Neutral meson properties under an external magnetic field in nonlocal chiral quark models

D. Gómez Dumm,[1,2] M. F. Izzo Villafañe,[1,2] and N. N. Scoccola[2,3,4]

[1]*IFLP, CONICET-Departamento de Física, Facultad de Ciencias Exactas,*
Universidad Nacional de La Plata, C.C. 67, 1900 La Plata, Argentina
[2]*CONICET, Rivadavia 1917, 1033 Buenos Aires, Argentina*
[3]*Physics Department, Comisión Nacional de Energía Atómica,*
Avenida del Libertador 8250, 1429 Buenos Aires, Argentina
[4]*Universidad Favaloro, Solís 453, 1078 Buenos Aires, Argentina*

We study the behavior of neutral meson properties in the presence of a static uniform external magnetic field in the context of nonlocal chiral quark models. The formalism is worked out introducing Ritus transforms of Dirac fields, which allow to obtain closed analytical expressions for π^0 and σ meson masses and for the π^0 decay constant. Numerical results for these observables are quoted for various parametrizations. In particular, the behavior of the π^0 meson mass with the magnetic field is found to be in good agreement with lattice QCD results. It is also seen that the Goldberger-Treiman and Gell-Mann-Oakes-Renner chiral relations remain valid within these models in the presence of the external magnetic field.

I. INTRODUCTION

The study of the behavior of strongly interacting matter under intense external magnetic fields has gained increasing interest in the last few years, especially due to its applications to the analysis of relativistic heavy ion collisions [1] and the description of compact objects like magnetars [2]. From the theoretical point of view, addressing this subject requires to deal with quantum chromodynamics (QCD) in nonperturbative regimes, therefore present analyses are based either in the predictions of effective models or in the results obtained through lattice QCD (LQCD) calculations. In this work we focus on the effect of an intense external magnetic field on π^0 and σ meson properties. This issue has been studied in the last years following various theoretical approaches for low-energy QCD, such as Nambu-Jona-Lasinio (NJL)-like models [3–8], chiral perturbation theory (ChPT) [9,10] and path integral Hamiltonians (PIH) [11,12]. In addition, results for the light meson spectrum under background magnetic fields have been recently obtained from LQCD calculations [13,14].

We will study in particular the behavior of the mass and decay constant of the π^0 meson in the presence of a uniform static magnetic field, within a relativistic chiral quark model in which quarks interact through a nonlocal four-fermion coupling [15]. This so-called "nonlocal NJL (nlNJL) model" can be viewed as a sort of extension of the NJL model that intends to provide a more realistic effective approach to QCD. Actually, nonlocality arises naturally in the context of successful descriptions of low-energy quark dynamics [16,17], and it has been shown [18] that nonlocal models can lead to a momentum dependence in quark propagators that is consistent with LQCD results. Moreover, in this framework it is possible to obtain an adequate description of the properties of light mesons at both zero and finite temperature [18–28].

The basic theoretical formalism required for the study of nlNJL models in the presence of a uniform static magnetic field B has been introduced in Refs. [29,30], where both zero and finite temperature cases have been considered. Noticeably, in these articles it is shown that nlNJL models naturally allow to reproduce the effect of inverse magnetic catalysis (IMC) observed from LQCD results—that is, the fact that the chiral restoration critical temperature turns out to be a decreasing function of B. In fact, the observation of IMC in LQCD calculations [31,32] represents a challenge from the point of view of theoretical models, since most naive effective approaches to low energy QCD (NJL model, chiral perturbation theory, MIT bag model, quark-meson models) predict that the chiral transition temperature should grow when the magnetic field is increased [33–35]. As shown in Refs. [36,37], this problem can be overcome (e.g. in the case of the local NJL model) by allowing for a B

dependence in the coupling constants. In the present paper we show that nlNJL models not only provide a natural description of the IMC effect but also lead to a B dependence of the π^0 mass that is found to be in good agreement with LQCD results.

This article is organized as follows. In Sec. II we show how to obtain the analytical equations required to determine the values of the π^0 mass and decay constant in the presence of the magnetic field. Our calculations are based on the formalism developed in Refs. [29,30], which makes use of Ritus eigenfunctions [38]. From this analysis it is also immediate to obtain an equation for the σ scalar meson mass. In the last subsection of Sec. II we prove within our model the validity of the Goldberger-Treiman and Gell-Mann-Oakes-Renner relations in the presence of the magnetic field. Previous checks of these relations have been carried out in Refs. [10,11] in the framework of ChPT and PIH, respectively. In Sec. III we quote and discuss our numerical results, comparing our findings with those obtained in LQCD. Our conclusions are presented in Sec. IV. Finally, in Appendices A and B we outline the derivation of some expressions quoted in the main text.

II. THEORETICAL FORMALISM

Let us start by stating the Euclidean action for our nonlocal NJL-like two-flavor quark model,

$$S_E = \int d^4x \left\{ \bar{\psi}(x)(-i\partial\!\!\!/ + m_c)\psi(x) - \frac{G}{2} j_a(x) j_a(x) \right\}. \quad (1)$$

Here m_c is the current quark mass, which is assumed to be equal for u and d quarks. The currents $j_a(x)$ are given by

$$j_a(x) = \int d^4z \,\mathcal{G}(z) \bar{\psi}\left(x + \frac{z}{2}\right) \Gamma_a \psi\left(x - \frac{z}{2}\right), \quad (2)$$

where $\Gamma_a = (\mathbb{1}, i\gamma_5\vec{\tau})$, and the function $\mathcal{G}(z)$ is a nonlocal form factor that characterizes the effective interaction. We introduce now in the effective action Eq. (1) a coupling to an external electromagnetic gauge field $\mathcal{A}_\mu$. For a local theory this can be done by performing the replacement

$$\partial_\mu \to D_\mu \equiv \partial_\mu - i\hat{Q}\mathcal{A}_\mu(x), \quad (3)$$

where $\hat{Q} = \text{diag}(q_u, q_d)$, with $q_u = 2e/3$, $q_d = -e/3$, is the electromagnetic quark charge operator. In the case of the nonlocal model under consideration, the inclusion of gauge interactions implies a change not only in the kinetic terms of the Lagrangian but also in the nonlocal currents in Eq. (2). One has

$$\psi(x - z/2) \to \mathcal{W}(x, x - z/2)\,\psi(x - z/2), \quad (4)$$

and a related change holds for $\bar{\psi}(x + z/2)$ [18,24,27]. Here the function $\mathcal{W}(s, t)$ is defined by

$$\mathcal{W}(r, s) = \text{P} \exp\left[-i \int_r^s d\ell_\mu \hat{Q}\mathcal{A}_\mu(\ell) \right], \quad (5)$$

where r runs over an arbitrary path connecting r with s. As it is usually done, we take it to be a straight line path.

Since we are interested in studying light meson properties, it is convenient to bosonize the fermionic theory, introducing scalar and pseudoscalar fields $\sigma(x)$ and $\vec{\pi}(x)$ and integrating out the fermion fields. The bosonized action can be written as [18,27]

$$S_{\text{bos}} = -\log \det \mathcal{D} + \frac{1}{2G} \int d^4x [\sigma(x)\sigma(x) + \vec{\pi}(x) \cdot \vec{\pi}(x)], \quad (6)$$

with

$$\begin{aligned}
\mathcal{D}(x, x') &= \delta^{(4)}(x - x')(-i\partial\!\!\!/ + m_c) \\
&+ \mathcal{G}(x - x')\gamma_0 W(x, \bar{x})\gamma_0[\sigma(\bar{x}) + i\gamma_5\vec{\tau} \cdot \vec{\pi}(\bar{x})] \\
&\times W(\bar{x}, x'),
\end{aligned} \quad (7)$$

where we have defined $\bar{x} = (x + x')/2$ for the neutral mesons. We will consider the particular case of a constant and homogenous magnetic field orientated along the positive direction of the 3 axis. Then, in the Landau gauge, one has $\mathcal{A}_\mu = Bx_1\delta_{\mu2}$.

A. Mean field fermion propagator

We proceed by expanding the operator in Eq. (7) in powers of the fluctuations $\delta\pi_i$ and $\delta\sigma$ around the corresponding mean field values. We assume that the field $\sigma(x)$ has a nontrivial translation invariant mean field value $\bar{\sigma}$, while the vacuum expectation values of pseudoscalar fields are zero. Thus, we write

$$\sigma(x) = \bar{\sigma} + \delta\sigma(x), \qquad \pi_i(x) = \delta\pi_i(x),$$

and

$$\mathcal{D}(x, x') = \mathcal{D}^{\text{MFA}}(x, x') + \delta\mathcal{D}(x, x'). \quad (8)$$

It is easy to see that the mean field piece is flavor diagonal. One has

$$\mathcal{D}^{\text{MFA}}(x, x') = \text{diag}(\mathcal{D}_u^{\text{MFA}}(x, x'), \mathcal{D}_d^{\text{MFA}}(x, x')), \quad (9)$$

where

$$\begin{aligned}
\mathcal{D}_f^{\text{MFA}}(x, x') &= \delta^{(4)}(x - x')(-i\partial\!\!\!/ - q_f Bx_1\gamma_2 + m_c) \\
&+ \bar{\sigma}\mathcal{G}(x - x') \exp[i\Phi_f(x, x')]. \quad (10)
\end{aligned}$$

Here a direct product to an identity matrix in color space is understood. It is seen that the operator $\mathcal{D}_f^{\text{MFA}}(x, x')$ includes

a translational invariant piece, plus a term carrying the nonlocal form factor and the so-called Schwinger phase $\Phi_f(x, x') = q_f B(x_2 - x'_2)(x_1 + x'_1)/2$. The mean field quark propagators $S_f^{\mathrm{MFA}}(x, x')$ are defined now as

$$S_f^{\mathrm{MFA}}(x, x') = [\mathcal{D}_f^{\mathrm{MFA}}(x, x')]^{-1}. \tag{11}$$

Their explicit form can be obtained by following the Ritus eigenfunction method [38]. As shown in Ref. [30] (see also the analysis carried out within the Schwinger-Dyson formalism in Refs. [39,40]), the propagators can be written in terms of the Schwinger phase $\Phi_f(x, x')$ and a translational invariant function, namely

$$S_f^{\mathrm{MFA}}(x, x') = \exp[i\Phi_f(x, x')] \int \frac{d^4 p}{(2\pi)^4} e^{ip\cdot(x-x')} \tilde{S}_f(p_\perp, p_\parallel), \tag{12}$$

where $p_\perp = (p_1, p_2)$ and $p_\parallel = (p_3, p_4)$. The expression of $\tilde{S}_f(p_\perp, p_\parallel)$ in the nlNJL model is found to be [30]

$$
\begin{aligned}
&\tilde{S}_f(p_\perp, p_\parallel) \\
&= 2\exp(-p_\perp^2/|q_f B|) \\
&\times \sum_{k=0}^\infty \sum_{\lambda=\pm} [(-1)^{k_\lambda}(\hat{A}_{k,p_\parallel}^{\lambda,f} - \hat{B}_{k,p_\parallel}^{\lambda,f} p_\parallel \cdot \gamma_\parallel) L_{k_\lambda}(2p_\perp^2/|q_f B|) \\
&+ 2(-1)^k (\hat{C}_{k,p_\parallel}^{\lambda,f} - \hat{D}_{k,p_\parallel}^{\lambda,f} p_\parallel \cdot \gamma_\parallel) p_\perp \cdot \gamma_\perp L_{k-1}^1(2p_\perp^2/|q_f B|)]\Delta^\lambda,
\end{aligned}
\tag{13}
$$

where the following definitions have been used. The perpendicular and parallel gamma matrices are collected in vectors $\gamma_\perp = (\gamma_1, \gamma_2)$ and $\gamma_\parallel = (\gamma_3, \gamma_4)$, while the matrices Δ^λ are defined as $\Delta^+ = \mathrm{diag}(1, 0, 1, 0)$ and $\Delta^- = \mathrm{diag}(0, 1, 0, 1)$. The integers k_λ are given by $k_\pm = k - 1/2 \pm s_f/2$, where $s_f = \mathrm{sign}(q_f B)$. The functions $\hat{X}_{k,p_\parallel}^{\pm,f}$, with $X = A, B, C, D$, are defined as

$$\hat{A}_{k,p_\parallel}^{\pm,f} = M_{k,p_\parallel}^{\mp,f} \hat{C}_{k,p_\parallel}^{\pm,f} + p_\parallel^2 \hat{D}_{k,p_\parallel}^{\pm,f}, \tag{14}$$

$$\hat{B}_{k,p_\parallel}^{\pm,f} = \hat{C}_{k,p_\parallel}^{\pm,f} - M_{k,p_\parallel}^{\mp,f} \hat{D}_{k,p_\parallel}^{\pm,f}, \tag{15}$$

$$\hat{C}_{k,p_\parallel}^{\pm,f} = \frac{2k|q_f B| + p_\parallel^2 + M_{k,p_\parallel}^{-,f} M_{k,p_\parallel}^{+,f}}{\Delta_{k,p_\parallel}^f}, \tag{16}$$

$$\hat{D}_{k,p_\parallel}^{\pm,f} = \frac{M_{k,p_\parallel}^{\pm,f} - M_{k,p_\parallel}^{\mp,f}}{\Delta_{k,p_\parallel}^f}, \tag{17}$$

where

$$
\begin{aligned}
\Delta_{k,p_\parallel}^f &= (2k|q_f B| + p_\parallel^2 + M_{k,p_\parallel}^{+,f} M_{k,p_\parallel}^{-,f})^2 \\
&+ p_\parallel^2 (M_{k,p_\parallel}^{+,f} - M_{k,p_\parallel}^{-,f})^2,
\end{aligned}
\tag{18}
$$

whereas the functions $M_{k,p_\parallel}^{\lambda,f}$ play the role of effective (momentum-dependent) dynamical quark masses in presence of the magnetic field. They are given by

$$
\begin{aligned}
M_{k,p_\parallel}^{\lambda,f} &= \frac{4\pi}{|q_f B|}(-1)^{k_\lambda} \int \frac{d^2 p_\perp}{(2\pi)^2} M(p_\perp^2 + p_\parallel^2) \\
&\times \exp(-p_\perp^2/|q_f B|) L_{k_\lambda}(2p_\perp^2/|q_f B|),
\end{aligned}
\tag{19}
$$

where

$$M(p^2) = m_c + \bar{\sigma} g(p^2), \tag{20}$$

$g(p^2)$ being the Fourier transform of the nonlocal form factor $\mathcal{G}(x)$. In Eqs. (13) and (19), $L_k(x)$, $L_k^1(x)$ stand for generalized Laguerre polynomials, with the convention $L_{-1}(x) = L_{-1}^1(x) = 0$. The relation in Eq. (19) can be understood as a Laguerre-Fourier transform of the function $M(p^2)$. It is also convenient to introduce Laguerre-Fourier transforms of the form factor $g(p^2)$,

$$
\begin{aligned}
g_{k,p_\parallel}^{\lambda,f} &= \frac{4\pi}{|q_f B|}(-1)^{k_\lambda} \int \frac{d^2 p_\perp}{(2\pi)^2} g(p_\perp^2 + p_\parallel^2) \\
&\times \exp(-p_\perp^2/|q_f B|) L_{k_\lambda}(2p_\perp^2/|q_f B|),
\end{aligned}
\tag{21}
$$

thus one has

$$M_{k,p_\parallel}^{\lambda,f} = [1 - \delta_{(k_\lambda+1)0}] m_c + \bar{\sigma} g_{k,p_\parallel}^{\lambda,f}. \tag{22}$$

Let us also quote the expressions for the quark condensates, $\langle \bar{u} u \rangle$ and $\langle \bar{d} d \rangle$, which can be obtained from

$$\langle \bar{f} f \rangle = -\frac{1}{V^{(4)}} \mathrm{Tr} S_f^{\mathrm{MFA}} = -N_C \int \frac{d^4 p}{(2\pi)^4} \mathrm{tr}_D \tilde{S}_f(p_\perp, p_\parallel). \tag{23}$$

Given the result for the propagators in Eq. (13) one gets

$$
\begin{aligned}
\langle \bar{f} f \rangle &= -4 \sum_{k=0}^\infty \int_{q_\parallel} \sum_{\lambda=\pm} (-1)^{k_\lambda} \hat{A}_{k,p_\parallel}^{\lambda,f} \int \frac{d^2 p_\perp}{(2\pi)^2} \\
&\times \exp(-p_\perp^2/|q_f B|) L_{k_\lambda}(2p_\perp^2/|q_f B|) \\
&= -\frac{N_C |q_f B|}{\pi} \sum_{k=0}^\infty \int_{q_\parallel} \sum_{\lambda=\pm} \hat{A}_{k,q_\parallel}^{\lambda,f}.
\end{aligned}
\tag{24}
$$

As usual in this type of models, it is seen that the chiral condensates turn out to be divergent away from the chiral limit, and they have to be regularized. We follow a prescription similar as that considered e.g. in Ref. [41],

in which we subtract the corresponding free quark contribution and then we add it in a regularized form. Here by "free" we mean in the absence of the four fermion effective coupling, but keeping the interaction with the magnetic field. Thus, we have

$$\langle \bar{f}f \rangle_{\text{reg}} = \langle \bar{f}f \rangle - \langle \bar{f}f \rangle_{\text{free}} + \langle \bar{f}f \rangle_{\text{free,reg}}. \tag{25}$$

To regularize the free contribution we separate it into two pieces, namely a divergent $B = 0$ term and a finite term that depends on the magnetic field. By dropping the divergent piece one gets [30]

$$\langle \bar{f}f \rangle_{\text{free,reg}}(B) = -\frac{N_C m_c^3}{4\pi^2} \left[\frac{\ln \Gamma(x_f)}{x_f} - \frac{\ln 2\pi}{2x_f} \right.$$

$$\left. + 1 - \left(1 - \frac{1}{2x_f} \right) \ln x_f \right], \tag{26}$$

with $x_f = m_c^2/(2|q_f B|)$. Notice that an expression similar to Eq. (26) is obtained for the explicit B-dependent contribution to the quark condensate in the case of the local NJL model, just replacing the current quark mass m_c by a dressed effective quark mass M_f, which in the local NJL model turns out to be constant [42,43].

B. π^0 and σ meson masses

The expression of the quark propagator in Eq. (13) can be used to obtain the theoretical expressions for the π^0 and σ meson masses within the nlNJL model. Let us first concentrate on the π^0 mass, which follows from the terms in the expansion of the bosonized action S_{bos} that are quadratic in $\delta\pi_3$. Expanding the first term in Eq. (6) around the mean field values of the meson fields one has

$$-\log \det \mathcal{D} = -\text{Tr} \log \mathcal{D}_0 - \text{Tr} \log(1 + \mathcal{D}_0^{-1}\delta\mathcal{D})$$

$$= -\text{Tr} \log \mathcal{D}_0 - \text{Tr}(\mathcal{D}_0^{-1}\delta\mathcal{D})$$

$$+ \frac{1}{2}\text{Tr}(\mathcal{D}_0^{-1}\delta\mathcal{D})^2 + \ldots \tag{27}$$

From Eq. (7), it is seen that the quadratic piece is given by

$$\frac{1}{2}\text{Tr}(\mathcal{D}_0^{-1}\delta\mathcal{D})^2|_{(\delta\pi_3)^2} = -\frac{1}{2}\int \mathcal{G}(x'-x'')\mathcal{G}(x'''-x)$$

$$\times \text{tr}_{cfD}[\mathcal{D}_0^{-1}(x,x')\gamma_5 \exp[\Phi(x',x'')]$$

$$\times \mathcal{D}_0^{-1}(x'',x''')\gamma_5 \exp[\Phi(x''',x)]]$$

$$\times \delta\pi_3\left(\frac{x'+x''}{2}\right)\delta\pi_3\left(\frac{x'''+x}{2}\right), \tag{28}$$

where the integral extends over coordinate spaces x, x', x'' and x''', and the trace acts on color, flavor and Dirac spaces.

To determine the π^0 mass it is convenient to write the trace in Eq. (28) in momentum space. In this way the $(\delta\pi_3)^2$ piece of the bosonized action in Eq. (6) can be written as

$$S_{\text{bos}}|_{(\delta\pi_3)^2} = \frac{1}{2}\text{Tr}(\mathcal{D}_0^{-1}\delta\mathcal{D})^2\Big|_{(\delta\pi_3)^2} + \frac{1}{2G}\int \frac{d^4t}{(2\pi)^4}\delta\pi_3(t)\delta\pi_3(-t)$$

$$= \frac{1}{2}\int \frac{d^4t}{(2\pi)^4}\left[F(t_\perp^2, t_\parallel^2) + \frac{1}{G} \right]\delta\pi_3(t)\delta\pi_3(-t), \tag{29}$$

and, choosing the frame in which the π^0 meson is at rest, its mass can be obtained as the solution of the equation

$$F(0, -m_{\pi^0}^2) + \frac{1}{G} = 0. \tag{30}$$

Thus, our task is to obtain within our model the function $F(t_\perp^2, t_\parallel^2)$ in the limit $t_\perp = 0$. After some straightforward calculation, from Eq. (28) one gets

$$F(0, t_\parallel^2) = 16\pi^2 N_C \sum_{f=u,d} \frac{1}{(q_f B)^2}$$

$$\times \int_{q_\perp p_\perp p_\perp' q_\parallel} g(q_\perp^2 + q_\parallel^2)g[(p_\perp' + p_\perp - q_\perp)^2 + q_\parallel^2]$$

$$\times \exp[i2\phi(q_\perp, p_\perp, p_\perp')/(q_f B)]$$

$$\times \text{tr}_D[\tilde{S}_f(p_\perp, q_\parallel^+)i\gamma_5\tilde{S}_f(p_\perp', q_\parallel^-)i\gamma_5], \tag{31}$$

where we have defined $q_\parallel^\pm = q_\parallel \pm t_\parallel/2$, and the function ϕ in the exponential is given by

$$\phi(q_\perp, p_\perp, p_\perp') = p_2 p_1' + q_1(p_2' - p_2) - p_1 p_2'$$

$$- q_2(p_1' - p_1). \tag{32}$$

For the integrals over two-dimensional momentum vectors we have used the notation

$$\int_{pq\ldots} \equiv \int \frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\ldots \tag{33}$$

The evaluation of the trace in Eq. (31) leads to

$$\text{tr}_D[\tilde{S}_f(p_\perp, q_\parallel^+)i\gamma_5\tilde{S}_f(p_\perp', q_\parallel^-)i\gamma_5]$$

$$= -8e^{-(p_\perp^2 + p_\perp'^2)/B_f}\sum_{k,k'=0}^{\infty}(-1)^{k+k'}$$

$$\times \left[\sum_{\lambda=\pm}F_{kk',q_\parallel^+ q_\parallel^-}^{\lambda,f(AB)}L_{k_\lambda}(2p_\perp^2/B_f)L_{k_\lambda'}(2p_\perp'^2/B_f) \right.$$

$$\left. + 8F_{kk',q_\parallel^+ q_\parallel^-}^{+,f(CD)}(p \cdot p')L_{k-1}^1(2p_\perp^2/B_f)L_{k'-1}^1(2p_\perp'^2/B_f) \right], \tag{34}$$

where

$$F^{\lambda,f(XY)}_{kk',q^+_\| q^-_\|} = \hat{X}^{\lambda,f}_{k,q^+_\|} \hat{X}^{\lambda,f}_{k',q^-_\|} + (q^+_\| \cdot q^-_\|) \hat{Y}^{\lambda,f}_{k,q^+_\|} \hat{Y}^{\lambda,f}_{k',q^-_\|}. \quad (35)$$

For simplicity we have introduced here the notation $B_f = |q_f B|$.

To work out the integrals over $p_\perp$, $p'_\perp$ and $q_\perp$, which involve the Laguerre polynomials, it is convenient to introduce the Laguerre-Fourier transforms of the nonlocal form factors. It is seen that Eq. (21) can be inverted to get

$$g(p^2_\perp + p^2_\|) = 2e^{-p^2_\perp/B_f} \sum_{k=0}^{\infty} (-1)^{k_\lambda} g^{\lambda,f}_{k,p_\|} L_{k_\lambda}(2p^2_\perp/B_f), \quad (36)$$

for either $\lambda = +$ or $\lambda = -$. Using this relation to transform the functions $g(q^2_\perp + q^2_\|)$ and $g[(p'_\perp + p_\perp - q_\perp)^2 + q^2_\|]$ in Eq. (31), it can be shown that the integrals over perpendicular momenta can be performed analytically. The corresponding calculation, sketched in Appendix A, leads to a relatively brief expression for $F(0, t^2_\|)$, namely

$$F(0, t^2_\|) = -\frac{N_C}{\pi} \sum_{f=u,d} B_f \sum_{k=0}^{\infty} \int \frac{d^2 q_\|}{(2\pi)^2} \left[\sum_{\lambda=\pm} g^{\lambda,f\ 2}_{k,q_\|} F^{\lambda,f(AB)}_{kk,q^+_\| q^-_\|} \right.$$
$$\left. + 4k B_f g^{+,f}_{k,q_\|} g^{-,f}_{k,q_\|} F^{+,f(CD)}_{kk,q^+_\| q^-_\|} \right], \quad (37)$$

which is one of the main analytical results of this article. In the limit $B \to 0$, it can be shown that Eq. (37) reduces, as it should, to the expression quoted e.g. in Ref. [24],

$$F(t^2)\Big|_{B=0} = -8N_C \int \frac{d^4 q}{(2\pi)^4} g(q^2)^2$$
$$\times \frac{(q^+ \cdot q^-) + M(q^{+2}) M(q^{-2})}{[q^{+2} + M(q^{+2})][q^{-2} + M(q^{-2})]}. \quad (38)$$

In the case of the σ meson, the mass can be determined from a relation similar to Eq. (30). The corresponding function $G(0, t^2_\|)$ is obtained by following basically the same steps as for the π^0 case. The essential difference is that one has to remove the factors $i\gamma_5$ in the trace in Eq. (31). When calculating this trace one arrives at a result analogous to that in Eq. (34), where the new functions $G^{\lambda,f(XY)}_{kk',q^+_\| q^-_\|}$ are given by

$$G^{\lambda,f(AB)}_{kk',q^+_\| q^-_\|} = -\hat{A}^{\lambda,f}_{k,q^+_\|} \hat{A}^{\lambda,f}_{k',q^-_\|} + (q^+_\| \cdot q^-_\|) \hat{B}^{\lambda,f}_{k,q^+_\|} \hat{B}^{\lambda,f}_{k',q^-_\|}$$
$$G^{\lambda,f(CD)}_{kk',q^+_\| q^-_\|} = \hat{C}^{\lambda,f}_{k,q^+_\|} \hat{C}^{\lambda,f}_{k',q^-_\|} - (q^+_\| \cdot q^-_\|) \hat{D}^{\lambda,f}_{k,q^+_\|} \hat{D}^{\lambda,f}_{k',q^-_\|}. \quad (39)$$

The final expression for $G(0, t^2_\|)$ has then the same form as the left-hand side (lhs) of Eq. (37), just replacing $F^{\pm,f(XY)}_{kk',q^+_\| q^-_\|} \to G^{\pm,f(XY)}_{kk',q^+_\| q^-_\|}$.

C. π^0 decay constant

In the absence of external fields, the π^0 decay constant is defined through the matrix element of the axial current $\mathcal{J}^\mu_{A3}$ between the vacuum and the physical pion state, taken at the pion pole. One has

$$\langle 0|\mathcal{J}^\mu_{A3}(x)|\tilde{\pi}_3(t)\rangle = ie^{-i(t\cdot x)} f(t^2) t^\mu, \quad (40)$$

where $\tilde{\pi}_3(t) = Z^{-1/2}_{\pi^0} \pi_3(t)$ is the renormalized field associated with the π^0 meson state, with $t^2 = -m^2_{\pi^0}$. In what follows we will obtain an analytical expression for the form factor $f(t^2)$ under a static uniform magnetic field, defining the π^0 decay constant $f_{\pi^0}(B)$ as the value of this form factor at $t^2 = -m^2_{\pi^0}(B)$. Notice, however, that in the presence of the magnetic field further Lorentz structures are allowed for the matrix element in Eq. (40), and consequently other form factors could be nonzero [44].

The wave function renormalization factor $Z^{1/2}_{\pi^0}$ is given by the residue of the pion propagator at $t^2 = -m^2_{\pi^0}$. Therefore, in the rest frame of the π^0 meson, from Eq. (29) we have

$$Z^{-1}_{\pi^0} = \frac{dF(0, t^2_\|)}{dt^2_\|}\Bigg|_{t^2_\|=-m^2_{\pi^0}}, \quad (41)$$

where $F(0, t^2_\|)$ is the function given by Eq. (37). The matrix element in Eq. (40) can be obtained by introducing a coupling between the current $\mathcal{J}^\mu_{A3}$ and an auxiliary axial gauge field W^μ_3, and taking the corresponding functional derivative of the effective action. In the same way as discussed at the beginning of this section, gauge invariance requires the couplings to this auxiliary gauge field to be introduced through the covariant derivative and the parallel transport of the fermion fields, see Eqs. (3) and (4). In the presence of the external magnetic field one has

$$D_\mu = \partial_\mu - i\hat{Q}\mathcal{A}_\mu(x) - \frac{i}{2}\gamma_5\tau_3 W_{3\mu}(x), \quad (42)$$

$$\mathcal{W}(r,s) = \text{P}\exp\left\{-i\int_r^s d\ell_\mu \left[\hat{Q}\mathcal{A}_\mu(\ell) + \frac{1}{2}\gamma_5\tau_3 W_{3\mu}(\ell)\right]\right\}. \quad (43)$$

Assuming that the mean field value of the π_3 field vanishes, the pion decay constant can be obtained by expanding the bosonized action up to first order in $W_{3\mu}$ and $\delta\pi_3$. Writing

$$S_{\text{bos}}|_{W_3\delta\pi_3} = \int \frac{d^4 t}{(2\pi)^4} F_\mu(t) W_{3\mu}(t)\delta\pi_3(-t), \quad (44)$$

one finds

$$f_{\pi^0} = f(-m_{\pi^0}^2) = iZ_{\pi^0}^{1/2} \frac{t_\mu F_\mu(t)}{t^2}\bigg|_{t_\perp^2=0, t_\parallel^2=-m_{\pi^0}^2}. \quad (45)$$

To find the function $F_\mu(t)$ we consider once again the expansion in Eq. (27). In addition, we expand $\delta \mathcal{D}$ in powers of $\delta\pi_3$ and W_3,

$$\delta\mathcal{D} = \delta\mathcal{D}_W + \delta\mathcal{D}_\pi + \delta\mathcal{D}_{W\pi} + \ldots, \quad (46)$$

which leads to

$$S_{\text{bos}}|_{W_3\delta\pi_3} = -\text{Tr}(\mathcal{D}_0^{-1}\delta\mathcal{D}_{W\pi}) + \text{Tr}(\mathcal{D}_0^{-1}\delta\mathcal{D}_W \mathcal{D}_0^{-1}\delta\mathcal{D}_\pi). \quad (47)$$

The operators in the right-hand side (rhs) of Eq. (46) explicitly read

$$\delta\mathcal{D}_\pi(x, x') = i\gamma_5\tau_3 \exp[\Phi(x, x')]\mathcal{G}(x - x')\delta\pi_3(\bar{x}), \quad (48)$$

$$\begin{aligned}\delta\mathcal{D}_W(x, x') &= \delta^{(4)}(x - x')\frac{\tau_3}{2}\gamma_5\gamma_\mu W_{3\mu}(\bar{x}) \\ &\quad + i\bar{\sigma}\gamma_5\frac{\tau_3}{2}\exp[\Phi(x, x')] \\ &\quad \times \mathcal{G}(x - x')[a_3(x, \bar{x}) - a_3(\bar{x}, x)], \quad (49)\end{aligned}$$

$$\begin{aligned}\delta\mathcal{D}_{W\pi}(x, x') &= -\frac{1}{2}\exp[\Phi(x, x')]\mathcal{G}(x - x')[a_3(x, \bar{x}) \\ &\quad - a_3(\bar{x}, x)]\delta\pi_3(\bar{x}), \quad (50)\end{aligned}$$

where we have introduced the definitions $\bar{x} = (x + x')/2$ and

$$a_3(x, y) = \int_x^y d\ell_\mu W_{3\mu}(\ell). \quad (51)$$

A direct product to an identity matrix in color space is understood.

The first and second terms in the rhs of Eq. (47) can be diagrammatically represented as a tadpole and a two-propagator contribution, respectively. Let us start by discussing the tadpole piece. After some straightforward calculation we get

$$-\text{Tr}(\mathcal{D}_0^{-1}\delta\mathcal{D}_{W\pi}) = \int \frac{d^4t}{(2\pi)^4} F_\mu^{(\text{I})}(t)W_{3\mu}(t)\delta\pi_3(-t), \quad (52)$$

where

$$\begin{aligned}F_\mu^{(\text{I})}(t) &= i\frac{N_C}{2}\sum_{f=u,d}\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\{g[(p - q/2)^2] \\ &\quad - g[(p - q/2 + t/2)^2]\} \\ &\quad \times \text{tr}_D[\tilde{S}_f(p_\perp, p_\parallel)]h_\mu(q, t - q), \quad (53)\end{aligned}$$

with

$$h_\mu(q, q') = -i\int d^4z e^{iq'\cdot z}\int_0^z d\ell_\mu e^{i(q+q')\cdot\ell}. \quad (54)$$

Since we are interested in the scalar product $t \cdot F^{(\text{I})}(t)$, we can use the relation

$$t_\mu h_\mu(q, t - q) = (2\pi)^4[\delta^{(4)}(t - q) - \delta^{(4)}(q)], \quad (55)$$

which holds independently of the integration path chosen in Eq. (54). Taking into account the expression for $\tilde{S}_f(p_\perp, p_\parallel)$ in Eq. (13) we obtain

$$\begin{aligned}k_\mu &F_\mu^{(\text{I})}(t)|_{t_\perp=0} \\ &= i2N_C\sum_{f=u,d}\sum_{k=0}^\infty\int_{p_\perp p_\parallel}[g(p^{+2}) + g(p^{-2}) - 2g(p^2)] \\ &\quad \times \exp(-p_\perp^2/B_f) \\ &\quad \times \sum_{\lambda=\pm}(-1)^{k_\lambda}\hat{A}_{k,p_\parallel}^{\lambda,f}L_{k_\lambda}(2p_\perp^2/B_f), \quad (56)\end{aligned}$$

where $p^{\pm 2} = p_\perp^2 + (p_\parallel \pm t_\parallel/2)^2$. Now, as in the case of the meson masses, we can perform the integral over $p_\perp$ after taking the Laguerre-Fourier transform of the nonlocal form factors, Eq. (36). We have

$$\begin{aligned}t_\mu &F_\mu^{(\text{I})}(t)|_{t_\perp=0} = i4N_C\sum_{f=u,d}\sum_{k,k'=0}^\infty(-1)^{k+k'}\int_{p_\perp p_\parallel}\exp(-2p_\perp^2/B_f) \\ &\quad \times \sum_{\lambda=\pm}(g_{k',p^+}^{\lambda,f} + g_{k',p^-}^{\lambda,f} - 2g_{k',p_\parallel}^{\lambda,f}) \\ &\quad \times \hat{A}_{k,p_\parallel}^{\lambda,f}L_{k_\lambda'}(2p_\perp^2/B_f)L_{k_\lambda}(2p_\perp^2/B_f) \\ &= i\frac{N_C}{2\pi}\sum_{f=u,d}B_f\sum_{k=0}^\infty\int_{p_\parallel}\sum_{\lambda=\pm} \\ &\quad \times (g_{k,p^+}^{\lambda,f} + g_{k,p^-}^{\lambda,f} - 2g_{k,p_\parallel}^{\lambda,f})\hat{A}_{k,p_\parallel}^{\lambda,f}, \quad (57)\end{aligned}$$

where we have made use of the orthogonality property of Laguerre polynomials.

To analyze the two-propagator piece we write

$$\begin{aligned}\text{Tr}&(\mathcal{D}_0^{-1}\delta\mathcal{D}_W \mathcal{D}_0^{-1}\delta\mathcal{D}_\pi) \\ &= \int \frac{d^4t}{(2\pi)^4}[F_\mu^{(\text{II})}(t) + F_\mu^{(\text{III})}(t)]W_{3\mu}(t)\delta\pi_3(-t), \quad (58)\end{aligned}$$

where $F_\mu^{(\text{II})}(t)$ and $F_\mu^{(\text{III})}(t)$ correspond to the contributions arising from the first and second terms of $\delta\mathcal{D}_W$ in Eq. (49), respectively. For the first term we obtain

$$F_\mu^{(II)}(t)$$

$$= i8\pi^2 N_C \sum_{f=u,d} \frac{1}{B_f^2}$$

$$\times \int_{q_\parallel q_\perp p_\perp p_\perp'} g(q^2) \exp[i2\varphi(q_\perp, p_\perp, p_\perp', t_\perp)/(q_f B)]$$

$$\times \mathrm{tr}_D[\tilde{S}_f(p_\perp, q_\parallel^+)\gamma_5\gamma_\mu\tilde{S}_f(p_\perp', q_\parallel^-)\gamma_5], \tag{59}$$

where $q_\parallel^\pm = q_\parallel \pm t_\parallel/2$, and the function φ in the exponential is given by

$$\varphi(q_\perp, p_\perp, p_\perp', t_\perp) = p_2(q_1 - t_1/2) - p_2'(q_1 + t_1/2)$$
$$- q_1 t_2 - p_2 p_1' - (1 \leftrightarrow 2). \tag{60}$$

Since we are interested in the product $t_\mu F_\mu^{(II)}(t)$ for $t_\perp = 0$, we calculate the trace

$$\mathrm{tr}_D[\tilde{S}_f(p_\perp, q_\parallel^+)\gamma_5(t_\parallel \cdot \gamma_\parallel)\tilde{S}_f(p_\perp', q_\parallel^-)\gamma_5]$$

$$= 8\exp[-(p_\perp^2 + p_\perp'^2)/B_f] \sum_{k,k'=0}^\infty (-1)^{k+k'} \left\{ \sum_{\lambda=\pm} [(t_\parallel \cdot q_\parallel^-)\hat{A}_{k,q_\parallel^+}^{\lambda,f}\hat{B}_{k',q_\parallel^-}^{\lambda,f} - (t_\parallel \cdot q_\parallel^+)\hat{A}_{k',q_\parallel^-}^{\lambda,f}\hat{B}_{k,q_\parallel^+}^{\lambda,f}] L_{k_\lambda}(2p_\perp^2/B_f) L_{k_\lambda'}(2p_\perp'^2/B_f) \right.$$

$$\left. + 8i(p_1 p_2' - p_2 p_1')[(t_\parallel \cdot q_\parallel^-)\hat{C}_{k,q_\parallel^+}^{+,f}\hat{D}_{k',q_\parallel^-}^{+,f} - (t_\parallel \cdot q_\parallel^+)\hat{C}_{k',q_\parallel^-}^{+,f}\hat{D}_{k,q_\parallel^+}^{+,f}] L_{k-1}^1(2p_\perp^2/B_f) L_{k-1}^1(2p_\perp'^2/B_f) \right\}. \tag{61}$$

One can now introduce the transformation in Eq. (36) for $g(q^2)$ in order to integrate over transverse momenta and express the result in terms of Laguerre-Fourier transforms of the form factors. This calculation, outlined in Appendix B, leads to

$$t_\mu F_\mu^{(II)}(t)|_{t_\perp=0} = -i\frac{N_C}{\pi} \sum_{f=u,d} B_f \sum_{k=0}^\infty \int_{q_\parallel} (t_\parallel \cdot q_\parallel^+) \left[\sum_{\lambda=\pm} g_{k,q_\parallel}^{\lambda,f} \hat{A}_{k,q_\parallel}^{\lambda,f} \hat{B}_{k,q_\parallel^+}^{\lambda,f} + 2kB_f(g_{k,q_\parallel}^{+,f} - g_{k,q_\parallel}^{-,f})\hat{C}_{k,q_\parallel^-}^{+,f}\hat{D}_{k,q_\parallel^+}^{+,f} \right]. \tag{62}$$

Finally, for the second term in Eq. (58) we find

$$F_\mu^{(III)}(t) = i8\pi^2 N_C\bar{\sigma} \sum_{f=u,d} \frac{1}{B_f^2} \int \frac{d^4r}{(2\pi)^4} h_\mu(r, t-r) \int_{q_\parallel q_\perp p_\perp p_\perp'} g(q^2)$$

$$\times \{g[(p_\perp - r_\perp/2 - t_\perp/2)^2 + (p_\parallel + p_\parallel' - q_\parallel - r_\parallel/2)^2]$$

$$- g[(p_\perp - r_\perp/2)^2 + (p_\parallel + p_\parallel' - q_\parallel - r_\parallel/2 + t_\parallel/2)^2]\}$$

$$\times \exp[i2\varphi(q_\perp, p_\perp, p_\perp', k_\perp)/(q_f B)]\mathrm{tr}_D[\tilde{S}_f(p_\perp, q_\parallel^+)i\gamma_5\tilde{S}_f(p_\perp', q_\parallel^-)i\gamma_5], \tag{63}$$

where the function $\varphi(q_\perp, p_\perp, p_\perp', k_\perp)$ is that given in Eq. (60). Using the relation in Eq. (55) we obtain

$$t_\mu F_\mu^{(III)}(t)|_{t_\perp=0} = i8\pi^2 N_C\bar{\sigma} \sum_{f=u,d} \frac{1}{B_f^2} \int_{q_\parallel q_\perp p_\perp p_\perp'} [g(s_\perp^2 + q_\parallel^{+2}) + g(s_\perp^2 + q_\parallel^{-2})$$

$$- 2g(s_\perp^2 + q_\parallel^2)]g(q^2) \exp[-i2\phi(q_\perp, p_\perp, p_\perp')/(q_f B)]$$

$$\times \mathrm{tr}_D[\tilde{S}_f(p_\perp, q_\parallel^+)i\gamma_5\tilde{S}_f(p_\perp', q_\parallel^-)i\gamma_5], \tag{64}$$

where $\phi(q_\perp, p_\perp, p_\perp')$ is given by Eq. (32), and we have defined $s_\perp = p_\perp' + p_\perp - q_\perp$. Comparing with Eq. (31), it is seen that the calculation to be done is basically the same as that carried out in the case of the analysis of the π^0 mass, described in Appendix A. In this way we obtain

$$t_\mu F_\mu^{(III)}(t)|_{t_\perp=0} = -i\frac{N_C}{2\pi}\bar{\sigma} \sum_{f=u,d} B_f \sum_{k=0}^\infty \int_{q_\parallel} \left[\sum_{\lambda=\pm} g_{k,q_\parallel}^{\lambda,f} \tilde{g}_{k,q_\parallel}^{\lambda,f} F_{kk,q_\parallel^+ q_\parallel^-}^{\lambda,f(AB)} + 2kB_f(g_{k,q_\parallel}^{+,f}\tilde{g}_{k,q_\parallel t_\parallel}^{-,f} + g_{k,q_\parallel}^{-,f}\tilde{g}_{k,q_\parallel t_\parallel}^{+,f})F_{kk,q_\parallel^+ q_\parallel^-}^{+,f(CD)} \right], \tag{65}$$

where we have defined

$$\tilde{g}^{\lambda,f}_{k,q_\parallel t_\parallel} = g^{\lambda,f}_{k,q^+_\parallel} + g^{\lambda,f}_{k,q^-_\parallel} - 2g^{\lambda,f}_{k,q_\parallel}. \tag{66}$$

Notice that one has

$$\bar{\sigma}\tilde{g}^{\lambda,f}_{k,q_\parallel t_\parallel} = M^{\lambda,f}_{k,q^+_\parallel} + M^{\lambda,f}_{k,q^-_\parallel} - 2M^{\lambda,f}_{k,q_\parallel}. \tag{67}$$

When summing the contributions given by Eqs. (57), (62) and (65) it is seen that some cancellations help to simplify the final expression for $t \cdot F(t)|_{t_\perp=0}$. After some algebra one gets

$$\begin{aligned}
t_\mu F_\mu(t)|_{t_\perp=0} = &\, i\frac{N_C}{\pi} \sum_{f=u,d} B_f \\
&\times \sum_{k=0}^\infty \int_{q_\parallel} \left[\sum_{\lambda=\pm} g^{\lambda,f}_{k,q_\parallel} \left(F^{\lambda,f(AB)}_{kk,q^+_\parallel q^-_\parallel} M^{\lambda,f}_{k,q_\parallel} - \hat{A}^{\lambda,f}_{k,q_\parallel} \right) \right. \\
&\left. + 2kB_f \left(g^{+,f}_{k,q_\parallel} M^{-,f}_{k,q_\parallel} + g^{-,f}_{k,q_\parallel} M^{+,f}_{k,q_\parallel} \right) F^{+,f(CD)}_{kk,q^+_\parallel q^-_\parallel} \right].
\end{aligned} \tag{68}$$

Moreover, the expression for f_{π^0} can be further simplified by making use of the gap equation and the relation (30) obtained for the π^0 mass. According to the result previously obtained in Ref. [30], the gap equation can be written as

$$\frac{\bar{\sigma}}{G} = \frac{N_C}{\pi} \sum_{f=u,d} B_f \sum_{k=0}^\infty \int_{q_\parallel} \sum_{\lambda=\pm} g^{\lambda,f}_{k,q_\parallel} \hat{A}^{\lambda,f}_{k,q_\parallel}, \tag{69}$$

while for the pion mass we have

$$\frac{1}{G} = -F(0, -m^2_{\pi^0}), \tag{70}$$

with $F(0,t^2_\parallel)$ given by Eq. (37). Taking into account these equations and the relation in Eq. (22), it is easy to see that for $t^2_\parallel = -m^2_{\pi^0}$ there are some additional cancellations in Eq. (68). Thus, we arrive to our final expression

$$m^2_{\pi^0} f_{\pi^0} = m_c Z^{1/2}_{\pi^0} J(-m^2_{\pi^0}), \tag{71}$$

where the function $J(t^2_\parallel)$ is given by

$$\begin{aligned}
J(t^2_\parallel) = &\, \frac{N_C}{\pi} \sum_{f=u,d} B_f \\
&\times \sum_{m=0}^\infty \int_{q_\parallel} \sum_{\lambda=\pm} g^{\lambda,f}_{k,q_\parallel} (F^{\lambda,f(AB)}_{kk,q^+_\parallel q^-_\parallel} + 2kB_f F^{\lambda,f(CD)}_{kk,q^+_\parallel q^-_\parallel}),
\end{aligned} \tag{72}$$

with $q^\pm_\parallel = q_\parallel \pm t_\parallel$. Taking the limit $B \to 0$ one arrives at the expression given e.g. in Ref. [24],

$$\begin{aligned}
J(t^2)\Big|_{B=0} = &\, 8N_C \int \frac{d^4q}{(2\pi)^4} g(q^2) \\
&\times \frac{(q^+ \cdot q^-) + M(q^{+2})M(q^{-2})}{[q^{+2} + M(q^{+2})][q^{-2} + M(q^{-2})]}.
\end{aligned} \tag{73}$$

D. Chiral relations

In this subsection we show that the Goldberger-Treiman (GT) and Gell-Mann-Oakes-Renner (GOR) relations remain valid in our model in the presence of the external magnetic field. For this purpose, following the line of the analysis in Ref. [24], it is useful to define the function

$$K(t^2_\parallel) = m_c J(t^2_\parallel) - \bar{\sigma}F(0, t^2_\parallel), \tag{74}$$

where $J(t^2_\parallel)$ and $F(0, t^2_\parallel)$ are given by Eqs. (72) and (37), respectively. From Eq. (68), taking into account the relation in Eq. (22) it is easy to show that

$$-it_\mu F_\mu(t)|_{t_\perp=0} = K(t^2_\parallel) - \frac{N_C}{\pi} \sum_{f=u,d} B_f \sum_{k=0}^\infty \int_{q_\parallel} \sum_{\lambda=\pm} g^{\lambda,f}_{k,q_\parallel} \hat{A}^{\lambda,f}_{k,q_\parallel}. \tag{75}$$

The second term in the rhs is a constant, equal to $-\bar{\sigma}/G$ according to the gap equation. Moreover, taking into account the relations

$$\begin{aligned}
F^{\lambda,f(AB)}_{kk,q_\parallel q_\parallel} + 2kB_f F^{\lambda,f(CD)}_{kk,q_\parallel q_\parallel} &= \hat{B}^{\lambda,f}_{k,q_\parallel}, \\
(M^{\pm,f}_{k,q_\parallel} - M^{\mp,f}_{k,q_\parallel})F^{\lambda,f(CD)}_{kk,q_\parallel q_\parallel} &= \hat{D}^{\pm,f}_{k,q_\parallel}, \\
\hat{B}^{\lambda,f}_{k,q_\parallel} M^{\lambda,f}_{k,q_\parallel} - 2kB_f \hat{D}^{\lambda,f}_{k,q_\parallel} &= \hat{A}^{\lambda,f}_{k,q_\parallel},
\end{aligned} \tag{76}$$

it is seen that

$$m_c J(0) - \bar{\sigma}F(0,0) = \frac{N_C}{\pi} \sum_{f=u,d} B_f \sum_{k=0}^\infty \int_{q_\parallel} \sum_{\lambda=\pm} g^{\lambda,f}_{k,q_\parallel} \hat{A}^{\lambda,f}_{k,q_\parallel}, \tag{77}$$

hence we can write

$$-it_\mu F_\mu(t)|_{t_\perp=0} = K(t^2_\parallel) - K(0). \tag{78}$$

Thus, from Eq. (45) we obtain

$$f_{\pi^0} = -Z^{1/2}_{\pi^0} \frac{[K(-m^2_{\pi^0}) - K(0)]}{-m^2_{\pi^0}}. \tag{79}$$

In the chiral limit one has $m_c \to 0$, $m^2_\pi \to 0$, therefore the pion decay constant is given by

$$f_{\pi^0,0} = -Z_{\pi^0,0}^{1/2} \frac{dK_0(t_\parallel^2)}{dt_\parallel^2}\bigg|_{t_\parallel=0}$$

$$= Z_{\pi^0,0}^{1/2} \bar{\sigma}_0 \frac{dF_0(0, t_\parallel^2)}{dt_\parallel^2}\bigg|_{t_\parallel=0}$$

$$= Z_{\pi^0,0}^{-1/2} \bar{\sigma}_0, \tag{80}$$

where we have taken into account the relation between Z_{π^0} and the derivative of $F(0, t_\parallel^2)$ in Eq. (41). Subindices 0 indicate that all quantities have to be evaluated in the chiral limit. Noticing that $Z_{\pi^0}^{1/2}$ turns out to be the effective coupling constant $g_{\pi q\bar{q}}$ between the π_3 field and the quark-antiquark pseudoscalar currents, we arrive at

$$f_{\pi^0,0} g_{\pi q\bar{q},0} = \bar{\sigma}_0, \tag{81}$$

which is the expression for the Goldberger-Treiman relation at the quark level.

Finally, let us consider the quark condensates, $\langle \bar{u}u \rangle$ and $\langle \bar{d}d \rangle$, which in the presence of the magnetic field are given by Eq. (24). Taking into account the relations (76), it is easy to see that in the chiral limit one has

$$\langle \bar{u}u + \bar{d}d \rangle_0 = -\bar{\sigma}_0 J_0(0) \tag{82}$$

[notice that away from the chiral limit the integrals in Eq. (24) are in general divergent, and need to be regularized]. In addition, we can perform a chiral expansion at both sides of Eq. (71), keeping only the lowest nonzero order. This leads to

$$m_{\pi^0}^2 f_{\pi^0,0} = m_c Z_{\pi^0,0}^{1/2} J_0(0). \tag{83}$$

From this relation, together with Eq. (80), we obtain the Gell-Mann-Oakes-Renner relation for the π^0 meson,

$$m_c \langle \bar{u}u + \bar{d}d \rangle_0 = -m_{\pi^0}^2 f_{\pi^0,0}^2. \tag{84}$$

III. NUMERICAL RESULTS

To obtain definite numerical predictions for the behavior of the above defined quantities as functions of the external magnetic field, it is necessary to specify the particular shape of the nonlocal form factor $g(p^2)$. We consider here two often-used forms [23,24,45], namely a Gaussian function

$$g(p^2) = \exp(-p^2/\Lambda^2) \tag{85}$$

and a "5-Lorentzian" function

$$g(p^2) = \frac{1}{1 + (p^2/\Lambda^2)^5}. \tag{86}$$

Notice that in the form factors we introduce an energy scale Λ, which acts as an effective momentum cut-off. This has to be taken as a free parameter of the model, together with the current quark mass m_c and the coupling constant G in the effective Lagrangian. In the particular case of the Gaussian form factor one has the advantage that the integral in Eq. (19) can be performed analytically, allowing to a dramatic reduction of the computer time needed for numerical calculations of the relevant quantities.

As in Refs. [29,30] (see also the discussion on different parameterizations in Ref. [24]), we determine the free parameters by requiring the model to reproduce the empirical values of the pion mass and decay constant, as well as some phenomenologically adequate value of the quark condensate $\langle \bar{f}f \rangle_{\text{reg}}$, at $B = 0$ [the pion mass and decay constant in the limit $B = 0$ can be calculated from Eqs. (38) and (73)]. The parameter sets obtained for Gaussian and 5-Lorentzian form factors, considering different values of the condensate, can be found in Ref. [30]. In that article, the behavior of the chiral quark condensates with the magnetic field has been analyzed, showing that at zero temperature the condensates grow monotonically with B (magnetic catalysis). Moreover, it is seen that these curves turn out to be in good quantitative agreement with the results obtained from LQCD calculations. The agreement is found to be particularly accurate for the parameter sets $m_c = 6.5$ MeV, $\Lambda = 678$ MeV, $G\Lambda^2 = 23.66$ and $m_c = 6.5$ MeV, $\Lambda = 857$ MeV, $G\Lambda^2 = 9.700$, corresponding to $\langle \bar{f}f \rangle_{\text{reg}} = (-230$ MeV$)^3$ for Gaussian and 5-Lorentzian form factors, respectively.

Our results for the behavior of the pion mass $m_{\pi^0}(B)$ and the squared pion decay constant $f_{\pi^0}^2(B)$ for the above mentioned parameter sets are shown in Figs. 1 and 2, respectively. In both cases the curves have been normalized to $B = 0$ values $m_{\pi^0}(0) = 139$ MeV and $f_{\pi^0}^2 = (92.4$ MeV$)^2$. As shown in Fig. 1, the π^0 mass is found to decrease when eB gets increased, reaching a value of about 65% of $m_{\pi^0}(0)$ at $eB \simeq 1.5$ GeV^2, which corresponds to a magnetic field of about 2.5×10^{20} G. We also include in Fig. 1 a gray band that corresponds to recently quoted results from lattice QCD [14]. The latter have been obtained from a continuum extrapolation of lattice spacing, considering a relatively large quark mass for which $m_\pi = 415$ MeV. For comparison, we also quote the results obtained within our model by shifting m_c to 56.3 MeV, which leads to this enhanced pion mass. In general it is seen from the figure that our predictions turn out to be in good agreement with LQCD calculations. It is worth remarking that our results have been obtained directly from model parametrizations used in previous works (where external magnetic fields have not been taken into account) [24], i.e. no extra adjustments have been performed to fit LQCD data. This is in contrast to the situation in the local NJL model, in which comparable results for the pion mass
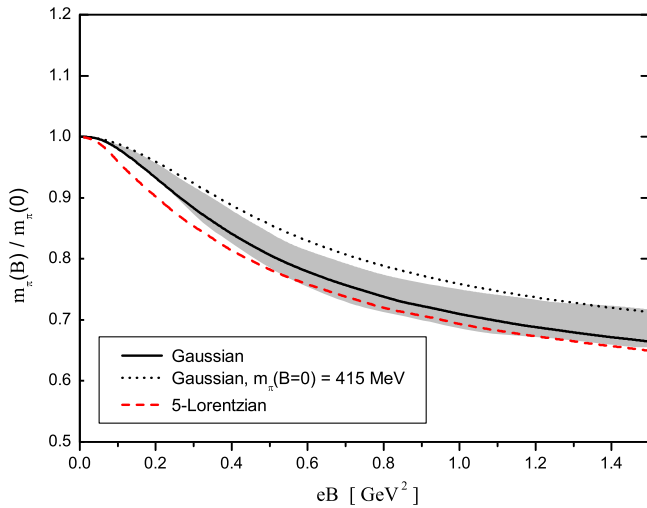
FIG. 1. Mass of the π^0 meson as a function of eB, normalized to its value for $B = 0$. Solid and dashed lines correspond to Gaussian and 5-Lorentzian form factors, respectively. The dotted line is obtained for a parametrization in which $m_\pi = 415$ MeV, while the gray band corresponds to the results of lattice QCD calculations quoted in Ref. [14].
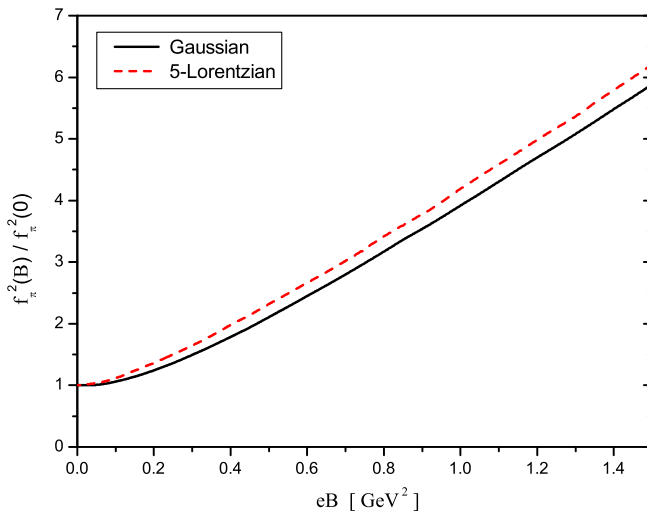

FIG. 2. Normalized squared pion decay coupling $f_{\pi^0}^2$ as a function of the external magnetic field, for Gaussian and 5-Lorentzian form factors.

behavior are obtained after introducing a B-dependent coupling constant adjusted to reproduce LQCD results for the quark condensates [7]. Concerning the pion decay constant f_{π^0}, as shown in Fig. 2 we find that it behaves as an increasing function of B. This is fully consistent with the approximate validity of the Gell-Mann-Oakes-Renner relation for a small value of the constituent mass m_c. In fact, taking into account the behavior of the π^0 mass, from Eq. (84) it is seen that $f_{\pi^0}^2$ should grow somewhat more rapidly than the condensates, which is in agreement with the results in Fig. 2 (the curves showing the behavior of the
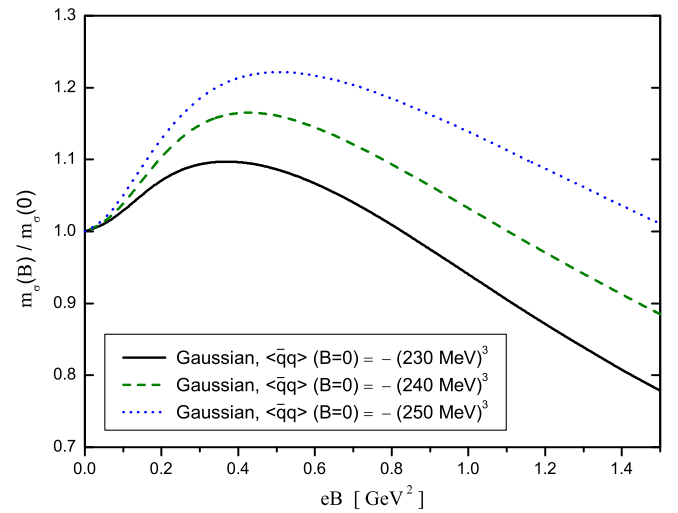

FIG. 3. Mass of the σ meson as a function of eB, normalized to its value for $B = 0$, for three different parametrizations (all of them corresponding to a Gaussian form factor).

condensates can be found in Ref. [30]). For example, at $eB = 1.5$ GeV^2 one gets $m_c \langle \bar{u}u + \bar{d}d \rangle / (m_{\pi^0}^2 f_{\pi^0}^2) \simeq -0.98$, both for Gaussian and 5-Lorentzian form factors. It is also worth mentioning that the curves in Figs. 1 and 2 are found to remain practically unchanged when the value of the $B = 0$ condensate used to fix the parametrization is varied within the range from $-(220$ MeV$)^3$ to $-(250$ MeV$)^3$. On the other hand, if our results are extended to larger values of eB, it is seen that the curves for the π^0 mass keep going down and start deviating from the band obtained from lattice QCD calculations, which is found to be basically flat for $eB \gtrsim 2$ GeV^2. Anyway, the range of validity of nlNJL models should not be extended well beyond a typical energy scale of the order of 1 GeV, above which gluon degrees of freedom are expected to start showing up.

Finally, in Fig. 3 we quote the values of the sigma meson mass as a function of eB, normalized to $m_\sigma(0)$. In the case of the sigma mass the results turn out to be more dependent on the parameter set, therefore we consider here three different parameterizations leading to $\langle \bar{f}f \rangle_{\text{reg}}|_{(B=0)}^{1/3} = -230, -240$ and -250 MeV, for the Gaussian form factor. The corresponding values of m_σ for $B = 0$ are 771, 683 and 616 MeV, respectively. For lower values of the $B = 0$ condensates, as well as for the case of 5-Lorentzian form factors, the determination of the σ mass becomes problematic since it exceeds a threshold of formation of two on-shell quarks, which requires an additional regularization prescription. This problem is usually found in NJL-like theories when one deals with relatively large meson masses. From Fig. 3 we observe that for all the cases considered the σ meson mass shows a nonmonotonic behavior as a function of B. Namely, it gets increased for low B, reaching a maximum at about $eB = 0.4$ GeV^2, after which it shows a steady decrease. It is worth noticing that a qualitative similar behavior is obtained

in Ref. [7] within the local NJL model for a B-dependent coupling constant $G(B)$. On the other hand, as it is also shown in Ref. [7], for a constant G the local NJL model predicts a monotonic increase of m_σ, which indicates that the B dependence of the coupling plays a crucial role in this sense. In our framework, although G is kept constant, such a dependence arises through the nonlocal form factor, which turns out to be a function of the magnetic field, as it is seen from the convolution equation (21).

IV. SUMMARY AND CONCLUSIONS

We have studied the behavior of neutral meson properties in the presence of a uniform static external magnetic field B in the context of a nonlocal chiral quark model. In this approach, which can be viewed as an extension of the local Nambu-Jona-Lasinio model, the effective couplings between quark-antiquark currents include nonlocal form factors that regularize ultraviolet divergences in quark loop integrals and lead to a momentum-dependent effective mass in quark propagators. We have worked out the formalism introducing Ritus transforms of Dirac fields, which allow to obtain closed analytical expressions for meson polarization functions and for the pion decay constant. In addition, we have shown that the Goldberger-Treiman and Gell-Mann-Oakes-Renner chiral relations remain valid within this model in the presence of the external magnetic field. In our numerical calculations we have considered the case of Gaussian and Lorentzian form factors, choosing some sets of model parameters that allow to reproduce the empirical values of the pion mass and decay constants and lead to acceptable values of the quark condensate for $B = 0$. Our results for the neutral pion mass behavior with the magnetic field display a very mild dependence on the parametrization and/or form factor and turn out to be in good quantitative agreement with the available lattice QCD calculations. In the case of the pion decay constant, our results are also quite independent of the chosen parametrization, displaying a rather strong increase of f_{π^0} with eB that implies, for example, $f_{\pi^0}(1 \text{ GeV}^2) \simeq 2 f_{\pi^0}(0)$. On the other hand, our results for the sigma mass behavior with the magnetic field show a stronger dependence on the parametrization. Nonetheless, in all the cases considered it is seen that m_σ shows a nonmonotonic behavior as a function of B. A qualitative similar behavior is obtained within the local NJL model when a B-dependent coupling constant is introduced [7].

We conclude by noting that, given the present results for the neutral pion mass and the fact that nonlocal chiral quark models naturally lead to the inverse magnetic catalysis effect [29,30], an extension of the present work to finite temperature appears to be very interesting. The study of the behavior of the charged pion properties within the present framework, although more involved due to the corresponding Schwinger phase structure, also deserves further attention. We expect to report on these issues in forthcoming articles.

APPENDIX A: CALCULATION OF THE π^0 POLARIZATION FUNCTION

We outline here the derivation of the relation in Eq. (37). It is easy to see that the expression in Eq. (31) can be rearranged in the form

$$
F(0, k_\parallel^2) = -128\pi^2 N_C \sum_{f=u,d} \frac{1}{B_f^2}
$$
$$
\times \sum_{k,k'=0}^{\infty} \int_{q_\parallel} \left[\sum_{\lambda=\pm} F_{kk',q_\parallel^+ q_\parallel^-}^{\lambda,f(AB)} I_{kk',q_\parallel}^{\lambda,f(0)} + F_{kk',q_\parallel^+ q_\parallel^-}^{+,f(CD)} I_{kk',q_\parallel}^{f(1)} \right],
$$
(A1)

where

$$
I_{kk',q_\parallel}^{\lambda,f(0)} = (-1)^{k+k'} \int_{q_\perp p_\perp p_\perp'} \exp[i2\phi(q_\perp, p_\perp, p_\perp')/(q_f B)]
$$
$$
\times \exp[-(p_\perp^2 + p_\perp'^2)/B_f]
$$
$$
\times g(q_\perp^2 + q_\parallel^2) g[(p_\perp' + p_\perp - q_\perp)^2
$$
$$
+ q_\parallel^2] L_{k_\lambda}(2p_\perp^2/B_f) L_{k_\lambda'}(2p_\perp'^2/B_f),
$$
(A2)

$$
I_{kk',q_\parallel}^{f(1)} = 8(-1)^{k+k'} \int_{q_\perp p_\perp p_\perp'} \exp[i2\phi(q_\perp, p_\perp, p_\perp')/(q_f B)]
$$
$$
\times \exp[-(p_\perp^2 + p_\perp'^2)/B_f](p_\perp \cdot p_\perp') g(q_\perp^2 + q_\parallel^2)
$$
$$
\times g[(p_\perp' + p_\perp - q_\perp)^2 + q_\parallel^2]
$$
$$
\times L_{k-1}^1(2p_\perp^2/B_f) L_{k'-1}^1(2p_\perp'^2/B_f).
$$
(A3)

These integrals can be worked out by taking the Laguerre-Fourier transforms of the nonlocal form factors given by Eq. (36). We obtain in this way

$$
I_{kk',q_\parallel}^{\lambda,f(0)} = 4(-1)^{k+k'} \sum_{m,m'=0}^{\infty} (-1)^{m+m'} g_{m,q_\parallel}^{\lambda,f} g_{m',q_\parallel}^{\lambda,f}
$$
$$
\times \int_{q_\perp p_\perp p_\perp'} \exp[i2\phi(q_\perp, p_\perp, p_\perp')/(q_f B)]
$$
$$
\times \exp[-(p_\perp^2 + p_\perp'^2 + q_\perp^2 + (p_\perp' + p_\perp - q_\perp)^2)/B_f]
$$
$$
\times L_{k_\lambda}(2p_\perp^2/B_f) L_{k_\lambda'}(2p_\perp'^2/B_f)
$$
$$
\times L_{m_\lambda}(2q_\perp^2/B_f) L_{m_\lambda'}[2(p_\perp' + p_\perp - q_\perp)^2/B_f],
$$
(A4)

$$I_{kk',q_\parallel}^{f(1)} = 32(-1)^{k+k'} \sum_{m,m'=0}^{\infty} (-1)^{m+m'} g_{m,q_\parallel}^{+,f} g_{m',q_\parallel}^{-,f} \int_{q_\perp p_\perp p'_\perp} \exp[i2\phi(q_\perp, p_\perp, p'_\perp)/(q_f B)]$$
$$\times \exp[-(p_\perp^2 + p_\perp'^2 + q_\perp^2 + (p'_\perp + p_\perp - q_\perp)^2)/B_f](p_\perp \cdot p'_\perp)$$
$$\times L_{k-1}^1(2p_\perp^2/B_f) L_{k'-1}^1(2p_\perp'^2/B_f) L_{m_+}(2q_\perp^2/B_f) L_{m'}[2(p'_\perp + p_\perp - q_\perp)^2/B_f]. \tag{A5}$$

Let us now change the integration variables, defining dimensionless two dimensional vectors $u = -\sqrt{(2/B_f)}p_\perp$, $v = \sqrt{(2/B_f)}p'_\perp$, $w = \sqrt{(2/B_f)}(p_\perp - q_\perp)$. The integrals read

$$I_{kk',q_\parallel}^{\lambda,f(0)} = \frac{B_f^3}{2}(-1)^{k+k'} \sum_{m,m'=0}^{\infty} (-1)^{m+m'} g_{m,q_\parallel}^{\lambda,f} g_{m',q_\parallel}^{\lambda,f} K_{kk'mm'}^{\lambda,f(0)},$$

$$I_{kk',q_\parallel}^{f(1)} = 2B_f^4(-1)^{k+k'} \sum_{m,m'=0}^{\infty} (-1)^{m+m'-1} g_{m,q_\parallel}^{+,f} g_{m',q_\parallel}^{-,f} K_{kk'mm'}^{f(1)}, \tag{A6}$$

where

$$K_{kk'mm'}^{\lambda,f(0)} = \int_{uvw} \exp[-w^2] \exp[-u^2 - u \cdot w - is_f(u_1 w_2 - u_2 w_1)] L_{k_\lambda}(u^2) L_{m_\lambda}[(u+w)^2]$$
$$\times \exp[-v^2 - v \cdot w - is_f(v_1 w_2 - v_2 w_1)] L_{k'_\lambda}(v^2) L_{m'_\lambda}[(v+w)^2],$$

$$K_{kk'mm'}^{f(1)} = -\int_{uvw} \exp[-w^2] \exp[-u^2 - u \cdot w - is_f(u_1 w_2 - u_2 w_1)] L_{k-1}^1(u^2) L_{m_+}[(u+w)^2]$$
$$\times (u \cdot v) \exp[-v^2 - v \cdot w - is_f(v_1 w_2 - v_2 w_1)] L_{k'-1}^1(v^2) L_{m'}[(v+w)^2]. \tag{A7}$$

Notice that $K_{kk'mm'}^{\lambda,f(0)}$ and $K_{kk'mm'}^{f(1)}$ do not depend on momenta, nor on the magnetic field. Their calculation can be performed with the aid of the following useful relations,

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta L_n(x^2 + y^2 + 2xy\cos\theta) \exp[-xy\exp(\pm i\theta)] = L_n(x^2) L_n(y^2), \tag{A8}$$

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta \cos\theta L_n(x^2 + y^2 + 2xy\cos\theta) \exp[-xy\exp(\pm i\theta)] = -\frac{xy}{2}\left[\frac{L_n^1(x^2)L_n^1(y^2)}{n+1} + \frac{L_{n-1}^1(x^2)L_{n-1}^1(y^2)}{n}\right], \tag{A9}$$

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta \sin\theta L_n(x^2 + y^2 + 2xy\cos\theta) \exp[-xy\exp(\pm i\theta)] = \mp\frac{ixy}{2}\left[\frac{L_n^1(x^2)L_n^1(y^2)}{n+1} - \frac{L_{n-1}^1(x^2)L_{n-1}^1(y^2)}{n}\right], \tag{A10}$$

together with the orthogonality properties of the generalized Laguerre polynomials. In the case of $K_{kk'mm'}^{\lambda,f(0)}$, usage of Eq. (A8) leads to

$$K_{kk'mm'}^{\lambda,f(0)} = \frac{1}{(4\pi)^2} \int_w \exp(-w^2) \int_0^{\infty} du^2 \exp(-u^2) L_{k_\lambda}(u^2) L_{m_\lambda}(u^2) L_{m_\lambda}(w^2) \int_0^{\infty} dv^2 \exp(-v^2) L_{k'_\lambda}(v^2) L_{m'_\lambda}(v^2) L_{m'_\lambda}(w^2)$$
$$= \frac{1}{(4\pi)^3} \delta_{km} \delta_{k'm'} \delta_{mm'}, \tag{A11}$$

and consequently

$$I_{kk',q_\parallel}^{\lambda,f(0)} = \frac{B_f^3}{128\pi^3} g_{k,q_\parallel}^{\lambda,f} g_{k,q_\parallel}^{\lambda,f} \delta_{kk'}. \tag{A12}$$

Finally, using Eqs. (A9) and (A10) we obtain

$$K_{kk'mm'}^{f(1)} = -\frac{1}{128\pi^3} k\delta_{kk'}(\delta_{m+1k_-}\delta_{m'k_+} + \delta_{mk_-}\delta_{m'-1k_+}), \tag{A13}$$

which leads to

$$I^{f(1)}_{kk',q_\parallel} = \frac{kB_f^4}{32\pi^3} g^{+,f}_{k,q_\parallel} g^{-,f}_{k,q_\parallel} \delta_{kk'}. \tag{A14}$$

Replacing the results in Eqs. (A12) and (A14) in Eq. (A1) one arrives at our final expression, quoted in Eq. (37).

APPENDIX B CALCULATION OF $t_\mu F^{(\mathrm{II})}_\mu(t)|_{t_\perp = 0}$

Let us discuss here the derivation of the result in Eq. (62). We start from the expression in Eq. (59). Introducing the Laguerre-Fourier transform of $g(q^2)$ and changing the order of integrals and sums one gets

$$
\begin{aligned}
t_\mu F^{(\mathrm{II})}_\mu(t)|_{t_\perp = 0} = {}& i128\pi^2 N_C \sum_{f=u,d} \frac{1}{B_f^2} \sum_{k,k',m=0}^{\infty} \int_{q_\parallel} \left\{ \sum_{\lambda=\pm} g^{\lambda,f}_{m,q_\parallel} [(t_\parallel \cdot q_\parallel^-) \hat{A}^{\lambda,f}_{k,q_\parallel^+} \hat{B}^{\lambda,f}_{k',q_\parallel^-} - (t_\parallel \cdot q_\parallel^+) \hat{A}^{\lambda,f}_{k',q_\parallel^-} \hat{B}^{\lambda,f}_{k,q_\parallel^+}] \tilde{K}^{\lambda,f(0)}_{kk'm} \right. \\
& \left. + 8i g^{+,f}_{m,q_\parallel} [(t_\parallel \cdot q_\parallel^-) \hat{C}^{+,f}_{k,q_\parallel^+} \hat{D}^{+,f}_{k',q_\parallel^-} - (t_\parallel \cdot q_\parallel^+) \hat{C}^{+,f}_{k',q_\parallel^-} \hat{D}^{+,f}_{k,q_\parallel^+}] \tilde{K}^{f(1)}_{kk'm} \right\},
\end{aligned} \tag{B1}
$$

where

$$
\begin{aligned}
\tilde{K}^{\lambda,f(0)}_{kk'm} = {}& (-1)^{k+k'+m_\lambda} \int_{q_\perp p_\perp p'_\perp} \exp[-i2\phi(q_\perp, p_\perp, p'_\perp)/(q_f B)] \\
& \times \exp[-(p_\perp^2 + p_\perp'^2 + q_\perp^2)/B_f] L_{k_\lambda}(2p_\perp^2/B_f) L_{k'_\lambda}(2p_\perp'^2/B_f) L_{m_\lambda}(2q_\perp^2/B_f),
\end{aligned} \tag{B2}
$$

$$
\begin{aligned}
\tilde{K}^{f(1)}_{kk'm} = {}& (-1)^{k+k'+m_+} \int_{q_\perp p_\perp p'_\perp} \exp[-i2\phi(q_\perp, p_\perp, p'_\perp)/(q_f B)](p_1 p_2' - p_2 p_1') \\
& \times \exp[-(p_\perp^2 + p_\perp'^2 + q_\perp^2)/B_f] L^1_{k-1}(2p_\perp^2/B_f) L^1_{k'-1}(2p_\perp'^2/B_f) L_{m_+}(2q_\perp^2/B_f).
\end{aligned} \tag{B3}
$$

Now we change the integration variables, defining dimensionless two dimensional vectors $u = \sqrt{(2/B_f)} q_\perp$, $v = \sqrt{(2/B_f)} p_\perp$, $w = \sqrt{(2/B_f)}(p'_\perp - p_\perp)$. The integrals read

$$
\begin{aligned}
\tilde{K}^{\lambda,f(0)}_{kk'm} = {}& (-1)^{k+k'+m_\lambda} \frac{B_f^3}{8} \int_{vw} \exp[is_f(v_1 w_2 - v_2 w_1)] \exp[-(v^2 + v \cdot w + w^2/2)] \\
& \times L_{k_\lambda}(v^2) L_{k'_\lambda}[(v+w)^2] \int_u \exp(-u^2/2) L_{m_\lambda}(u^2) \exp[is_f(w_1 u_2 - w_2 u_1)],
\end{aligned} \tag{B4}
$$

$$
\begin{aligned}
\tilde{K}^{f(1)}_{kk'm} = {}& (-1)^{k+k'+m_+} \frac{B_f^4}{16} \int_{vw} \exp[is_f(v_1 w_2 - v_2 w_1)] \exp[-(v^2 + v \cdot w + w^2/2)] \\
& \times (v_1 w_2 - v_2 w_1) L^1_{k-1}(v^2) L^1_{k'-1}[(v+w)^2] \\
& \times \int_u \exp(-u^2/2) L_{m_+}(u^2) \exp[is_f(w_1 u_2 - w_2 u_1)].
\end{aligned} \tag{B5}
$$

To evaluate the integrals over u, let us fix the external vector w along the 1 direction. We get

$$
\begin{aligned}
\int_u \exp(-u^2/2) L_{m_\lambda}(u^2) \exp[is_f(w_1 u_2 - w_2 u_1)] &= \frac{1}{(2\pi)^2} \int_0^\infty d|u||u| \exp(-u^2/2) L_{m_\lambda}(u^2) \int_0^{2\pi} d\theta \exp(is_f|wu|\sin\theta) \\
&= \frac{1}{2\pi} \int_0^\infty d|u||u| \exp(-u^2/2) L_{m_\lambda}(u^2) J_0(|wu|) \\
&= \frac{(-1)^{m_\lambda}}{2\pi} \exp(-w^2/2) L_{m_\lambda}(w^2),
\end{aligned} \tag{B6}
$$

where we have used the relations

$$\int_0^{2\pi} d\theta \exp(\pm iy \sin\theta) = 2\pi J_0(y) \tag{B7}$$

and

$$\int_0^\infty dx\, x^{\nu+1} e^{-\beta x^2} L_n^\nu(\alpha x^2) J_\nu(xy) = \frac{(1-\alpha/\beta)^n}{(2\beta)^{\nu+1}} y^\nu e^{-y^2/(4\beta)} L_n^\nu\left[\frac{\alpha y^2}{4\beta(\alpha-\beta)}\right], \tag{B8}$$

$J_\nu(x)$ being Bessel functions of the first kind. Now, taking into account Eq. (A8), together with the orthogonality property of the Laguerre polynomials, we find

$$
\begin{aligned}
\tilde{K}_{kk'm}^{\lambda,f(0)} &= (-1)^{k+k'} \frac{B_f^3}{128\pi^4} \int_0^\infty d|w||w| \exp(-w^2) L_{m_\lambda}(w^2) \int_0^\infty d|v||v| \exp(-v^2) \\
&\quad \times L_{k_\lambda}(v^2) \int_0^{2\pi} d\psi L_{k'_\lambda}(v^2 + w^2 + 2|vw|\cos\psi) \exp[-|vw|\exp(is_f\psi)] \\
&= (-1)^{k+k'} \frac{B_f^3}{64\pi^3} \int_0^\infty d|w||w| \exp(-w^2) L_{m_\lambda}(w^2) L_{k'_\lambda}(w^2) \int_0^\infty d|v||v| \exp(-v^2) L_{k_\lambda}(v^2) L_{k'_\lambda}(v^2) \\
&= \frac{B_f^3}{256\pi^3} \delta_{kk'}\delta_{k'm}.
\end{aligned}
\tag{B9}
$$

For the evaluation of $\tilde{K}_{kk'm}^{f(1)}$ we use the result in Eq. (B6) and then change to new variables $\bar{v} = -v$ and $\bar{w} = w + v$. We have

$$
\begin{aligned}
\tilde{K}_{kk'm}^{f(1)} &= (-1)^{k+k'} \frac{B_f^4}{256\pi^4} \int_0^\infty d|\bar{w}||\bar{w}|^2 \exp(-\bar{w}^2) L_{k'-1}^1(\bar{w}^2) \int_0^\infty d|\bar{v}||\bar{v}|^2 \exp(-\bar{v}^2) \\
&\quad \times L_{k-1}^1(\bar{v}^2) \int_0^{2\pi} d\psi \sin\psi L_{m_+}(\bar{v}^2 + \bar{w}^2 + 2|\bar{v}\,\bar{w}|\cos\psi) \exp[-|\bar{v}\,\bar{w}|\exp(-is_f\psi)] \\
&= (-1)^{k+k'} is_f \frac{B_f^4}{256\pi^3} \left[\frac{1}{m_+ + 1} \int_0^\infty d|\bar{w}||\bar{w}|^3 \exp(-\bar{w}^2) L_{k'-1}^1(\bar{w}^2) L_{m_+}^1(\bar{w}^2) \right. \\
&\quad \left. \times \int_0^\infty d|\bar{v}||\bar{v}|^3 \exp(-\bar{v}^2) L_{k-1}^1(\bar{v}^2) L_{m_+}^1(\bar{v}^2) - (m_+ \leftrightarrow m_+ - 1) \right] \\
&= is_f k \frac{B_f^4}{1024\pi^3} \delta_{kk'}(\delta_{m_+k-1} - \delta_{m_+k}),
\end{aligned}
\tag{B10}
$$

where we have made use of the relation in Eq. (A10). Finally, noting that

$$\sum_{m=0}^\infty s_f(\delta_{m_+k-1} - \delta_{m_+k}) g_{m,q_\parallel}^{+,f} = g_{k,q_\parallel}^{-,f} - g_{k,q_\parallel}^{+,f}, \tag{B11}$$

it is easy to see that Eqs. (B1), (B9) and (B10) lead to our result in Eq. (62).

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