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Quantum state space dimension as a quantum resource

A. Plastino* and G. Bellomo

Instituto de Física La Plata (IFLP-CONICET)
and Departamento de Física, Facultad de Ciencias Exactas,
Universidad Nacional de La Plata, 115 y 49,
C.C. 67, 1900 La Plata, Argentina
**angeloplastino@gmail.com*

A. R. Plastino

CeBio and Secretaría de Investigaciones,
Universidad Nacional del Noroeste de la,
Provincia de Buenos Aires (UNNOBA-CONICET),
R. Saenz Peña 456, Junín, Argentina

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We argue that the dimensionality of the space of quantum systems' states should be considered as a legitimate resource for quantum information tasks. The assertion is supported by the fact that quantum states with discord-like capacities can be obtained from classically-correlated states in spaces of dimension large enough. We illustrate things with some simple examples that justify our claim.

Keywords: Hilbert space; quantum information; quantum discord.

The Hilbert space dimension has been related to physical resources for different physical systems, playing a fundamental role in quantum computation. Basically, the idea is that "if you want to avoid supplying an amount of some physical resource that grows exponentially with problem's size, the computer must be made up of parts whose number grows nearly linearly with the number of qubits required in an equivalent quantum computer. This thus becomes a fundamental requirement for a system to be a scalable quantum computer".[1] Moreover, some recent results show that quantum dimensionality could be regarded as a physical entity. For example, Brunner *et al.* defined what they call "dimension witnesses": observable quantities to estimate the minimum dimension of a given system state-space consistent with a number of measured correlations.[2–4] In the same spirit, Wehner *et al.* found a lower bound that gives a fundamental limit on the dimension of the state to implement certain measurement strategies.[5] Here we propose to consider the dimension of the Hilbert space as a legitimate resource for quantum information processing. Our main argument lies in the observation, due to Li and Luo,[6] that quantum separable states can be obtained from reductions of classically-correlated ones. Although some authors have suggested the possibility of understanding the size of the Hilbert space as a resource by itself,[7] the assertion that it is a quantum-better-than-classical resource was never technically analyzed, as far as we know.

Under the discord paradigm, a classically-correlated state (or simply, a classical state) is one that is information-wise accessible to local observers. Given a discord-like measure δ and a classical state σ^{AB} of a composite system $A+B$, one knows that $\delta(\sigma^{AB}) = 0$. The following theorem, due to Li and Luo, demonstrates a notable relation between separable states and classical states.[6]

Theorem. *A state ρ^{ab}, of a composite system $a+b$, is separable over $\mathcal{H}^{ab} = \mathcal{H}^a \otimes \mathcal{H}^b$ if and only if there is a classical state σ^{AB} over $\mathcal{H}^{AB} = \mathcal{H}^A \otimes \mathcal{H}^B$, with $\mathcal{H}^A = \mathcal{H}^a \otimes \mathcal{H}^{\bar{a}}$ and $\mathcal{H}^B = \mathcal{H}^b \otimes \mathcal{H}^{\bar{b}}$, such that*

$$\rho^{ab} = tr_{\bar{a}\bar{b}}[\sigma^{AB}]. \tag{1}$$

Here, the state σ^{AB} of the composite $A+B$ should be regarded as a classical extension of the separable ρ^{ab}.

The proof is given by Li and Luo in Ref. 6. The next result follows directly from the above theorem:

Proposition. *Any quantum task carried out using un-entangled states can also be undertaken using classically-correlated states.*

Indeed, if a quantum task needs appealing to a given un-entangled state ρ^{ab}, then there exists a classical extension σ^{AB} from whose reduction ρ^{ab} can be obtained (Fig. 1). The scheme is straightforwardly generalized to tasks requiring several input quantum states.

Un-entangled quantum correlations, discord-ones in particular, have proved their usefulness both in the interpretation of foundational quantum issues and in applications to quantum information/computation problems (see, for example, the excellent review in Refs. 8 and 9).

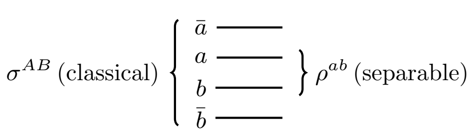

Fig. 1. Every quantum state that is separable (within a given bipartition of the full system) is, in a formal sense, the reduction of a classical state of a system defined over a larger state-space (and preserves the original bipartition).

We will explicitly illustrate here just how classically-correlated states can replace discord-possessing separable states in two specific jobs: Remote state preparation (RSP) and entanglement distribution (ED).

Remark. Not all classically-correlated states are equally useful for performing a given task in a better-than-classical manner. The key lies in the structure of its reductions: a classically-correlated state would be a good resource if the selected reduction is quantum-correlated. In other words, we are dealing with states that have some "hidden" quantum correlation. Let us see, for example, the following simple illustration. We start with the bipartite state $\sigma^{AB} = \Pi^A_\alpha \otimes \Pi^B_\beta$, where Π^A_α (Π^B_β) are eigenprojectors of two Bell (maximally entangled) states. Then, if we look at the $A|B$-correlations, the state is a product state and, in particular, a non-discordant one. A and B are both composite, say $A = a + \bar{a}$ and $B = b + \bar{b}$. Tracing out $\bar{a}$ and $\bar{b}$ yields $\sigma^{ab} \propto \mathbb{1}$, which is virtually useless for performing any informational task. Instead, we could trace out the whole B-part to end up with a maximally discordant state, but in such a case we would be violating the original $A|B$-partition in the sense of Luo's Theorem, where it is of the essence to obtain discord from the non-discordant part with respect to a given fixed bipartition.

Remote state preparation

As a first illustration, consider the RSP-protocol, a variant of the well-known teleportation-one, in which the emitter knows the state being sent to the recipient (for details see, for instance, Ref. 10). Dakić *et al.* showed that, for certain two-qubits states' family (those with maximally-mixed marginals), the protocol's fidelity coincides with the geometric discord of such states. Girolami and Adesso singled out certain separable states that maximize the geometric discord,[11,12] although such states do not possess maximally-mixed marginals. In fact, it is easy to see that the RSP-fidelity for these states vanishes. Instead, the state defined by the density matrix (standard basis)

$$\rho_{\mathrm{RSP}} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

does maximize both the geometric discord and the RSP-fidelity. This state, defined in $\mathcal{H}^a \otimes \mathcal{H}^b$, can be obtained (save for discord-preserving local unitary transformations), as the reduction of the classical state (in $\mathbb{C}^6 \otimes \mathbb{C}^6$)[13]:

$$\sigma_{\mathrm{RSP}} = \frac{1}{3}\sum_{i=1}^{3} |w_k, k\rangle\langle w_k, k| \otimes |w_k, k\rangle\langle w_k, k|, \tag{3}$$

with $|w_k, k\rangle := |w_k\rangle \otimes |k\rangle$, respectively for $k = 1,2,3$. $|w_k\rangle := |\theta,\phi\rangle$, $|\theta,\phi = \cos(\frac{\theta}{2})\times |0\rangle + \exp(i\phi)\sin(\frac{\theta}{2})$, where the pairs (θ,ϕ) take the values $(0,0)$, $(\frac{2\pi}{3},0)$, and $(\frac{2\pi}{3},\pi)$. The states $|w_k\rangle$ correspond to the parties a and b, while $\{|k\rangle\}_{1\leq k\leq 3}$ are three orthogonal states in $\mathbb{C}^3$, corresponding to the extended parties $\bar{a}$ and $\bar{b}$. Thus, the same task can be performed with identical efficiency by use of the classical extension.

Note that ρ_{RSP} maximizes the geometric discord but not the conventional one. This last discord, in $\mathbb{C}^2 \otimes \mathbb{C}^2$, is maximized in the subset of separable states by a different states-family.[11,13,14]

Finally, in the spirit of the discussion of the previous section (see the Remark there), in this case it is straightforward to show that the better-than-classical performance depends on the geometric discord of the selected reduction.

Entanglement distribution

Another example of the classical states' ability to perform quantum tasks is that of ED.[15] We use the following scheme. One starts with a system in which two classically-correlated, composite parties can be identified, A and B, represented by σ^{AB}. For A we have the subparts a-$\bar{a}$, and for B, b-$\bar{b}$. The reduction is $\rho^{Ab} := tr_{\bar{b}}[\sigma^{AB}]$. It permits to tackle the job. In order to do so, consider two partitions of the same state: $ab|\bar{a}$ is the initial partition and $a|\bar{a}b$ the final one. Entanglement distribution consists of the entanglement-increase in passing from the initial to the final configurations. In such process, the subsystem b is taken from being a partner of a to being a partner of $\bar{a}$. So as to succeed, the protocol does not need entangling A with b. Discord is necessary in our partition, but not sufficient.[15,16]

As an example, we start from a four-qubits classical state:

$$\gamma^{AB} = \sum_{k=1}^{4} p_k \Pi_k^A \otimes \Pi_k^B, \tag{4}$$

where $\{\Pi_k^A\}_{1\leq k\leq 4}$ and $\{\Pi_k^B\}_{1\leq k\leq 4}$ are basis of orthogonal, rank 1 projectors in $\mathbb{C}^4$, and $\{p_k\}_{1\leq k\leq 4}$ is a probability distribution. So as to find ED-optimal classical states, we generate random $\mathbb{C}^4$-basis for the A-B parties. Specifically, we restrict our search to states such that $p_k = \frac{1}{4}$ and $\Pi_k^A = \Pi_k^B \forall k$. If $E^{X|Y}$ is the entanglement measure given by the negativity in the partition $X|Y$ of the system, we find that γ^{AB}'s ED is $E^{a|\bar{a}b} - E^{ab|\bar{a}} \leq 0.0915$.

We now replace the initial state γ^{AB} by another one in which both a and b are composed by qu*d*its, while $\bar{a}$ and $\bar{b}$ retain their two-qubits character. The new state γ_d^{AB} operates on $\mathbb{C}^{2d} \otimes \mathbb{C}^{2d}$, with $d = 1,2,\ldots$. As before, we can look for classical states maximizing ED for each value of d. We numerically did this for $2 \leq d \leq 6$, again restricting the search to states with $p_k = \frac{1}{2d}$ and $\Pi_k^A = \Pi_k^B \forall k$. We encounter that the ED augments with the dimension of the initial classical state (Table 1).

Table 1. Maximal ED by classical states γ_d^{AB} in $\mathbb{C}^{2d} \otimes \mathbb{C}^{2d}$.

d	Maximal ED
2	0.0915
3	0.1269
4	0.1681
5	0.1744
6	0.3326

Accordingly, classically-correlated states γ_d^{AB} allow us to improve ED as long as we augment the Hilbert space dimension.

Work extraction from classical extensions

The distinction between classical and quantal can be made in different ways. The discord establishes a division in the capacity to locally 'interrogate' a composite state. Another way, introduced by Oppenheim *et al.*,[17] revolves around the work that can be extracted from the state by quantum Maxwell demons.[18] If the whole work can be extracted by local demons, then the state is classically-correlated. The "work-deficit" between global and local demons is a measure of the correlations' "quantumness". Optimizing over all possible local measurements determines the *thermal discord*, which differs from the conventional discord. The equivalence between information and work[19,20] is of the essence to compare both types of discord.[21–24]

We have thus far shown that any separable state can be extended to other, classically-correlated states, and that such extensions allow one to perform the same quantum tasks. From a thermodynamical viewpoint, from the classical extension of a given separable state one can always extract more work than from the original quantum state. Indeed, we will now demonstrate as an important new result, the validity of the relation

$$W^Q(\sigma^{AB}) = W^Q(\rho^{ab}) + W^Q(\rho^{aux}) + I(ab|aux). \tag{5}$$

Here, $\rho^{ab} = \sum_k p_k \rho_k^a \otimes \rho_k^b$ ia a quantum-correlated state, σ^{AB} is a classical extension of ρ^{ab} (see Eq. (1)) and $\rho^{aux} := \mathrm{tr}_{ab}[\sigma^{AB}]$ the marginal state of the ancilla. $I(x|y) := S(x) + S(y) - S(x, y)$ is the mutual quantum information between the parties x and y, with $S(x) := -\mathrm{tr}[\rho^x \log_2 \rho^x]$. $W^Q(\sigma^{AB}) := \log_2 d_{AB} - S(\sigma^{AB})$ is the maximum extractable work from σ^{AB}, when in contact with a reservoir of temperature T, with k_B Boltzmann's constant, in units of $k_B T = 1$. We determine in similar fashion W^Q (ρ^{ab}) and $W^Q(\rho^{aux})$.

Equation (5) tells us that the extractable work from the classical extension is bounded below by the sum of the extractable works from the original state and that of the auxiliary part. This is simply demonstrated. It suffices to appeal to the von Neumann entropy's subadditivity and using it in the preceding definition of $W^Q(\sigma^{AB})$. Equation (5) is important because it establishes a relation between local (classical) resources and global (quantum) ones, given that the whole extractable work from σ^{AB} can be locally acceded (the thermal discord of σ^{AB} vanishes in the partition $A|B$). This can be done, for instance, by a local Maxwell demon (in A) that makes a measurement in the eigenbasis of local projectors, $\{\Pi_k^A\}$, and communicates then with B, extracting work $W^C(\sigma^{AB}) = W^Q(\sigma^{AB})$. Thus, from the above equality it follows that

$$W^C(\sigma^{AB}) \geq W^Q(\rho^{ab}) + W^Q(\rho^{aux}). \tag{6}$$

Given that there exist a panoply of possible classical extensions for a given separable ρ^{ab}, it makes sense to ask for the optimal extension: that σ^{AB} with the least possible dimension.[13] This kind of extension can not be generally encountered by recourse to the algorithm of Li and Luo. What is peculiar in the extractable work from such optimal extension?

Consider the state ρ_{RSP} of Eq. (2). In addition to the classical extension given by σ_{RSP} defined in $\mathbb{C}^6 \otimes \mathbb{C}^6$ (Eq. (3)), one can also find an extension $\tilde{\sigma}_{\text{RSP}}$ in $\mathbb{C}^8 \otimes \mathbb{C}^8$. None of them is the optimal one. A Monte Carlo numerical search suggests that the optimal extension, $\sigma^{\text{opt}}_{\text{RSP}}$, acts on $\mathbb{C}^4 \otimes \mathbb{C}^4$.[13] Computing the classical extractable works associated to each of the extensions, one sees that the least-dimension extension is the one that also minimizes the extractable work (Table 2). Such a result stimulates inquiry concerning whether the optimal extension defined as the least dimension one does always coincide with that of minimum extractable work.

The globally extractable work from a state σ^{AB} is a measure of the ability of distinguishing with respect to the totally mixed state, since $W^Q(\sigma^{AB}) = S(\sigma^{AB} || \mathbb{1}^{AB}/d_{AB})$, with $\mathbb{1}^{AB}$ the identity in $\mathcal{H}^{AB}$. In this sense, the classical extension of ρ^{ab} that minimizes the extractable work would be given by that state σ^{AB} closest to $\mathbb{1}^{AB}/d_{AB}$. The question remains concerning whether the minimization of $W^Q(\sigma^{AB})$ (or $S(\sigma^{AB}||\mathbb{1}^{AB}/d_{AB})$) is equivalent to the minimization of d_{AB}, i.e. if both conditions indistinctly determine the optimal classical extension.

Monogamy of correlations

Keeping in mind the equivalence between information and work discussed above, relation (6) can be regarded as a monogamy one for the bi-partition $ab|aux$ of a

Table 2. Extractable work from different classical extensions of the separable state ρ_{RSP}. The minimum-dimension extension corresponds to minimum work.

Extension	Dimension	Extractable work
σ_{RSP}	64	4.00
$\tilde{\sigma}_{\text{RSP}}$	36	3.58
$\sigma^{\text{opt}}_{\text{RSP}}$	16	2.00

classical extension AB. The function $i(\sigma^{AB}) := \log d_{AB} - S(\sigma^{AB})$ can be seen as the accessible information in the state σ^{AB}.[22] Thus, inequality (6) is equivalent to the inequality $i(\sigma^{AB}) \geq i(\rho^{ab}) + i(\rho^{aux})$, that determines a "hybrid" monogamous behavior concerning the classical information of σ^{AB} and the quantum information from ρ^{ab} and ρ^{aux}.

It is of great interest to find monogamy relations for different measures of quantum and classical correlations. Except for peculiar instances, quantum correlation measures do not satisfy general monogamy relations. Even more, if an arbitrary measure Q possesses a few reasonable properties, it must vanish for separable states in order to fulfill monogamy relations of the type $Q^{ab|c} \geq Q^{a|c} + Q^{b|c}$.[25] The usual discord, for example, is not monogamous for general states.[25–28]

The classical extensions that we are advancing here constitute a clear example of monogamy violation because $0 = \delta^{ab|aux}(\sigma^{AB}) \leq \delta^{a|aux}$, and the same holds for $\delta^{b|aux}$. All classical extensions undergo discord-increase if some subsystem is discarded. Thus, *all classical extensions of separable states are polygamous in the usual sense.* This observation (i) constitutes the basic argument in demonstrating that quantum correlations are not monogamous in general[25] and (ii) underlies the violation of more general monogamy relations, even for multipartite correlation measures.[29,30] However, there exist monogamy relations valid even for classical extensions if we consider some generalized multipartite quantum correlations. Consider for instance the global quantum discord (GQD) — a symmetric discord extension for multipartite states– defined as[31,32] $\delta_g(a_1|\dots|a_N) := \min_\Phi[I(a_1|\dots|a_N) - I^\Phi(a_1|\dots|a_N)]$, where $a_1, \dots, a_N$ are the parties of an N-partite state $\rho^{a_1\cdots|a_N}$, where $I(a_1|\dots|a_N) := \sum_k S(a_k) - S(a_1\dots|a_N)$ is the generalized mutual information and $I^\Phi(a_1|\dots|a_N)$ is the mutual information after effecting a multilocal measurement $\Pi^{\mathbf{j}} := \{\Pi_{a_1}^{j_1} \otimes \cdots \otimes \Pi_{a_N}^{j_N}\}$, such that the post-measurement state becomes $\Phi(\rho^{a_1\cdots|a_N}) = \sum_{\mathbf{j}} \Pi^{\mathbf{j}} \rho^{a_1\cdots a_N} \Pi^{\mathbf{j}}$. For the GQD of any N-partite state, it is true that[29]

$$\delta_g(a_1|\dots|a_N) \geq \sum_{k=1}^{N-1} \delta_g(a_1\dots a_k|a_{k+1}). \tag{7}$$

For example, for the classical state σ^{AB}, if we consider the partition $a|\bar{a}|B$, one has $\delta_g(a|\bar{a}|B) \geq \delta_g(a|\bar{a}) + \delta_g(a\bar{a}|B)$, but $\delta_g(a\bar{a}|B) = \delta_g(A|B) = 0$ and then $\delta_g(a|\bar{a}|B) \geq \delta_g(a|\bar{a})$. Alternatively, we can consider the partition $a|b|aux$. We have then $\delta_g(a|b|aux) \geq \delta_g(a|b) + \delta_g(ab|aux)$. Equation (7) suggests that so as to obtain a monogamy relation valid for discord-like measures we need to appeal to generalized multipartite measures that account for the partition's internal structure.

Our examples strongly validate our initial thesis: classically-correlated states in Hilbert spaces of large-enough dimension constitute quantum resources for undertaking processing information tasks. We showed how RSP and ED can be carried out using classically-correlated states. These results are relevant, for example, in the decoherence process: for generic system-environment interactions, the system ends up in a classically-correlated — i.e. non-discordant — state.[33] We show here that those states might be, nonetheless, useful for quantum information processing.

Additionally, we exhibited two important aspects of classical extensions of separable quantum states. First, from a thermodynamic viewpoint concerned with extracting work from the extension, we showed that the minimal extension of a given state is related to the minimum possible work extraction from the extended state. Second, we suggested that the possibility of classically extending any separable state is strongly linked to the non-monogamous nature of the discord-type correlations. We can only recover generalized monogamy relations by considering genuine multipartite correlations.

We have discussed classically correlated states of bipartite systems that have discordant reductions and are useful for implementing quantum information tasks. Two basic ingredients lie behind this effect. They are (i) the (large enough) dimensionality of the Hilbert spaces of the two constituent systems and (ii) the classical correlations between these two parts. Both of them are necessary: bipartite systems of zero (global) discord having minimal dimensionality (two qubits) or having null classical correlations, do not admit discordant reduction. Consequently, the quantum resources involved here are a combination of large dimensional Hilbert spaces and classical correlations.

Our present considerations suggest that a redefinition of the concept of classically correlates states may be needed. Bipartite quantum states with vanishing (global) discord-like correlations can still exhibit these kind of correlations in their reductions. The original classically correlated, discord-free state, can be regarded as actually having quantum correlations, but in a highly diluted fashion. This possible interpretation certainly deserves investigation. Any further developments along these lines of inquire would be warmly welcome.

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