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Journal of Algebra 275 (2004) 615–628
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

On the representation dimension of some classes of algebras

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Received 8 April 2003
Communicated by Kent R. Fuller

Abstract

We show that the representation dimension of the following classes of algebras is at most 3: (a) Artin algebras A such that the functor $\operatorname{Hom}_A(D(A), -)$ has finite length (or dually, $\operatorname{Hom}_A(-, A)$ has finite length). These algebras coincide with the right (left) glued algebras, as introduced in [I. Assem, F.U. Coelho, J. Pure Appl. Algebra 96 (3) (1994) 225]; and (b) Trivial extensions of iterated tilted algebras.

Keywords: Artin algebras; Representation dimension; Trivial extensions

The concept of representation dimension of an Artin algebra A denoted by $\operatorname{rep.dim} A$, was introduced by M. Auslander [3] in the early 70s in an attempt to, paraphrasing him, *give a reasonable way of measuring how far A is from being representation-finite* (see Section 1 below for the appropriate definitions).

For some time, this notion stayed apart from the main lines of investigation in the representation theory of algebras. Recently, its interest has revived, and many interesting connections have been established with different problems in representation theory, as well as with other areas. For details see, for instance, [5,8–10,12,13].

It has been shown by Auslander that an algebra A is representation-finite if and only if $\operatorname{rep.dim} A \leqslant 2$. On the other hand, O. Iyama proved in [9] that the representation dimension

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2003.09.022

of an Artin algebra is always finite, using the relationship with quasihereditary algebras. An interesting connection with the finitistic dimension conjecture follows from the work of Igusa and Todorov in [8], and is given by the fact that if an Artin algebra has representation dimension at most three, then its finitistic dimension is finite.

The purpose of the present work is to calculate the representation dimension for some classes of algebras. We will show, for instance, that it is at most three for the following algebras:

(a) Artin algebras A such that the functor $\mathrm{Hom}_A(D(A), -)$ has finite length (or dually, $\mathrm{Hom}_A(-, A)$ has finite length). These algebras coincide with the right (left) glued algebras, as introduced in [1]. The class of right glued algebras includes all the hereditary algebras as well as all tilted algebras with complete slices in their preinjective components.
(b) Trivial extensions of iterated tilted algebras.

For that, we shall prove a criterion which appears implicitly in the works of Auslander [3] and Xi [12]. Also, as a consequence of this criterion, we get a better insight of the relations between the representation dimension of an algebra which is a one-point extension $B[M]$ of an algebra B, and the representation dimension of B itself. This extends some results of [12].

This paper is organized as follows. In the first section, after recalling some preliminary notions needed along the work, we state and prove the above mentioned criterion. Sections 2 and 3 deal with the calculation of the representation dimension of the algebras mentioned in (a) and (b) above, while in Section 4, we show some results concerning one-point extension algebras.

1. Preliminaries

1.1. Throughout this paper, all our algebras are Artin algebras. For an algebra A, we denote by $\operatorname{mod} A$ its category of finitely generated left A-modules and by $\operatorname{ind} A$ a full subcategory of $\operatorname{mod} A$ having as objects a full set of representatives of the isomorphism classes of the indecomposable A-modules. Also, given $M \in \operatorname{mod} A$, we denote by $\operatorname{add} M$ the full subcategory of $\operatorname{mod} A$ having as objects the direct sums of indecomposable summands of M. We denote by $\mathrm{pd}_A M$ (or $\mathrm{id}_A M$) the projective dimension (or injective dimension, respectively) of M. Finally, we denote by $\mathrm{gl.dim}\, A$ the global dimension of A, that is, the supremum of the projective dimensions of modules in $\operatorname{mod} A$.

We recall that an A-module M is a *generator* (or a *cogenerator)* for mod A provided for each $X \in \operatorname{mod} A$, there exists an epimorphism $M' \to X$ (or a monomorphism $X \to M'$) with $M' \in \operatorname{add} M$.

For unexplained notions and facts needed on $\operatorname{mod} A$ we refer the reader to [4].

1.2. The notion of representation dimension of an algebra was introduced in [3] by Auslander. We refer to this work for the original definition. For us, it will be more convenient to use the following characterization, also proven in [3].

Definition. The *representation dimension* of an Artin algebra is the number $\operatorname{rep.dim} A = \inf\{\operatorname{gl.dim}(\operatorname{End}_A M)\colon M \text{ is a generator–cogenerator of } \operatorname{mod} A\}$.

1.3. The first aim is to show a criterion for the calculation of the representation dimension of an algebra. From now on, A will denote an Artin algebra, and let $\mathcal{C}$ be a full subcategory of $\operatorname{mod} A$. We recall that a map $f : C \to M$ is called a right $\mathcal{C}$-approximation of the A-module M if C is in $\mathcal{C}$ and the sequence $(-, C) \to (-, M) \to 0$ is exact in $\mathcal{C}$. Moreover, we will say that an exact sequence

$$0 \to C_r \xrightarrow{f_r} \cdots \xrightarrow{f_2} C_1 \xrightarrow{f_1} M \to 0$$

is a *$\mathcal{C}$-approximation resolution of M* if C_i is in $\mathcal{C}$ for all i and the sequence

$$0 \to (-, C_r) \xrightarrow{(-,f_r)} \cdots \xrightarrow{(-,f_2)} (-, C_1) \xrightarrow{(-,f_1)} (-, M) \to 0$$

is exact in $\mathcal{C}$. We say that r is the *length of the resolution.*

Definition. An A-module $\overline{X}$ is said to have the *r-resolution property* if each A-module M has an $\operatorname{add}\overline{X}$-approximation resolution of length r.

Remarks. (a) The condition of the above definition can be replaced by a similar one holding for indecomposable modules M.

(b) Clearly, the modules in $\operatorname{add}\overline{X}$ always have an $\operatorname{add}\overline{X}$-approximation resolution of length 1.

(c) Given a module M in a subcategory $\mathcal{C}$, one can, dually, define a *left $\mathcal{C}$-approximation of M*, and a *$\mathcal{C}$-approximation coresolution of M*. Also, one can look at the notion of *r-coresolution property*.

1.4. Examples. (a) Let A be a representation-finite algebra, and let $M_1, M_2, \ldots, M_s$ be a set of representatives of all isoclasses of indecomposable A-modules. Clearly, $\overline{X} = M_1 \oplus \cdots \oplus M_s$ has the 1-resolution property, since $\operatorname{add}\overline{X} = \operatorname{mod} A$. It is also not difficult to see that if an algebra A has a module M satisfying the 1-resolution property, then A is representation-finite, and $\operatorname{add} M = \operatorname{mod} A$.

(b) Let H be a non-semisimple hereditary algebra and let $\overline{X} = H \oplus D(H)$. We show that such $\overline{X}$ has the 2-resolution property. Let M be an indecomposable H-module not in $\operatorname{add}\overline{X}$, and consider the minimal projective resolution $0 \to P_1 \to P_0 \to M \to 0$ of M. Clearly, $\operatorname{Hom}_H(H \oplus D(H), M) = \operatorname{Hom}_H(H, M)$ because M is not in $\operatorname{add}\overline{X}$ and therefore is not injective, so the sequence

$$0 \to \big(H \oplus D(H), P_1\big) \to \big(H \oplus D(H), P_0\big) \to \big(H \oplus D(H), M\big) \to 0$$

is exact and we are done.

1.5. We will see in the sequel that if a module $\overline{X}$ satisfies the r-resolution property then $\mathrm{gl.dim}(\mathrm{End}_A \overline{X}) \leqslant r + 1$. If, moreover, $\overline{X}$ is a generator–cogenerator of $\operatorname{mod} A$, then $\mathrm{rep.dim}\, A \leqslant r + 1$. This result has been used by Auslander in [3] and by Xi in [12], in order to give bounds for the representation dimension of some classes of algebras.

In the following theorem, which is the main result of this section, we will prove that the converse statement also holds.

Theorem. *Let A be an Artin algebra. The following statements are equivalent for a positive integer r:*

(a) $\mathrm{rep.dim}\, A \leqslant r + 1$;
(b) *there exists a generator–cogenerator of* $\operatorname{mod} A$ *satisfying the r-resolution property.*

Though, as we mentioned above, the implication (b) $\Rightarrow$ (a) has been implicitly proven in [3,12], for the convenience of the reader we shall provide here a complete proof of this result. For that, it is convenient to recall some facts.

1.6. For a module Y, denote by $\mathcal{F}_Y$ the category of all coherent functors $F : (\operatorname{add} Y)^{\mathrm{op}} \to \mathcal{A}b$, where $\mathcal{A}b$ denotes the category of abelian groups. Recall that a functor $F : (\operatorname{add} Y)^{\mathrm{op}} \to \mathcal{A}b$ is called *coherent* provided there is a morphism $Y_1 \to Y_2$ in $\operatorname{add} Y$ such that the sequence

$$(-, Y_1) \to (-, Y_2) \to F \to 0$$

is exact in $\operatorname{add} Y$. Here we denote by $(-, C)$ the restriction of the functor

$$\mathrm{Hom}_A(-, C) : \operatorname{mod} A \to \mathcal{A}b$$

to $\operatorname{add} Y$. It follows from [3, Proposition, Ch. III, p. 104] that the categories $\mathcal{F}_Y$ and $\operatorname{mod}(\mathrm{End}_A Y)$ are equivalent and so, in particular, $\mathrm{gl.dim}(\mathrm{End}_A Y) = \mathrm{gl.dim}(\mathcal{F}_Y)$.

If an A-module M has an $\operatorname{add} Y$-approximation resolution of length s, then $\mathrm{pd}(-, M) \leqslant s - 1$. If, moreover, Y is a generator of $\operatorname{mod} A$, then the converse holds. In fact, if $\mathrm{pd}(-, M) \leqslant s - 1$, let

$$0 \to (-, Y_{s-1}) \to \cdots \to (-, Y_0) \to (-, M) \to 0$$

with $Y_i \in \operatorname{add} Y$, be a sequence which is exact in $\operatorname{add} Y$. Since, by hypothesis, A is in $\operatorname{add} Y$, by evaluating the above sequence at A, we infer that there exists an exact sequence

$$0 \to Y_{s-1} \cdots \to Y_0 \to M \to 0$$

inducing the above one, proving then that M has an $\operatorname{add} Y$-approximation resolution of length s, as desired.

1.7. Proof of Theorem 1.5

(b) $\Rightarrow$ (a) Let $\overline{X}$ be a module satisfying the r-resolution property. We will show that $\text{gl.dim}\,\mathcal{F}_{\overline{X}} \leqslant r+1$, leading to the required result. Let F be a functor in $\mathcal{F}_{\overline{X}}$. By definition, there exists a morphism $X'' \xrightarrow{f} X'$ in $\text{add}\,\overline{X}$ such that

$$(-, X'') \xrightarrow{(-,f)} (-, X') \to F \to 0 \tag{$*$}$$

is exact in $\text{add}\,\overline{X}$. Denote $M = \text{Ker}\, f$. Now, since $\overline{X}$ has the r-resolution property, there exists an exact sequence

$$0 \to X_r \to \cdots \to X_1 \to M \to 0$$

with $Xi \in \text{add}\,\overline{X}$ such that the induced sequence

$$0 \to (-, X_r) \to \cdots \to (-, X_1) \to (-, M) \to 0 \tag{$**$}$$

is exact in $\text{add}\,\overline{X}$. Glueing together ($*$) and ($**$) we end up with a sequence

$$0 \to (-, X_r) \to \cdots \to (-, X_1) \to (-, X'') \to (-, X') \to F \to 0$$

which is exact in $\text{add}\,\overline{X}$, showing that $\text{pd}(F) \leqslant r+1$. Therefore, $\text{gl.dim}\,\mathcal{F}_{\overline{X}} \leqslant r+1$ and $\text{rep.dim}\,A \leqslant r+1$, as required. This proves the implication (b) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b) Suppose $\text{rep.dim}\,A = s \leqslant r+1$. Then there exists a module $\overline{X}$ such that $A \oplus D(A)$ is in $\text{add}\,\overline{X}$ and $\text{gl.dim}(\text{End}_A\,\overline{X}) = s$. By the above remarks, $\text{gl.dim}\,\mathcal{F}_{\overline{X}} = s$. We claim that $\overline{X}$ has the $(s-1)$-resolution property. In fact, let $M \in \text{mod}\,A$ not in $\text{add}\,\overline{X}$, and consider a minimal injective copresentation $0 \to M \to I_0 \xrightarrow{f_0} I_1$ of M. Hence, for $F = \text{Coker}(-, f_0)$, we have that

$$0 \to (-, M) \to (-, I_0) \xrightarrow{(-,f_0)} (-, I_1) \to F \to 0 \tag{$*$}$$

is exact. Since $\overline{X}$ is a cogenerator of $\text{mod}\,A$ we get that I_0, I_1 are in $\text{add}\,\overline{X}$, thus $F \in \mathcal{F}_{\overline{X}}$. Now, M is not in $\text{add}\,\overline{X}$, so $(-, M)$ is not projective. Since $\text{gl.dim}\,\mathcal{F}_{\overline{X}} = s$ we then infer that

$$\text{pd}(-, M) = \text{pd}(F) - 2 \leqslant s - 2.$$

As observed before the proof of the theorem, this implies that M has a right $\text{add}\,\overline{X}$-approximation of length smaller than $s - 1 \leqslant r$. This ends the proof of the theorem.

1.8. Corollary. *Let A be an Artin algebra. Then* $\text{rep.dim}\,A = r+1$ *if and only if there exists a generator–cogenerator of* $\text{mod}\,A$ *satisfying the r-resolution property but there is none satisfying the s-resolution property for $s < r$.*

1.9. Corollary. *Let A be a representation-infinite algebra. Then* $\text{rep.dim}\,A = 3$ *if and only if there exists a generator–cogenerator of* $\text{mod}\,A$ *satisfying the* 2*-resolution property.*

2. Glued algebras

2.1. We will prove in this section that the algebras for which the length of $\mathrm{Hom}_A(D(A), -)$ is finite (or dually the length of $\mathrm{Hom}_A(-, A)$ is finite) have representation dimension at most three. These algebras were studied by Assem and Coelho, who introduced in [1] the right (left) glued algebras, which coincide with them. We refer the reader to this work for their original definition. We will prove the result for right glued algebras, the corresponding result for left glued algebras follows by duality. We shall use here a characterization whose proof can be found in [1,2]. Given $X, Y \in \mathrm{ind}\, A$, we say that X is a *predecessor of* Y or that Y is a *successor of* X provided there exists a sequence $X = X_0 \to X_1 \to \cdots \to X_t = Y$ of non-zero morphisms between indecomposable modules. For a given algebra A, define the subcategory

$$\mathcal{L}_A = \{X \in \mathrm{ind}\, A\colon \text{ for each predecessor } Y \text{ of } X,\ \mathrm{pd}_A\, Y \leqslant 1\}.$$

Theorem [1,2]. *The following statements are equivalent for an Artin algebra A:*

(a) *A is a right glued algebra*;
(b) *the length of* $\mathrm{Hom}_A(D(A), -)$ *is finite*;
(c) *all but finitely many indecomposable A-modules have projective dimension at most one*;
(d) $\mathcal{L}_A$ *is cofinite in* $\mathrm{ind}\, A$.

Clearly, the class of right glued algebras includes all the representation-finite ones. Not so immediate, but it also includes all the tilted algebras with complete slices in a preinjective component. Further examples can be found in [1].

2.2. Our main result of this section is the following.

Theorem. *Let A be a representation-infinite right glued algebra. Then* $\mathrm{rep.dim}\, A = 3$.

Proof. By 1.7, it is enough to exhibit a generator–cogenerator of $\mathrm{mod}\, A$ satisfying the 2-resolution property. Since A is right glued, the set $\mathcal{X}_1 = \mathrm{ind}\, A \setminus \mathcal{L}_A$ is finite. Also, by [2, (1.5)], the set

$$\mathcal{X}_2 = \{Y \in \mathcal{L}_A\colon\ Y \text{ is a successor of an injective in } \mathrm{ind}\, A\}$$

is finite. So, the set

$$\mathcal{X} = \mathcal{X}_1 \cup \{P\colon\ P \text{ is a projective in } \mathrm{ind}\, A\} \cup \mathcal{X}_2$$

is finite, say $\mathcal{X} = \{X_1, \ldots, X_s\}$. Write $\overline{X} = X_1 \oplus \cdots \oplus X_s$. Clearly, such module is a generator–cogenerator of $\mathrm{mod}\, A$, and we claim that it satisfies the 2-resolution property. Let now M be an indecomposable A-module not in $\mathrm{add}(\overline{X})$. Then $M \in \mathcal{L}_A$, $\mathrm{Hom}_A(\mathcal{X}_1, M) = 0$ because $\mathcal{L}_A$ is closed under predecessors, and $\mathrm{Hom}_A(\mathcal{X}_2, M) = 0$

because $\mathcal{X}_2$ is closed under successors. Moreover, since $M \in \mathcal{L}_A$ and is not projective, $\mathrm{pd}_A M = 1$.

Let now $0 \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ be the minimal projective resolution of M in $\mathrm{mod}\,A$. We will prove that this sequence is an $\mathrm{add}\,\overline{X}$-approximation resolution of M. This amounts to prove that $(X, P_0) \to (X, M) \to 0$ is exact for each indecomposable $X \in \mathrm{add}\,\overline{X}$. This clearly holds if X is projective, and it also holds if X is not projective, since then $\mathrm{Hom}_A(X, M) = 0$, as observed above. Thus the proof of the theorem is complete. □

2.3. The following result is the dual of Theorem 2.2. We leave the details of the proof to the reader.

Theorem. *Let A be a representation-infinite left glued algebra. Then* $\mathrm{rep.dim}\,A = 3$.

2.4. The above results imply Theorem 5.1 of [12]. Also, for reference, we mention the following corollary which improves Corollary 5.2 of [12].

Corollary. *Let A be a tilted algebra. If A has a complete slice in either a postprojective component or in a preinjective component then,* $\mathrm{rep.dim}\,A = 3$.

3. Trivial extensions of iterated tilted algebras

3.1. Along this section, H will denote a hereditary algebra. We will prove here that the representation dimension of the trivial extension $T(H)$ of H is at most 3. As a consequence, we will have the same bound for the representation dimension of the trivial extensions of iterated tilted algebras, using results by Happel [7] and by Xi in [13]. We shall first recall some background on this construction. For further details, we refer the reader to [6,11].

The *trivial extension of an algebra A* is the algebra $T(A) = A \ltimes D(A)$ defined as follows. As a vector space, $T(A) = A \oplus D(A)$, and the product is defined by $(a, f)(b, g) = (ab, ag + fb)$, for $a, b \in A$, $f, g \in D(A)$. The algebra $T(A)$ is selfinjective.

Let $\mathcal{A}$ be an additive category and $F : \mathcal{A} \to \mathcal{A}$ be an additive functor. The *trivial extension $\mathcal{A} \ltimes F$ of $\mathcal{A}$ by F*, defined in [6, Section 1], is the category whose objects are the maps $\alpha : F(A) \to A$ such that the composition $\alpha \cdot F(\alpha) = 0$. For objects $\alpha : F(A) \to A$ and $\beta : F(B) \to B$ in $\mathcal{A} \ltimes F$, a morphism $f : \alpha \to \beta$ is a morphism $f : A \to B$ such that $\beta F(f) = f\alpha$. When A is an Artin algebra and $F = D(A) \otimes_A - : \mathrm{mod}\,A \to \mathrm{mod}\,A$, then $\mathrm{mod}\,A \ltimes F$ is equivalent to $\mathrm{mod}\,T(A)$. For an A-module X and a morphism $f : D(A) \otimes_A X \to X$, the $T(A)$-module structure is defined on X by $(a, g) \cdot x = ax + f(g \otimes x)$, for $x \in X$, $a \in A$ and $g \in D(A)$ [6, p. 19].

In the case we are primarily interested, that is, the trivial extension of the hereditary algebra H, the modules in $\mathrm{mod}\,T(H)$ can be seen as triples (X_1, X_2, f) with $X_1, X_2 \in \mathrm{mod}\,H$, and $f : D(H) \otimes X_1 \to X_2$ a surjective H-map. A morphism from (X_1, X_2, f) to (Y_1, Y_2, g) is a triple $(\alpha_{11}, \alpha_{22}, \alpha_{21})$ of morphisms in $\mathrm{mod}\,H$, $\alpha_{11} : X_1 \to Y_1$, $\alpha_{22} : X_2 \to Y_2$ and $\alpha_{21} : X_1 \to Y_2$ such that $\alpha_{22} f = g(1 \otimes \alpha_{11})$. This description of the $T(H)$-modules was given by Tachikawa in [11]. To see that the morphisms are appropriately defined we use the equivalence between mod $T(H)$ and mod $A \ltimes F$, considering the triples as elements in mod $A \ltimes F$ in the following way.

The element in mod $A \ltimes F$ corresponding to the triple (X_1, X_2, f) is the map

$$\begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix} : D(H) \otimes (X_1 \oplus X_2) \simeq D(H) \otimes X_1 \oplus D(H) \otimes X_2 \to X_1 \oplus X_2$$

Then a morphism from (X_1, X_2, f) to (Y_1, Y_2, g) corresponds to a map $\left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 0 & 0 \\ g & 0 \end{smallmatrix}\right)$ in mod $A \ltimes F$, that is, a map $\alpha = \left(\begin{smallmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{smallmatrix}\right) : X_1 \oplus X_2 \to Y_1 \oplus Y_2$ such that

$$\begin{pmatrix} 0 & 0 \\ g & 0 \end{pmatrix} \left(\mathrm{id}_{D(H)} \otimes \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \right) = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix}.$$

This is satisfied precisely when $\alpha_{12} = 0$ and the triple $(\alpha_{11}, \alpha_{22}, \alpha_{21})$ satisfies the above stated condition.

Since H is hereditary then $D(H) \otimes X$, being a homomorphic image of $D(H)^n$ for some n, is injective. Thus, $D(H)X$ is also injective, and $X \simeq D(H)X \oplus X/D(H)X$ in mod H. Observe that the triple associated to the $T(H)$-module X is $(X/D(H)X, D(H)X, f)$, where $f : D(H) \otimes X/D(H)X \to D(H)X$ is the multiplication map.

From now on, we will write the adjoint functors $D(H) \otimes_H -$ and $\mathrm{Hom}_H(D(H), -)$ by F and G, respectively, and by $\varepsilon : FG \to Id$ and $\eta : Id \to GF$ the adjunction morphisms.

Following [11], to a given indecomposable H-module X, two indecomposable $T(H)$-modules can be assigned as follows. The $T(H)$-module $(X, 0, 0)$, called module of the *first type* and which we shall also denote by X. On the other hand, we consider a fixed minimal injective coresolution $0 \to X \to I_0(X) \xrightarrow{f} I_1(X) \to 0$ of X, and assign to X the indecomposable $T(H)$-module $\widetilde{X} = (G(I_0(X)), I_1(X), f\varepsilon_{I_0(X)})$. We say that $\widetilde{X}$ is a module of the *second type*. It follows from [11] that an indecomposable nonprojective $T(H)$-module can be identified either with an H-module or with a module of the second type.

We extend the above notation to arbitrary H-modules $X = X_1 \oplus \cdots \oplus X_n$, with X_i in ind A by writing $\widetilde{X} = \widetilde{X}_1 \oplus \cdots \oplus \widetilde{X}_n$.

3.2. Given $X, Y \in \mathrm{ind}\, H$, there is naturally a morphism

$$\psi : \mathrm{Hom}_{T(H)}\big(\widetilde{X}, \widetilde{Y}\big) \to \mathrm{Hom}_H(X, Y)$$

defined by the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & I_0(X) & \xrightarrow{f} & I_1(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi(\alpha)} & & \downarrow & & \downarrow{\scriptstyle \alpha_2} & & \\ 0 & \longrightarrow & Y & \longrightarrow & I_0(Y) & \xrightarrow{g} & I_1(Y) & \longrightarrow & 0 \end{array} \qquad (*)$$

for $\alpha = (\alpha_1, \alpha_2, \alpha_{21}) : \widetilde{X} \to \widetilde{Y}$, where the middle vertical map is $\varepsilon_{I_0(Y)} F(\alpha_1) \varepsilon^{-1}_{I_0(X)}$.

Then ψ is surjective, functorial in X and Y, and induces an isomorphism

$$\underline{\psi}:\underline{\mathrm{Hom}}_{T(H)}\big(\widetilde{X},\widetilde{Y}\big)\to \mathrm{Hom}_H(X,Y)$$

as follows from the following lemma. Recall that

$$\underline{\mathrm{Hom}}(X,Y)=\mathrm{Hom}(X,Y)/P(X,Y),$$

where $P(X,Y)$ denotes the set of morphisms from X to Y which factor through a projective module.

Lemma. *Let $\alpha=(\alpha_1,\alpha_2,\alpha_{21}):(X_1,X_2,f)\to(Y_1,Y_2,g)$ be a morphism in $\mathrm{mod}\,T(H)$ and assume that there exists $\rho:X_2\to F(Y_1)$ such that $F(\alpha_1)=\rho f$. Then*:

(a) $(0,0,\alpha_{21}):(X_1,X_2,f)\to(Y_1,Y_2,g)$ *factors through the projective module* $(G(Y_2), Y_2,\varepsilon_{Y_2})$;
(b) $(\alpha_1,\alpha_2,0):(X_1,X_2,f)\to(Y_1,Y_2,g)$ *factors through* $(Y_1,F(Y_1),\mathrm{id})$;
(c) *if Y_1 is a projective H-module then α factors through a projective module.*

Proof. (a) Observe that $(0,0,\alpha_{21})$ is the composition of

$$(0,0,\alpha_{21}):(X_1,X_2,f)\to\big(G(Y_2),Y_2,\varepsilon_{Y_2}\big)$$

and

$$(0,\mathrm{id},0):\big(G(Y_2),Y_2,\varepsilon_{Y_2}\big)\to(Y_1,Y_2,g).$$

(b) Since $F(\alpha_1)=\rho f$, using that f is surjective we get that $\alpha_2=gp$. Then we can write $(\alpha_1,\alpha_2,0)=(\mathrm{id},g,0)(\alpha_1,\rho,0)$.

Finally, (c) follows from (a) and (b), observing that $(Y_1,F(Y_1),\mathrm{id})$ is projective in $\mathrm{mod}\,T(H)$ when Y_1 is projective in $\mathrm{mod}\,H$. □

3.3. The main result of this section is the following.

Theorem. *Let H be a hereditary Artin algebra. Then* $\mathrm{rep.dim}\,T(H)\leqslant 3$.

Proof. Let $\overline{X}=H\oplus D(H)\oplus T(H)\oplus\widetilde{H}$ be in $\mathrm{mod}\,T(H)$. We shall prove that the generator–cogenerator module $\overline{X}$ satisfies the 2-resolution property. Let $N'\in\mathrm{ind}\,T(H)$.

Case 1. $N=N'$ is a module of the first type (that is, $N'\in\mathrm{mod}\,H$). As seen in (1.4)(b), $H\oplus D(H)$ satisfies the 2-resolution property in $\mathrm{mod}\,H$, and so, there exists an exact sequence

$$0\to Y_2\to Y_1\xrightarrow{g}N\to 0$$

with Y_1, Y_2 in $\mathrm{add}(H \oplus D(H))$ and such that

$$0 \to {}_H(-, Y_2) \to {}_H(-, Y_1) \to {}_H(-, N) \to 0$$

is exact in $\mathrm{add}(H \oplus D(H))$. Clearly, also

$$0 \to {}_{T(H)}(-, Y_2) \to {}_{T(H)}(-, Y_1) \to {}_{T(H)}(-, N) \to 0 \qquad (*)$$

is exact in $\mathrm{add}(H \oplus D(H))$ (as $T(H)$-modules). Since $T(H)$ is projective, then $(*)$ is also exact in $\mathrm{add}(T(H))$. The proof will be complete in this case once we prove that $\mathrm{Hom}_{T(H)}(\widetilde{H}, -)$ preserves the exactness of $0 \to Y_2 \to Y_1 \xrightarrow{g} N \to 0$. So, let P be an indecomposable projective H-module. Then $\widetilde{P} = (Q, I_1(P), h\varepsilon_{I_0(P)})$, where

$$0 \to P \to I_0(P) \xrightarrow{h} I_1(P) \to 0$$

is a minimal injective resolution, and $Q = G(I_0(P))$. Observe that a map $f : \widetilde{P} \to N$ is given by $(\alpha_0, 0, 0)$, where $\alpha_0 : Q \to N$. Since Q is projective, there exists $\beta : Q \to Y_1$ such that $g\beta = \alpha_0$, and $(\beta, 0, 0) : \widetilde{P} \to Y_1$ lifts f, as desired.

Case 2. N' is not a module of the first type (i.e., $N' \notin \mathrm{mod}\, H$). So $N' = \widetilde{N}$, for some $N \in \mathrm{ind}\, H$. As above, consider an exact sequence

$$0 \to Y_2 \to Y_1 \to N \to 0 \quad \text{with } Y_1, Y_2 \in \mathrm{add}\big(H \oplus D(H)\big), \qquad (*)$$

which remains exact under $\mathrm{Hom}_H(H \oplus D(H), -)$. Let I_0, I_1 be injective modules so that

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y_2 & \longrightarrow & Y_1 & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_0(Y_2) & \longrightarrow & I_0 & \longrightarrow & I_0(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_1(Y_2) & \longrightarrow & I_1 & \longrightarrow & I_1(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \qquad (*)$$

is exact and commutes. Observe that $I_0 = I_0(Y_1) \oplus I$, $I_1 = I_1(Y_1) \oplus I$, where I is an injective module and $f : I_0 \to I_1$ is $f = \begin{pmatrix} f_0 & 0 \\ 0 & \sigma \end{pmatrix}$ in the above decomposition with σ being an isomorphism. Clearly, we get a sequence

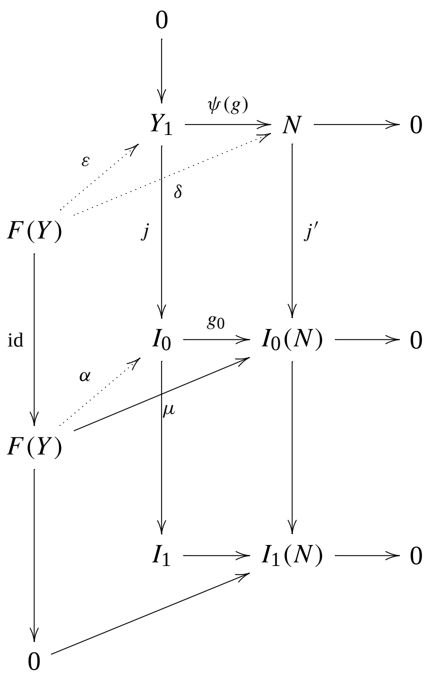
$$0 \to \widetilde{Y}_2 \to \widetilde{Y}_1 \oplus Q \to \widetilde{N} \to 0 \qquad (\tilde{*})$$

with $Q = (G(I), I, \varepsilon_I)$, which is projective in mod $T(H)$. To prove that this sequence is exact, we observe two facts:

(a) The exact sequence $0 \to I_0(Y_2) \to I_0 \to I_0(N) \to 0$ splits, and therefore it remains exact under G.
(b) All the maps in $(\tilde{*})$ are of the form $(\alpha_1, \alpha_2, 0)$, with α_i in mod H.

Then, using that a short exact sequence is exact in $\mathcal{A} \ltimes F$ when the corresponding sequence in $\mathcal{A}$ is exact [6, Corollary 1.2] we obtain that $(\tilde{*})$ is exact. Also, each of $\widetilde{Y}_1$, $\widetilde{Y}_2$ belongs to add$(\widetilde{H} \oplus \widetilde{D(H)})$, and therefore to add $\overline{X}$, because $\widetilde{D(H)} \simeq (H, 0, 0)$. We shall now prove that $\mathrm{Hom}_{T(H)}(\overline{X}, -)$ keeps $(\tilde{*})$ exact.

Let Y in add $\overline{X}$. We proved in Lemma 3.2 that maps of the form $(0, 0, \theta_{21}) : Y \to \widetilde{N}$ factor through a projective module, and therefore they can be lifted to $\widetilde{Y}_1 \oplus Q \to \widetilde{N}$. Thus it is enough to prove that maps of the form $(\theta_0, \theta_1, 0) : Y \to \widetilde{N}$ can be lifted to $\widetilde{Y}_1 \oplus Q \to \widetilde{N}$. Consider first $Y \in \mathrm{add}(H \oplus D(H))$ and let $\theta = (\theta_0, 0, 0) : Y \to \widetilde{N}$ (so, $\varepsilon_{I_0(N)} F(\theta_0) : F(Y) \to I_0(N)$). We then have the following diagram, where $\mu = \varepsilon_{I_0(N)} F(\theta_0)$:



First observe that $\mu = \varepsilon_{I_0(N)} F(\theta_0)$ lifts to $\delta : F(Y) \to N$. Since ${}_H(D(H), -)$ keeps the sequence $(*)$ exact, and $F(Y)$ is injective (see (3.1)), we infer that δ lifts through $\psi(g) : Y_1 \to N$. So $j'\delta = \varepsilon_{I_0(N)} F(\theta_0)$ and $\delta = \psi(g)\varepsilon$, for some $\delta : F(Y) \to N$ and

$\varepsilon : F(Y) \to Y_1$. Let $\alpha = j\varepsilon$. Then $g_0\alpha = g_0(j\varepsilon) = j'\underset{\sim}{\psi}(g)\varepsilon = j'\delta = \varepsilon_{I_0(N)}F(\theta_0)$. Let $\alpha_0 = G(\alpha)\eta_Y$. Then $(\alpha_0, 0, 0) : Y \to (G(\underset{\sim}{I_0}), I_1, f\varepsilon_{I_0}) = \widetilde{Y}_1 \oplus Q$ lifts $(\theta_0, 0, 0)$, as required.

It remains to show that $\mathrm{Hom}_{T(H)}(\widetilde{P}, -)$ keeps $(\tilde{*})$ exact for each projective H-module P. We have a commutative diagram

$$\begin{array}{ccccc} {}_{T(H)}\underline{(\widetilde{P}, \widetilde{Y}_1)} & \longrightarrow & {}_{T(H)}\underline{(\widetilde{P}, \widetilde{N})} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ {}_H(P, Y_1) & \longrightarrow & {}_H(P, N) & \longrightarrow & 0, \end{array} \qquad (*)$$

where the lower sequence is exact and the vertical arrows arc the isomorphisms defined in (3.2). Then we have that the upper sequence is exact. Consequently,

$$(\widetilde{P}, \underline{g}) : {}_{T(H)}\underline{(\widetilde{P}, \widetilde{Y}_1 \oplus Q)} \to {}_{T(H)}\,\underline{(\widetilde{P}, \widetilde{N})}$$

is surjective. It follows then that

$$(\widetilde{P}, g) : {}_{T(H)}(\widetilde{P}, \widetilde{Y}_1 \oplus Q) \to {}_{T(H)}(\widetilde{P}, \widetilde{N})$$

is also surjective, because $g : \widetilde{Y}_1 \oplus Q \to \widetilde{N}$ is an epimorphism. This ends the proof of the theorem. □

3.4. Corollary. *Let A be an iterated tilted algebra. Then* $\mathrm{rep.dim}\, T(A) \leqslant 3$.

Proof. It follows from [7] that such an A is derived equivalent to a hereditary algebra H. So, by [13], $\mathrm{rep.dim}\, T(A) = \mathrm{rep.dim}\, T(H)$ and the result follows using our theorem above. □

4. One-point extension algebras

4.1. In this section we compare the representation dimension of an Artin algebra B and the representation dimension of a one point extension of B, under appropriate hypothesis. More precisely, we will prove the following proposition, extending results proven in [12] for one point extensions of finite-dimensional algebras by simple injective modules. We refer the reader to [4] for an account on the one-point extension construction. We also observe that a dual version of this result holds for one-point coextensions. We leave to the reader the details of the corresponding proof.

Proposition. *Let B be an Artin algebra, D a division ring, M a $B - D$ bimodule and $A = B[M]$ the one-point extension of B by M. Assume that the set of successors of M in* $\mathrm{ind}\, B$ *is finite. Then*:

(a) $\mathrm{rep.dim}\, B \leqslant \mathrm{rep.dim}\, A$;
(b) *if* $\mathrm{ind}\, B$ *is cofinite in* $\mathrm{ind}\, A$ *then* $\mathrm{rep.dim}\, B = \mathrm{rep.dim}\, A$.

Proof. (a) Let $\{Z_1, \dots, Z_t\}$ be the set of successors of M in $\operatorname{ind} B$. We will consider the A-modules as triples (D^n, X, f), with X in $\operatorname{mod} B$ and $f : M \otimes D^n \to X$ a B-morphism, in the usual way (see, e.g., [4, III, 2]). We start by observing that any A-module K can be written in the form $K = (D^n, Z, g) \oplus (0, N, 0)$ with $Z \in \operatorname{add}(Z_1 \oplus \cdots \oplus Z_t)$. In fact, let $K = (D^n, K_1, h)$, and let N' be an indecomposable summand of K_1 which is not a successor of M. Then ${}_B(M, N') = 0$ and therefore $(0, N', 0)$ is a summand of K.

Let $\operatorname{rep.dim} A = r + 1$ and let $\overline{X}$ be a generator–cogenerator of $\operatorname{mod} A$ satisfying the r-resolution property, $\overline{X} = (D^m, X, f)$ with X in $\operatorname{mod} B$ and $f : M \otimes D^n \to X$ a morphism of B-modules.

Let $\overline{Y} = \bigoplus_{i=1}^t Z_i \oplus X$. If follows from the description of the projective and the injective modules in $\operatorname{mod} A$, [4, III, Prop. 2.5], that $B \oplus D(B)$ is in $\operatorname{add} \overline{Y}$ because $A \oplus D(A)$ is in $\operatorname{add} \overline{X}$. So $\overline{Y}$ is generator–cogenerator of $\operatorname{mod} B$, and we will prove that it satisfies the r-resolution property, and thus $\operatorname{rep.dim} B \leqslant r + 1 = \operatorname{rep.dim} A$. This amounts to prove that $\operatorname{pd}_B(-, N) \leqslant r - 1$ for each N in $\operatorname{ind} B$, where ${}_B(-, N)$ is considered as an element of $\mathcal{F}_Y$, as observed in (1.6).

We will also consider ${}_A(-, N) \in \mathcal{F}_{\overline{X}}$, and show that $\operatorname{pd}_B(-, N) \leqslant \operatorname{pd}_A(-, N) \leqslant r - 1$. This will end the proof of (a). We will prove the inequality by induction on $k = \operatorname{pd}_A(-, N)$. We may assume that $N \notin \operatorname{add} \overline{Y}$.

If $k = 0$, then the result clearly holds. So let $k > 0$ and consider a right $\overline{X}$-approximation $X_1 \to N$ of N. Let $0 \to K \to X_1 \to N \to 0$ be exact. Then

$$0 \to {}_A(-, K) \to {}_A(-, X_1) \to {}_A(-, N) \to 0$$

is exact in $\operatorname{add} \overline{X}$, and $\operatorname{pd}_A(-, K) < k = \operatorname{pd}_A(-, N)$.

We write $K = (D^n, K_1, f)$ and $X_1 = (D^d, Y, g)$ with $Y \in \operatorname{add} X \subseteq \operatorname{add} \overline{Y}$. We have an exact sequence $0 \to K_1 \to Y \xrightarrow{\alpha} N \to 0$. Moreover, as we observed above, $K = (D^n, Z, h) \oplus (0, N_1, 0)$ with $Z \in \operatorname{add}(Z_1 \oplus \cdots \oplus Z_t) \subseteq \operatorname{add} \overline{Y}$. Since $\operatorname{pd}_A(-, N_1) \leqslant \operatorname{pd}_A(-, K) < k$ we can apply the induction hypothesis to conclude that $\operatorname{pd}_B(-, N_1) < k$. On the other hand, $K_1 = Z \oplus N_1$, and $Z \in \operatorname{add} \overline{Y}$. So $\operatorname{pd}_B(-, K_1) = \operatorname{pd}_B(-, N_1) < k$. Thus, to prove that $\operatorname{pd}_B(-, N) \leqslant k$, we only need to show that the sequence

$$0 \to {}_B(-, K_1) \to {}_B(-, Y) \xrightarrow{(-,\alpha)} {}_B(-, N) \to 0 \qquad (*)$$

is exact in $\operatorname{add} \overline{Y}$. Since $N \notin \operatorname{add} \overline{Y}$ we have that N is not a successor of M and therefore ${}_B(Z_i, N) = 0$ for each $i = 1, \dots, t$. So we only need to prove that $(*)$ is exact in $\operatorname{add} X$.

Let $\theta : X \to N$ be a map in $\operatorname{mod} A$. Then the composition $M^m \simeq M \otimes D^m \to X \to N$ is zero because ${}_B(M, N) = 0$. So $(0, \theta) : \overline{X} = (D^m, X, f) \to N$ is a morphism in $\operatorname{mod} A$, and thus it can be lifted through $X_1 \to N$, because ${}_A(-, X_1) \to {}_A(-, N)$ is surjective in $\operatorname{add} \overline{X}$. Since $X_1 = (D^d, Y, g)$, the map θ can be lifted through $Y \xrightarrow{\alpha} N$. This proves that $(-, \alpha)$ is surjective, as desired.

(b) Let $Z_1, \dots, Z_t$ be the successors of M in $\operatorname{ind} B$, let $\operatorname{ind} A \setminus \operatorname{ind} B = \{D_1, \dots, D_s\}$, and assume that $\operatorname{rep.dim} B = r + 1$.

Let $\overline{Y}$ be a generator–cogenerator of $\operatorname{mod} B$ with the r-resolution property, and let $\overline{X} = \overline{Y} \oplus \bigoplus_{i=1}^s D_i \oplus \bigoplus_{i=1}^t Z_i$.

Let N in $\operatorname{ind} A$ but not in $\operatorname{add} \overline{X}$. Then N is in $\operatorname{ind} B$. Considering again ${}_A(-,N) \in \mathcal{F}_{\overline{X}}$ and ${}_B(-,N) \in \mathcal{F}_{\overline{Y}}$, we will prove that $\operatorname{pd}_A(-,N) \leqslant \operatorname{pd}_B(-,N) \leqslant r-1$, by induction on $k = \operatorname{pd}_B(-,N)$. The result holds for $k=0$, so we assume $k>0$.

Let $0 \to K \to Y_1 \to N \to 0$ be an exact sequence in $\operatorname{mod} B$ such that $Y_1 \in \operatorname{add} \overline{Y}$ and

$$0 \to {}_B(-,K) \to {}_B(-,Y_1) \to {}_B(-,N) \to 0$$

is exact in $\operatorname{add} \overline{Y}$. Since ${}_B(-,N)$ is not projective we have that $\operatorname{pd}_B(-,K) < k = \operatorname{pd}_B(-,N)$ and then by the induction hypothesis we conclude that $\operatorname{pd}_A(-,K) < k$.

Let $D_i = (D^{n_i}, U_i, f_i)$. Then $(M, U_i) \neq 0$ for all i, and $U_i \in \operatorname{add}(Z_1 \oplus \cdots \oplus Z_t)$. Then ${}_B(U_i, N) = 0$ and consequently ${}_B(D_i, N) = 0$. Since ${}_A(-,Y_1) \to {}_A(-,N) \to 0$ is exact in $\operatorname{add} \overline{Y}$, it follows that it is also exact in $\operatorname{add} \overline{X}$. Since $\operatorname{pd}_A(-,K) < k$ we get that $\operatorname{pd}_A(-,N) \leqslant k$, as desired, ending the proof of the theorem. □

4.2. The next result extends [12, (6.1)].

Corollary. *Let B be an Artin algebra, D a division ring, M a $B-D$ bimodule and $A = B[M]$ the one-point extension of B by M. If M is a simple injective module then* $\operatorname{rep.dim} A = \operatorname{rep.dim} B$.

Acknowledgments

This work was done during reciprocal visits supported by the exchange project CAPES-SCyT between Argentina and Brazil and by FAPESP. The first named author also acknowledges support from CNPq and the second from SECyT, UNS.

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