (Q_\Lambda)_0$	1	2	3	4	5	6	7	8
$\partial_{\Lambda,\mathcal{C}_1}$	+	+	+	−	+	+	−	−
$h_{\Lambda,\mathcal{C}_1}$	1	1	6	5	2	2	1	1
$\partial_{\Lambda,\mathcal{C}_4}$	−	−	+	−	−	+	+	+
$h_{\Lambda,\mathcal{C}_4}$	1	1	3	4	2	7	1	1

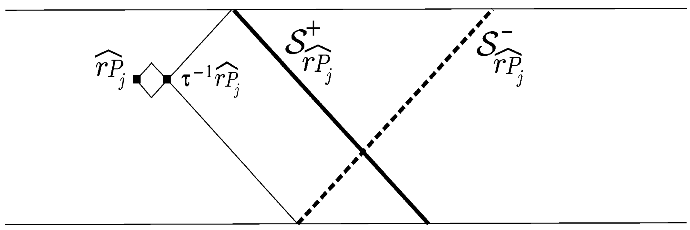
Let $\mathcal{C}$ be a cycle of reference of Q_Λ and t an insertion vertex belonging to $\mathcal{C}$. The pair $(\mathcal{C}, t)$ induces a *maximal tree* $\mathcal{T}_{\mathcal{C},t}$ in Q_Λ, which is obtained from Q_Λ by deleting exactly one arrow (chosen in appropriate way) from each minimal oriented cycle of Q_Λ. To obtain $\mathcal{T}_{\mathcal{C},t}$ we start by deleting the arrow of $\mathcal{C}$ starting at t. Let now $\mathcal{C}'$ and $\mathcal{C}''$ be different minimal oriented cycles of Q_Λ meeting at the vertex t', and assume that an arrow of $\mathcal{C}'$ has been deleted, then we delete the arrow of $\mathcal{C}''$ starting at t'.

Now we are in a position to define the set $r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{A}_n) = \{\widehat{rP_i} \in (\mathbb{Z}\mathbf{A}_n)_0\colon\ i \in (Q_\Lambda)_0\}$. Afterwards we will prove that $r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{A}_n)[\mathbb{Z}] = \bigcup_{i\in\mathbb{Z}} r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{A}_n)[i]$ is the desired configuration.

Definition 6.5. Let $\mathcal{C}$ be a cycle of reference in Q_Λ, t an insertion vertex belonging to $\mathcal{C}$, u a vertex of the top border of $\mathbb{Z}\mathbf{A}_n$, and $\mathcal{T}_{\mathcal{C},t}$ the tree defined above.

We define the set $r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{A}_n)$ of vertices of $\mathbb{Z}\mathbf{A}_n$ by the following rules:

(i) $\widehat{rP_t} = u$,
(ii) Let $i \to j$ be an arrow of $\mathcal{T}_{\mathcal{C},t}$ and assume that $\widehat{rP_j}$ is defined. Then $\widehat{rP_i}$ is the vertex in $\mathcal{S}^{\partial_\Lambda(i)}_{\widehat{rP_j}}$ with height $h_n^{\partial_\Lambda(i)}(\widehat{rP_i}) = h_\Lambda(i)$.



We observe that (ii) can be stated as follows: if $i \to j$ is an arrow of Q_Λ, $x_j = (a, b)$ and x_i has not been defined, then we set $x_i = (a + h_\Lambda(i), n - h_\Lambda(i) + 1)$ if $\partial_\Lambda(i) = +$, and $x_i = (a + b, h_\Lambda(i))$ otherwise.

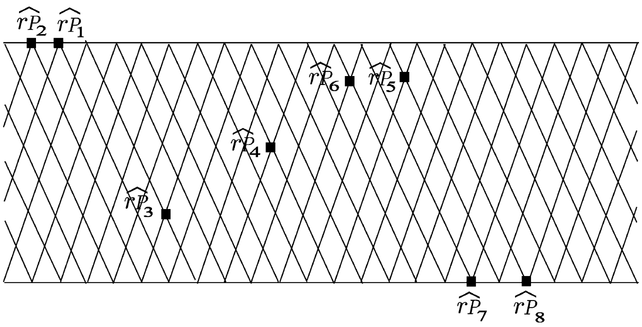
Remark 6.6. If x and y are insertion vertices of Q_Λ and there is an arrow $x \to y$ in $\mathcal{T}_{\mathcal{C},t}$, then $\widehat{rP_y}, \widehat{rP_x}$ are consecutive vertices in the corresponding border of $\mathbb{Z}\mathbf{A}_n$.

Example. Let Λ be the trivial extension of Cartan class $\mathbf{A}_8$ given after Definition 6.4. We choose $\mathcal{C}_1$ as a cycle of reference and we fix the vertex $t = 2$. The table gives the values of $\partial_{\Lambda,\mathcal{C}_1}$ and $h_{\Lambda,\mathcal{C}_1}$ on the vertices of Q_Λ.

$i \in (Q_\Lambda)_0$	1	2	3	4	5	6	7	8
$\partial_{\Lambda,\mathcal{C}_1}$	+	+	+	−	+	+	−	−
$h_{\Lambda,\mathcal{C}_1}$	1	1	6	5	2	2	1	1

The arrows of $\mathcal{T}_{\mathcal{C}_1,2}$ are: $2 \leftarrow 1, 1 \leftarrow 3, 3 \leftarrow 4, 4 \leftarrow 6, 6 \leftarrow 7, 6 \leftarrow 5, 5 \leftarrow 8$.

In the following picture we indicate the vertices of $r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{A}_8)$ with small black squares.

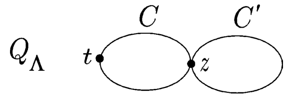


We state next our main result.

Theorem 6.7. *Let Λ be a trivial extension of Cartan class $\mathbf{A}_n$, C a cycle of reference, t an insertion vertex belonging to C, and u a vertex in the top border of $\mathbb{Z}\mathbf{A}_n$. Let $\pi : \mathbb{Z}\mathbf{A}_n \to {}_S\Gamma_\Lambda$ be the universal covering of ${}_S\Gamma_\Lambda$ which lifts the radical $r_\Lambda P_t$ to u, and ${}_S\Gamma_\Lambda[0]$ be the lifting of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\mathbf{A}_n$ at rP_t such that $rP_t[0] = u$. Let $r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{A}_n) = \{\widehat{rP_i} \in (\mathbb{Z}\mathbf{A}_n)_0 \colon i \in (Q_\Lambda)_0\}$ be the set associated to these data. Then*:

(a) $\widetilde{\mathcal{C}_\Lambda} = r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{A}_n)[\mathbb{Z}]$,
(b) $\pi(\widehat{rP_i}) = rP_i$ *for any* $i \in (Q_\Lambda)_0$,
(c) $h_n^{\partial_\Lambda(i)}(rP_i[0]) = h_\Lambda(i)$ *for any* $i \in (Q_\Lambda)_0$.

We will prove this theorem by induction on the number of minimal oriented cycles of Q_Λ. In order to do that, we delete a minimal oriented cycle of Q_Λ obtaining a trivial extension Γ, getting the functions: $h_\Lambda, h_\Gamma, \partial_\Lambda, \partial_\Gamma$. The restriction of ∂_Λ to $(Q_\Gamma)_0$ is ∂_Γ. However, the relationship between h_Λ and h_Γ is more complicated, as we can see in the following example.



If we eliminate the cycle C' of Q_Λ we obtain that the quiver Q_Γ is C. Let x be a vertex of C. If $x \neq z$ then $h_\Lambda(x) = 1 = h_\Gamma(x)$. On the other hand, $h_\Lambda(z)$ is equal to the number of vertices of C' and $h_\Gamma(z) = 1$. To get a closer relation between h_Λ and h_Γ we introduce the notion of free and linked vertices.

Definition 6.8. Let Λ be a trivial extension of Cartan class $\mathbf{A}_n$, and let C and C' be minimal oriented cycles of Q_Λ. Let $C = C_1, C_2, \ldots, C_t = C'$ be a chain of minimal oriented cycles of Q_Λ such that $(\underline{C_i})_0 \cap (\underline{C_{i+1}})_0 = \{x_i\}$ for any $i = 1, 2, \ldots, t-1$. We say that the vertices $x_1, x_2, \ldots, x_{t-1}$ are (C, C')-*linked* and that the remaining vertices of Q_Λ are (C, C')-*free*.

Example. In the example given after 6.4 the vertices (C_1, C_4)-linked are: 3, 4, 6.

Proposition 6.9. *Let Λ be a trivial extension of Cartan class $\mathbf{A}_n$ and let C be a cycle of reference in Q_Λ. Let Γ be the trivial extension obtained from Λ by eliminating a cycle C' different from C. For any vertex $z \in (Q_\Gamma)_0$ we have that $h_\Lambda(z) = h_{\Lambda,C}(z)$ and $h_\Gamma(z) = h_{\Gamma,C}(z)$ are related as follows*:

$$h_\Lambda(z) = \begin{cases} h_\Gamma(z) & \text{if } z \text{ is } (C, C')\text{-free}, \\ h_\Gamma(z) + |(\underline{C'})_0| - 1 & \text{if } z \text{ is } (C, C')\text{-linked}. \end{cases}$$

Proof. The proof is straightforward. □

We also need to know the relationship between the border and signature functions ∂_Λ and δ_n, which will be important in the inductive step in the proof of the theorem.

Proposition 6.10. *With the hypothesis of the theorem, let C' in Q_Λ be another minimal oriented cycle with at least one insertion vertex z. Then for any vertex x in Q_Λ the following conditions hold*:

$$\delta_n^{r_\Lambda P_z[0]}\big(rP_x[0]\big) = \begin{cases} \partial_\Lambda(x) & \text{if } x \text{ is } (C, C')\text{-free}, \\ -\partial_\Lambda(x) & \text{if } x \text{ is } (C, C')\text{-linked}. \end{cases}$$

Moreover, $h_n^{\partial_\Lambda(x)}(rP_x[0]) = 1$ if x is an insertion vertex of Q_Λ.

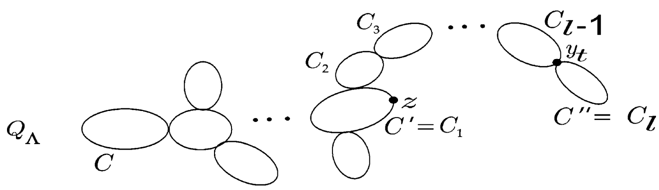
Proof. We assume that either $Q_\Lambda = C \cup C'$ or C' is not an elimination cycle, because otherwise the proof goes on likewise by considering the trivial extension Γ obtained from Λ by eliminating the cycle C'. We choose a cycle C'' of Q_Λ in the following way:

If $Q_\Lambda = C \cup C'$, then $C'' = C$. If Q_Λ is the union of more than two minimal oriented cycles and C' is not an elimination cycle, then C'' is an elimination cycle different from C.

Consider a chain of different minimal oriented cycles $C' = C_1, C_2, \dots, C_{\ell-1}, C_\ell = C''$ with $(\underline{C_i})_0 \cap (\underline{C_{i+1}})_0 \neq \emptyset$ for $i = 1, 2, \dots, \ell - 1$. Let

$$\underline{C''} = .^{y_t} \leftarrow .^{y_1} \leftarrow .^{y_2} \leftarrow \cdots \leftarrow .^{y_{t-1}} \leftarrow .^{y_t}$$

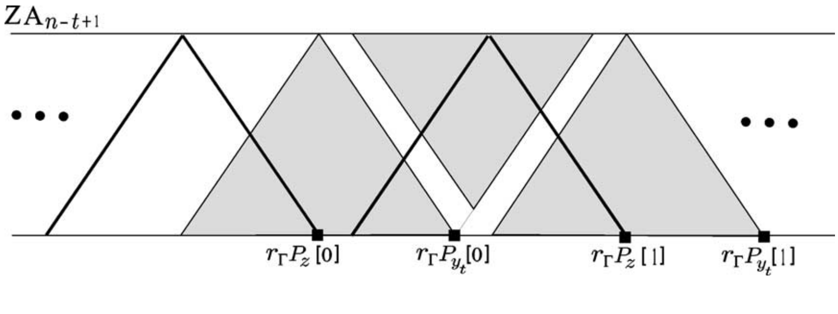
where $\{y_t\} = (\underline{C_{\ell-1}})_0 \cap (\underline{C_\ell})_0$. Then y_t is (C', C'')-linked.



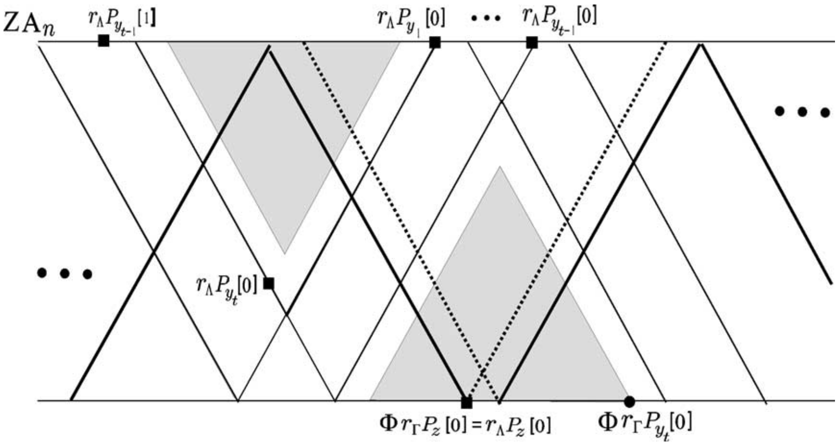
Let Γ be the trivial extension of Cartan class $\mathbf{A}_{n-t+1}$ obtained from Λ by eliminating the cycle C''. Then y_t is an insertion vertex in Q_Γ and therefore the radical $r_\Gamma P_{y_t}$ lifts to some border of $\mathbb{Z}\mathbf{A}_{n-t+1}$ (see 3.8) and induces the partition $\{C^-_{r_\Gamma P_{y_t}[0]}[i], C^+_{r_\Gamma P_{y_t}[0]}[i]\colon i \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{A}_{n-t+1}$. We will assume that the vertex $r_\Gamma P_{y_t}[0]$ is in the bottom border or $\mathbb{Z}\mathbf{A}_{n-t+1}$ (in the other case the proof is similar). Hence

$$\delta^{r_\Gamma P_z[0]}_{n-t+1}\big(r_\Gamma P_{y_t}[0]\big) = -.$$

The idea of the proof is to use the embedding $\Phi = \Phi_{r_\Gamma P_{y_t}[0]} : k(\mathbf{A}_{n-t+1}) \to k(\mathbf{A}_n)$ given in Section 5 to compare the partitions $\{C^-_{r_\Gamma P_z[0]}[i], C^+_{r_\Gamma P_z[0]}[i]\colon i \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{A}_{n-t+1}$ and $\{C^-_{r_\Lambda P_z[0]}[i], C^+_{r_\Lambda P_z[0]}[i]\colon i \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{A}_n$. We may assume that $\partial_\Gamma(z) = -$ (otherwise the proof is similar). Then $r_\Gamma P_z[0] \in C^-_{r_\Gamma P_{y_t}[0]}[\mathbb{Z}]$. The shaded regions in the following picture correspond to the partition $\{C^-_{r_\Gamma P_{y_t}[0]}[i], C^+_{r_\Gamma P_{y_t}[0]}[i]\colon i \in \mathbb{Z}\}$.



As observed before 5.1, by applying $\Phi = \Phi_{r_\Gamma P_{y_t}[0]}$ to this partition we obtain the indicated shaded regions.



Considering in the last picture the partition associated to the vertex $r_\Lambda P_z[0]$ we obtain

$$\delta_n^{r_\Lambda P_z[0]}\big(r_\Lambda P_{y_t}[0]\big) = -, \quad \delta_n^{r_\Lambda P_z[0]}\big(r_\Lambda P_{y_j}[0]\big) = +, \quad \text{for } 1 \leqslant j < t, \quad \text{and}$$
$$\delta_n^{r_\Lambda P_z[0]}\big(r_\Lambda P_z[0]\big) = -.$$

For this we use that $\Phi(r_\Gamma P_z[0]) = r_\Lambda P_z[0]$, by 5.2(a$_1$).

Now we prove the proposition by induction on the number of cycles of Q_Λ. If this number is two, then $C'' = C$ and $\partial_\Lambda(y_j) = +$ for all j. Comparing with the values of $\delta_n^{r_\Lambda P_z[0]}$ just obtained we have that the proposition holds for the vertices of C. On the other hand, $z \in C'$ and $\delta_n^{r_\Lambda P_z[0]}(r_\Lambda P_z[0]) = -$. Since all vertices in C' different from y_t are insertion vertices, the radicals of the corresponding projective modules lift to the same border or $\mathbb{Z}\mathbf{A}_n$. So $\delta_n^{r_\Lambda P_z[0]}$ coincides on them and takes therefore the value $-$, which is also the value of ∂_Λ on them. Thus the result holds also for the vertices of C, and therefore for all vertices of Q_Λ.

Suppose now that Q_Λ is the union of more than two minimal oriented cycles. By the inductive hypothesis we know that

$$\delta_{n-t+1}^{r_\Gamma P_z[0]}\big(r_\Gamma P_x[0]\big) = \begin{cases} \partial_\Gamma(x) & \text{if } x \text{ is } (C, C')\text{-free}, \\ -\partial_\Gamma(x) & \text{if } x \text{ is } (C, C')\text{-linked}. \end{cases}$$

By 5.1(b) we have that $\delta_{n-t+1}^{r_\Gamma P_z[0]}(r_\Gamma P_x[0]) = \delta_n^{\Phi(r_\Gamma P_z[0])}(\Phi(r_\Gamma P_x[0])) = \delta_n^{r_\Lambda P_z[0]}(r_\Lambda P_x)$, for any $x \in (Q_\Gamma)_0$, $x \neq y_t$. The last equality follows from 5.2(a$_1$), since $\Phi = \Phi_{r_\Gamma P_{y_t}}[0]$. On the other hand, ∂_Γ and ∂_Λ coincide in $(Q_\Gamma)_0$.This proves that

$$\delta_n^{r_\Lambda P_z[0]}\big(r_\Lambda P_x[0]\big) = \begin{cases} \partial_\Lambda(x) & \text{if } x \text{ is } (C, C')\text{-free}, \\ -\partial_\Lambda(x) & \text{if } x \text{ is } (C, C')\text{-linked} \end{cases}$$

for all $x \in (Q_\Gamma)_0 \setminus \{y_t\}$.

So we only have to prove that these equalities hold for vertices x in C'', this is, for $y_1, \ldots, y_{t-1}, y_t$. This follows from the following facts:

(a) We have that

$$\delta_{n-t+1}^{r_\Gamma P_z[0]}\big(r_\Lambda P y_t[0]\big) = -.$$

So by the induction hypothesis we get that $\partial_\Gamma(y_t) = -$ and consequently $\partial_\Lambda(y_t) = -$. This value coincides with $\delta_n^{r_\Lambda P_z[0]}(r_\Lambda P_{y_t}[0])$, and y_t is (C, C')-free. Therefore the first equality holds for y_t.

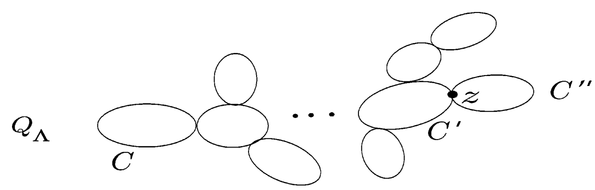
(b) $\delta_n^{r_\Lambda P_z[0]}(r_\Lambda P_{y_t}[0]) = -\partial_\Lambda(P_{y_j}[0]) = +$, and $\partial_\Lambda(y_t) = -\partial_\Lambda(y_j)$, for all $j = 1, \ldots, t-1$. $\square$

Now we are in a position to prove the main result of this section.

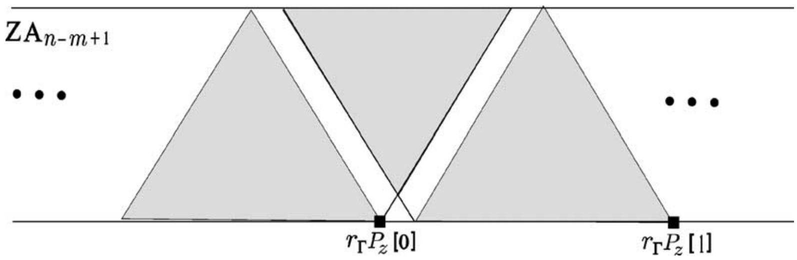
Proof of Theorem 6.7. It is enough to prove (b), which implies (a) and (c). The proof will be carried out by induction on the number of minimal oriented cycles of Q_Λ.

Case I. Suppose that $Q_\Lambda = C = .^1 \leftarrow .^2 \leftarrow \cdots .^n \leftarrow .^1$ is a minimal oriented cycle. We may assume that the fixed vertex in C is $t = 1$. Then by 3.9 we have that $rP_1[0], rP_2[0], \ldots, rP_n[0]$ are consecutive vertices in the top border of $\mathbb{Z}\mathbf{A}_n$, and $rP_1[0] = \widehat{rP_1}$. On the other hand, it follows from 6.6 that $\widehat{rP_1}, \widehat{rP_2}, \ldots, \widehat{rP_n}$ are also consecutive vertices in the top border of $\mathbb{Z}\mathbf{A}_n$. Thus $rP_i[0] = \widehat{rP_i}$ for any i, so (b) holds.

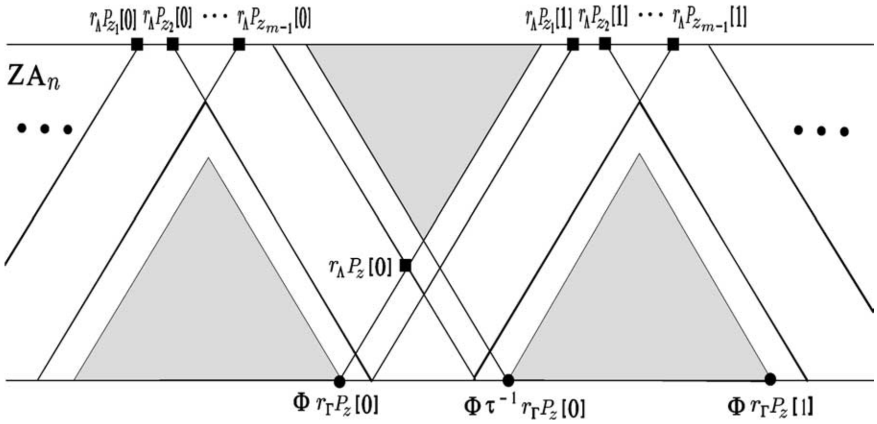
Case II. Suppose that Q_Λ has at least two minimal oriented cycles. Let $C'' = .^z \leftarrow .^{z_1} \leftarrow \cdots .^{z_{m-1}} \leftarrow .^z$ be an elimination cycle different from C, and let C' be a minimal oriented cycle such that $\underline{(C')}_0 \cap \underline{(C'')}_0 = \{z\}$.



Let Γ be the trivial extension of Cartan class $\mathbf{A}_{n-m+1}$ obtained from Λ by eliminating the cycle C''. We can assume that $r_\Gamma P_z[0]$ is in the bottom border of $\mathbb{Z}\mathbf{A}_{n-m+1}$, since otherwise the proof is similar. Let $\Phi = \Phi_{r_\Gamma P_z[0]} : k(\mathbb{Z}\mathbf{A}_{n-m+1}) \to k(\mathbb{Z}\mathbf{A}_n)$ be the embedding given in Section 5. Let $v \in \mathbb{Z}\mathbf{A}_{n-m+1}$ such that $\Phi(v) = u$. Using Φ we compare the sets $r\mathcal{P}(\Gamma, \mathbb{Z}\mathbf{A}_{n-m+1})$, relative to C, t and v, and $r\mathcal{P}(\Lambda, \mathbb{Z}\mathbf{A}_n)$, relative to C, t and u. The shaded regions of the following picture correspond to the partition $\{C^-_{r_\Gamma P_z[0]}[i], C^+_{r_\Gamma P_z[0]}[i]\colon i \in \mathbb{Z}\}$ of $\mathbb{Z}\mathbf{A}_{n-m+1}$.



The functor $\Phi = \Phi_{r_\Gamma P_z[0]}$ sends the above partition to the shaded regions of the following picture.



By 6.10 we have $\partial_\Gamma(z) = -$, since $r_\Gamma P_z[0]$ is in the bottom border of $\mathbb{Z}\mathbf{A}_{n-m+1}$. We assume that the theorem holds for algebras with less cycles than Λ. So it holds for Γ. In particular $\pi'(\widehat{r_\Gamma P_i}) = r_\Gamma P_i$ for any $i \in (Q_\Gamma)_0$, where $\pi' : \mathbb{Z}\mathbf{A}_{n-m+1} \to {}_S\Gamma_\Gamma$ denotes the universal covering of ${}_S\Gamma_\Gamma$.

In all that follows we use the notation: $x_i = \widehat{r_\Gamma P_i}$, $X_i = \widehat{r_\Lambda P_i}$. Thus, to prove the theorem we need to prove that $\pi(X_i) = r_\Lambda P_i$, for any $i \in (Q_\Lambda)_0$.

We start by proving that $\Phi(x_i) = X_i$ for a given $i \in (Q_\Gamma)_0 \setminus \{z\}$ implies that $\pi(X_i) = r_\Lambda P_i$. In fact, by the inductive hypothesis we know that $\pi'(\widehat{r_\Gamma P_i}) = r_\Gamma P_i$. Thus $\pi(X_i) = \pi\Phi(x_i) = \iota\pi'(x_i) = \iota(r_\Gamma P_i) = r_\Lambda P_i$ (see Section 5 and Theorem 2.8). So we will prove that $\Phi(x_i) = X_i$ for $i \in (Q_\Gamma)_0 \setminus \{z\}$. We start by proving two lemmas.

Lemma A. *With the preceding notations and hypothesis, let x be a vertex of $\mathbb{Z}\mathbf{A}_{n-m+1}$. Then*:

(a$_1$) $\Phi(\mathcal{S}_x^{\pm}) \subseteq \mathcal{S}_{\Phi(x)}^{\pm}$, *if x and $\tau^{-1}x$ belong to the same component of the partition* $\{C^-_{r_\Gamma P_z[0]}[i], C^+_{r_\Gamma P_z[0]}[i]: i \in \mathbb{Z}\}$ *of* $\mathbb{Z}\mathbf{A}_{n-m+1}$.

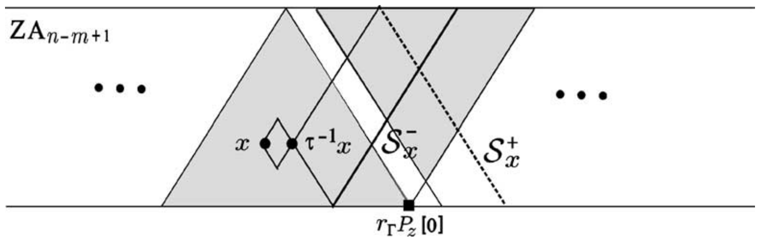
(a$_2$) $\Phi(\mathcal{S}^{\partial_\Gamma(z)}_{r_\Gamma P_z[0]}) \subseteq \mathcal{S}^{\partial_\Gamma(z)}_{r_\Lambda P_z[0]}$.

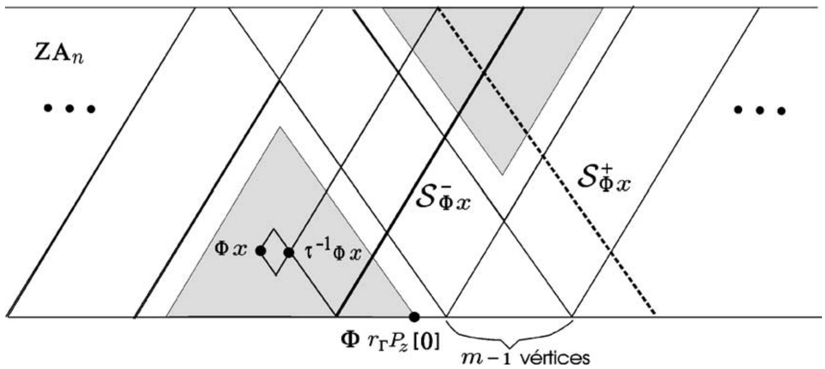
(a$_3$) *If* $x = r_\Gamma P_j[0]$ *for some* $j \in (Q_\Gamma)_0 \setminus \{z\}$*, then* x *and* $\tau^{-1}x$ *belong to the same component of the partition* $\{C^-_{r_\Gamma P_z[0]}[i], C^+_{r_\Gamma P_z[0]}[i]\colon i \in \mathbb{Z}\}$ *of* $\mathbb{Z}\mathbf{A}_{n-m+1}$.

(a$_4$) $\begin{cases} h_n^{\delta(x)}(\Phi(x)) = h_{n-m+1}^{\delta(x)}(x), \\ h_n^{-\delta(x)}(\Phi(x)) = h_{n-m+1}^{-\delta(x)}(x) + m - 1, \end{cases}$ *where* $\delta(x) = \delta_{n-m+1}^{r_\Gamma P_z[0]}(x)$.

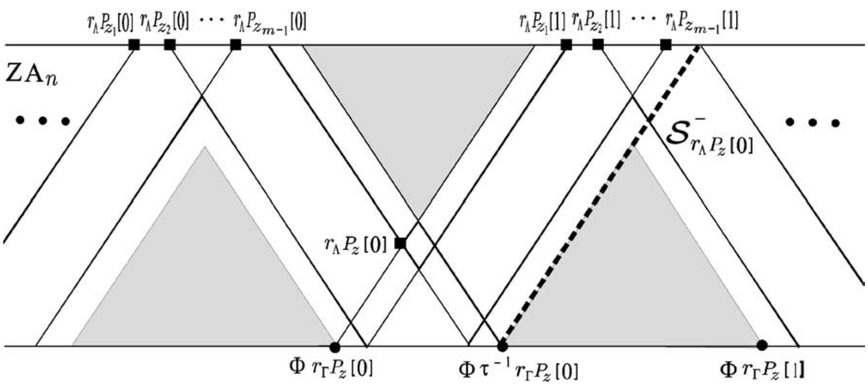
(a$_5$) $h_n^{\partial_\Lambda(i)}(\Phi(x_i)) = h_\Lambda(i)$*, for* $i \in (Q_\Gamma)_0$.

Proof. (a$_1$) and (a$_4$) follow easily from the definition of Φ (see Section 5) and the following two pictures.





(a$_2$) Follows from the next picture if $\partial_\Gamma(z) = -$. The other case is similar.



(a$_3$) Follows from the fact that

$$\underline{\mathrm{Hom}}_\Gamma(r_\Gamma P_k, r_\Gamma P_i) = 0 \quad \text{for } k \neq i.$$

(a$_5$) Since we assumed that the theorem holds for Γ we know that $h_{n-m+1}^{\partial_\Gamma(i)}(x_i) = h_\Gamma(i)$, and the result follows from a_4, using 6.9 and 6.10. □

Lemma B. *With the preceding notations and hypothesis, let $i \to j$ be an arrow of Q_Γ belonging to the maximal tree $\mathcal{T}_{C,t}$ (see 6.5). Then:*

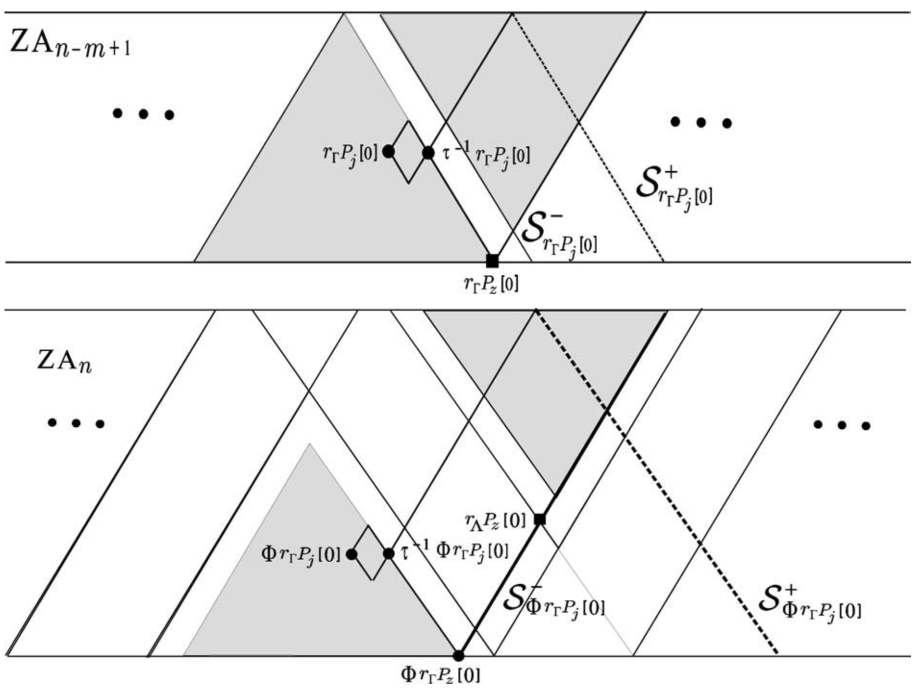
(b$_1$) *If $i, j \neq z$ then $\Phi(x_i) \in \mathcal{S}_{\Phi(x_j)}^{\partial_\Lambda(i)}$.*
(b$_2$) *If $i = z$ and $x_z = r_\Gamma P_z[d]$, then $x_j = r_\Gamma P_j[d]$, and $r_\Lambda P_z[d] \in \mathcal{S}_{\Phi(x_j)}^{\partial_\Lambda(z)}$. Moreover, $h_n^{\partial_\Lambda(z)}(r_\Lambda P_z[d]) = h_\Lambda(z)$.*

Proof. (b$_1$) By the definition of $r\mathcal{P}(\Gamma, \mathbb{Z}\mathbf{A}_{n-m+1})$ we have that $x_i \in \mathcal{S}_{x_j}^{\partial_\Gamma(i)}$. From Lemma A(a$_1$), (a$_3$) we know that

$$\Phi\big(\mathcal{S}_{x_j}^{\partial_\Gamma(i)}\big) \subseteq \mathcal{S}_{\Phi(x_j)}^{\partial_\Lambda(i)},$$

since $\partial_\Gamma(i) = \partial_\Lambda(i)$. So $\Phi(x_i) \in \mathcal{S}_{\Phi(x_j)}^{\partial_\Lambda(i)}$, proving that (b$_1$) holds.

(b$_2$) Assume that $i = z$, that is, we have an arrow $z \to j$ in Q_Γ with $\partial_\Gamma(z) = -$. First we assume that $x_z = r_\Gamma Pz[0]$. By induction we know that $r_\Gamma P_z[0] \in \mathcal{S}_{\Phi(x_j)}^-$. Since $j \neq z$ the situation is the following:



We know that $\pi'(x_j) = r_\Gamma P_j$, and the first picture shows that, more precisely, $x_j = r_\Gamma P_j[0]$. The second picture shows that

$$r_\Lambda P_z[0] \in \mathcal{S}^-_{\Phi(r_\Gamma P_j[0])} \quad \text{and} \quad h_n^-\bigl(r_\Lambda P_z[0]\bigr) = m = h_\Lambda(z),$$

proving (b_2) when $d = 0$. If d is an arbitrary integer, the result is proven using an appropriate shifting. □

We are now in a position to finish the proof of (b) in the Theorem 6.7. Let $i \in (Q_\Lambda)_0$. First we prove that $\Phi(x_i) = X_i$ for $i \in (Q_\Gamma)_0$, $i \neq z$. We observe that $\Phi(x_t) = X_t$, because $\Phi(v) = u$.

Let $i \in (Q_\Gamma)_0$, let $i \to j$ be an arrow of Q_Γ belonging to the maximal tree $\mathcal{T}_{C,t}$ (see 6.5) and assume that $\Phi(x_j) = X_j$.

If both i, j are different from z, then $\Phi(x_i) \in \mathcal{S}^{\partial_\Lambda(i)}_{X_j}$, by ($b_1$) of Lemma B. From ($a_5$) of Lemma A we know that

$$h_n^{\partial_\Lambda(i)}\bigl(\Phi(x_i)\bigr) = h_\Lambda(i),$$

so $\Phi(x_i) = X_i$ in this case.

Let now $i = z$, so that we are considering an arrow $z \to j$. Since we are assuming that $\pi'(x_z) = r_\Gamma P_z$, there is d so that $x_z = r_\Gamma P_z[d]$. We are assuming that $\Phi(x_j) = X_j$, so (b_2) of Lemma B states that

$$r_\Lambda P_z[d] \in \mathcal{S}^{\partial_\Gamma(z)}_{X_j} \quad \text{and} \quad h_n^{\partial_\Lambda(z)}\bigl(r_\Lambda P_z[d]\bigr) = h_\Lambda(z).$$

That is, $X_z = r_\Lambda P_z[d]$, and therefore (b) holds for z. Assume finally that $j = z$. Then the arrow considered is $i \to z$. So, the vertex i is (C, C')-free because $i \to z$ is an arrow of the maximal tree $\mathcal{T}_{C,t}$. Therefore $\partial_\Gamma(z) = \partial_\Gamma(i) = \partial_\Lambda(i)$, and using that $X_z = r_\Lambda P_z[d]$ we obtain that (a_2) of Lemma A means that

$$\Phi\bigl(\mathcal{S}^{\partial_\Gamma(i)}_{x_z}\bigr) \subseteq \bigl(\mathcal{S}^{\partial_\Lambda(i)}_{X_z}\bigr).$$

Since we are assuming that $x_i \in \mathcal{S}^{\partial_\Gamma(i)}_{x_z}$ we obtain $\Phi(x_i) \in \mathcal{S}^{\partial_\Lambda(i)}_{X_z}$. This, together with ($a_5$) of Lemma A, implies that $\Phi(x_i) = X_i$.

We finished the proof that $\pi(X_i) = r_\Lambda P_i$ for any $i \in (Q_\Gamma)_0$. So, to end the proof of the theorem we only need to prove this latter equality for the remaining vertices of Q_Λ. This is, for $z_1, z_2, \ldots, z_{m-1}$. We know that that $\widehat{r_\Lambda P_z} = r_\Lambda P_z[d]$, and we may assume that $d = 0$, since otherwise we apply an appropriate shifting. Then from the picture preceding Lemma A we obtain that $\widehat{r_\Lambda P_{z_1}} = r_\Lambda P_{z_1}[1]$, and therefore $\widehat{r_\Lambda P_i} = r_\Lambda P_i[1]$ for $i = z_1, z_2, \ldots, z_{m-1}$ (see also Remark 6.6), proving (b) in this case and ending the proof of the theorem. □

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