Dimensionally regularized Tsallis' statistical mechanics and two-body Newton's gravitation

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HIGHLIGHTS

- We study the q-partition function of the Newtonian gravitation.
- We study the poles of this q-partition function in phase space.
- We use Dimensional Regularization of Bollini and Giambiagi.
- We find interesting gravitational effects.

ARTICLE INFO

ABSTRACT

Typical Tsallis' statistical mechanics' quantifiers like the partition function and the mean energy exhibit poles. We are speaking of the partition function $\mathcal{Z}$ and the mean energy $\langle \mathcal{U} \rangle$. The poles appear for distinctive values of Tsallis' characteristic real parameter q, at a numerable set of rational numbers of the q-line. These poles are dealt with dimensional regularization resources. The physical effects of these poles on the specific heats are studied here for the two-body classical gravitation potential.

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1. Introduction

Tsallis' information measure S_T generalizes Shannon's one and is considered a very important statistical quantifier [1–13], etc. S_T reads [2]

$$S_T = \frac{1 - \int_{\mathcal{M}} P^q d\mu}{q - 1} \tag{1.1}$$

The dimensional regularization of S_T was discussed in [14] by investigating the poles that emerge in computing the partition function and the mean energy ($\mathcal{Z}$, and $\langle \mathcal{U} \rangle$, respectively), ascertaining the physical significance of these poles for the harmonic oscillator (HO). In this paper we do the same for the classical gravitation potential. To do so we appeal to the dimensional regularization approach of Bollini and Giambiagi [15], [16–18], generalized as explained in [19]. Dimensional regularization constitutes one of the most important theoretical physics' advances of the 20th century's second half. It is used in several of

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its disciplines [20–73]. In particular, we heavily rely on the article [74], a quite useful prerequisite. More details about this topic are given in Appendix of this paper.

The statistical mechanics of systems ruled by gravity is connected to aspects of condensed matter physics, fluid mechanics, re-normalization group, etc. It constitutes a challenge with regards to basic foundations. Associated notions may encounter application in variegated areas of astrophysics and cosmology. Among many statistical works of this kind one could recommend, for instance, [75–78]. Here we address our subject via dimensional regularization.

2. The theory for $q > 1$

Consider the two-body Newton's gravity and its Tsallis' statistical mechanics. For the partition function one deals with

$$\mathcal{Z}_\nu = \int_{\mathcal{M}} \left[1 + (1-q)\beta \left(\frac{p^2}{2m} - \frac{GmM}{r} \right) \right]_+^{\frac{1}{q-1}} d^\nu x d^\nu p \tag{2.1}$$

For effecting the integration process one uses hyper-spherical coordinates and two integrals, each in ν dimensions. Ones is left with just two radial coordinates (one in $r-$ space and the other in $p-$ space) and $2(\nu - 1)$ angles. Since the argument of the brackets must be positive, one has

$$\mathcal{Z}\nu = \left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \right]^2 [\beta(q-1)]^{\frac{1}{q-1}} \int_0^\infty r^{\nu-1} dr \otimes$$

$$\int_0^{\sqrt{2m\left(\frac{1}{\beta(q-1)} + \frac{GmM}{r}\right)}} p^{\nu-1} \left[\frac{1}{\beta(q-1)} + \frac{GmM}{r} - \frac{p^2}{2m} \right]^{\frac{1}{q-1}} dp, \tag{2.2}$$

where by $[\,]_+$ we mean that one considers only that p region where the bracket is positive, which entails that the p integration runs from 0 till $\sqrt{2m\left(\frac{1}{\beta(q-1)} + \frac{GmM}{r}\right)}$. This is called the Tsallis' cut-off. The two integrals above can be evaluated by recourse to Euler's Beta function $\mathcal{B}$ [79]). We give the result of the first one, let us call it I_1.

$$I_1 = \int_0^{\sqrt{2m\left(\frac{1}{\beta(q-1)} + \frac{GmM}{r}\right)}} p^{\nu-1} \left[\frac{1}{\beta(q-1)} + \frac{GmM}{r} - \frac{p^2}{2m} \right]^{\frac{1}{q-1}} dp =$$

$$\left[\frac{1}{\beta(q-1)} + \frac{GmM}{r} \right]^{\frac{\nu}{2} + \frac{1}{q-1}} \mathcal{B}\left(\frac{\nu}{2}, \frac{1}{q-1} + 1 \right). \tag{2.3}$$

Accordingly,

$$\mathcal{Z}_\nu = \frac{2\pi^\nu (2m)^{\frac{\nu}{2}}}{[\Gamma\left(\frac{\nu}{2}\right)]^2} [\beta(q-1)]^{\frac{\nu}{2}} (GmM)^\nu \mathcal{B}\left(\frac{\nu}{2}, \frac{1}{q-1} + 1 \right) \mathcal{B}\left(\frac{\nu}{2} + \frac{1}{1-q}, -\nu \right) \tag{2.4}$$

From (2.4) one gathers that poles appear for any dimension ν, $\nu = 3$ included. Thus, appeal to dimensional regularization (DR) is mandatory. To this effect we will use the DR-generalization given in [19] of Bollini–Giambiagi's original DR-technique.

To proceed further we face now

$$\mathcal{Z}_\nu \langle \mathcal{U} \rangle_\nu = \int_{\mathcal{M}} \left[1 + (1-q)\beta \left(\frac{p^2}{2m} - \frac{GmM}{r} \right) \right]_+^{\frac{1}{q-1}} \left(\frac{p^2}{2m} - \frac{GmM}{r} \right) d^\nu x d^\nu p, \tag{2.5}$$

and

$$\mathcal{Z}_\nu \langle \mathcal{U} \rangle_\nu = \left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \right]^2 [\beta(q-1)]^{\frac{1}{q-1}} \left\{ \int_0^\infty r^{\nu-1} \, dr \otimes \right.$$

$$\int_0^{\sqrt{2m\left(\frac{1}{\beta(q-1)} + \frac{GmM}{r}\right)}} \frac{p^{\nu+1}}{2m} \left[\frac{1}{\beta(q-1)} + \frac{GmM}{r} - \frac{p^2}{2m} \right]^{\frac{1}{q-1}} dp -$$

$$\left. GmM \int_0^\infty r^{\nu-2} dr \int_0^{\sqrt{2m\left(\frac{1}{\beta(q-1)} + \frac{GmM}{r}\right)}} p^{\nu-1} \left[\frac{1}{\beta(q-1)} + \frac{GmM}{r} - \frac{p^2}{2m} \right]^{\frac{1}{q-1}} dp \right\}. \tag{2.6}$$

Beta functions were invented by Euler and we not give an explicit form of them here because they appear in almost all fields of physics. For more details see the Appendix. Via the Beta function one finds

$$\langle \mathcal{U} \rangle_\nu = \frac{2\pi^\nu (2m)^{\frac{\nu}{2}}}{\mathcal{Z}_\nu \left[\Gamma\left(\frac{\nu}{2}\right)\right]^2} [\beta(q-1)]^{\frac{\nu}{2}-1} (GmM)^\nu \left[\mathcal{B}\left(\frac{\nu}{2}+1, \frac{1}{q-1}+1\right) \otimes \right.$$
$$\left. \mathcal{B}\left(\frac{\nu}{2}+\frac{1}{1-q}, 1-\nu\right) - \mathcal{B}\left(\frac{\nu}{2}, \frac{1}{q-1}+1\right) \mathcal{B}\left(\frac{\nu}{2}+\frac{1}{1-q}-1, 1-\nu\right) \right) \tag{2.7}$$

3. The theory for $q < 1$

The treatment becomes more complicated in this instance. From (2.1) we find

$$\mathcal{Z}_\nu = \left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}\right]^2 [\beta(1-q)]^{\frac{1}{q-1}} \left\{ \int_0^{GmM\beta(1-q)} r^{\nu-1} dr \otimes \right.$$

$$\int_{\sqrt{2m\left(\frac{GmM}{r}-\frac{1}{\beta(q-1)}\right)}}^\infty p^{\nu-1} \left[\frac{p^2}{2m} - \frac{GmM}{r} + \frac{1}{\beta(1-q)}\right]_+^{\frac{1}{q-1}} dp +$$

$$\left. \int_{GmM\beta(1-q)}^\infty r^{\nu-1} dr \int_0^\infty p^{\nu-1} \left[\frac{p^2}{2m} - \frac{GmM}{r} + \frac{1}{\beta(1-q)}\right]_+^{\frac{1}{q-1}} dp \right\}. \tag{3.1}$$

We deal with four integrals that are evaluated by appeal to Beta functions.

$$\mathcal{Z}_\nu = \frac{2\pi^\nu (2m)^{\frac{\nu}{2}}}{\left[\Gamma\left(\frac{\nu}{2}\right)\right]^2} [\beta(q-1)]^{\frac{\nu}{2}} (GmM)^\nu \left[\mathcal{B}\left(\frac{\nu}{2}, \frac{1}{1-q}-\frac{\nu}{2}\right) \otimes \right.$$
$$\mathcal{B}\left(-\nu, \frac{\nu}{2}-\frac{1}{1-q}+1\right) + \mathcal{B}\left(\frac{1}{1-q}-\frac{\nu}{2}, \frac{1}{q-1}+1\right) \otimes$$
$$\left. \mathcal{B}\left(\frac{\nu}{2}+\frac{1}{q-1}+1, \frac{\nu}{2}+\frac{1}{1-q}\right). \right] \tag{3.2}$$

Looking for the mean energy we deal, from (2.5), with

$$\mathcal{Z}_\nu \langle \mathcal{U} \rangle_\nu = \left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}\right]^2 [\beta(1-q)]^{\frac{1}{q-1}} \left\{ \int_0^{GmM\beta(1-q)} r^{\nu-1} dr \otimes \right.$$

$$\int_{\sqrt{2m\left(\frac{GmM}{r}-\frac{1}{\beta(q-1)}\right)}}^\infty \frac{p^{\nu+1}}{2m} \left[\frac{p^2}{2m} - \frac{GmM}{r} + \frac{1}{\beta(1-q)}\right]_+^{\frac{1}{q-1}} dp +$$

$$\int_{GmM\beta(1-q)}^\infty r^{\nu-1} dr \int_0^\infty \frac{p^{\nu+1}}{2m} \left[\frac{p^2}{2m} - \frac{GmM}{r} + \frac{1}{\beta(1-q)}\right]_+^{\frac{1}{q-1}} dp -$$

$$GmM \int_0^{GmM\beta(1-q)} r^{\nu-2} dr \int_{\sqrt{2m\left(\frac{GmM}{r}-\frac{1}{\beta(q-1)}\right)}}^\infty p^{\nu-1} \left[\frac{p^2}{2m} - \frac{GmM}{r} + \frac{1}{\beta(1-q)}\right]_+^{\frac{1}{q-1}} dp -$$

$$GmM \int_{GmM\beta(1-q)}^\infty r^{\nu-1} dr \int_0^\infty p^{\nu-1} \left[\frac{p^2}{2m} - \frac{GmM}{r} + \frac{1}{\beta(1-q)}\right]_+^{\frac{1}{q-1}} dp \right\}, \tag{3.3}$$

involving eight integrals. Beta functions are again needed. We have

$$\langle \mathcal{U} \rangle_\nu = \frac{2\pi^\nu (2m)^{\frac{\nu}{2}}}{\mathcal{Z}_\nu \left[\Gamma\left(\frac{\nu}{2}\right)\right]^2} [\beta(q-1)]^{\frac{\nu}{2}-1} (GmM)^\nu \left[\mathcal{B}\left(\frac{1}{1-q}-\frac{\nu}{2}-1, \frac{1}{q-1}+1\right) \otimes \right.$$
$$\mathcal{B}\left(\frac{\nu}{2}+\frac{1}{q-1}+2, \frac{\nu}{2}+\frac{1}{1-q}-1\right) + \mathcal{B}\left(\frac{\nu}{2}+1, \frac{1}{1-q}-\frac{\nu}{2}-1\right) \otimes$$
$$\mathcal{B}\left(-\nu, \frac{\nu}{2}-\frac{1}{1-q}+2\right) - \mathcal{B}\left(\frac{1}{1-q}-\frac{\nu}{2}, \frac{1}{q-1}+1\right) \otimes$$

$$\mathcal{B}\left(\frac{\nu}{2} + \frac{1}{q-1} + 1, \frac{\nu}{2} + \frac{1}{1-q} - 1\right) - \mathcal{B}\left(\frac{\nu}{2}, \frac{1}{1-q} - \frac{\nu}{2}\right) \otimes$$

$$\mathcal{B}\left(1 - \nu, \frac{\nu}{2} - \frac{1}{1-q} + 1\right)\Bigg].$$

(3.4)

Dimensional regularization is needed.

4. The divergences of the theory

From (3.4) we gather that the mean energy cannot be regularized for some q values, those such that

$$1 + \frac{1}{q-1} = -n \quad \text{for} \quad n = 0, 1, 2, 3, \ldots \ldots,$$

(4.1)

DR can be attempted whenever

$$1 + \frac{1}{q-1} \neq -n \quad \text{for} \quad n = 0, 1, 2, 3, \ldots \ldots,$$

(4.2)

or, equivalently, for

$$q \neq \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \ldots, \frac{\nu-2}{\nu-1}, \frac{\nu-1}{\nu}, \frac{\nu}{\nu+1}, \ldots \ldots$$

(4.3)

We emphasize that here we have $q < 1$.

5. The three-dimensional scenario for $q > 1$

Let us deal with the $q = \frac{3}{2}$ instance. We go back to (2.4). The idea it to work out the dimensional regularization task and its corresponding Laurent expansion. We have

$$\mathcal{Z}_\nu = -\frac{8\pi^2(\beta G^2 m^3 M^2)^{\frac{3}{2}}}{3(\nu - 3)} + \frac{4\pi^2}{3}(\beta G^2 m^3 M^2)^{\frac{3}{2}}\left[\frac{23}{3} - \ln\left(16\pi^2\beta G^2 m^3 M^2\right)\right] +$$

$$\sum_{s=1}^{\infty} a_s(\nu - 3)^s.$$

(5.1)

In the present case $\nu - 3$, the independent term in the $\mathcal{Z}$-Laurent expansion yields the physical value of the series. Thus,

$$\mathcal{Z} = \frac{4\pi^2}{3}(\beta G^2 m^3 M^2)^{\frac{3}{2}}\left[\frac{23}{3} - \ln\left(16\pi^2\beta G^2 m^3 M^2\right)\right].$$

(5.2)

Since $\mathcal{Z}$ must be positive, one faces a temperature-lower bound

$$T > \frac{e^{-\frac{23}{3}}}{k_B} 16\pi^2 G^2 m^3 M^2.$$

(5.3)

Similarly, from (2.7), we have for $\langle \mathcal{U} \rangle$

$$\mathcal{Z}\langle \mathcal{U} \rangle_\nu = \frac{32\pi^2(\beta G^2 m^3 M^2)^{\frac{3}{2}}}{3(\nu - 3)} - \frac{32\pi^2}{3}(\beta G^2 m^3 M^2)^{\frac{3}{2}}\left[3 - \ln\left(\pi^2\beta G^2 m^3 M^2\right)\right] +$$

$$\sum_{s=1}^{\infty} a_s(\nu - 3)^s.$$

(5.4)

Accordingly,

$$\mathcal{Z}\langle \mathcal{U} \rangle = \frac{32\pi^2}{\beta}(\beta G^2 m^3 M^2)^{\frac{3}{2}}\left[\ln\left(\pi^2\beta G^2 m^3 M^2\right) - 3 - 2\mathbf{C}\right],$$

(5.5)

and

$$\langle \mathcal{U} \rangle = \frac{8[\ln\left(\pi^2\beta G^2 m^3 M^2\right) - 3 - 2\mathbf{C}]}{\beta[\frac{23}{3} - \ln\left(16\pi^2\beta G^2 m^3 M^2\right)]}.$$

(5.6)

where $\mathbf{C}$ is the Euler's constant [80].

6. The three-dimensional scenario for $q < 1$

Consider $q = \frac{1}{3}$. The concomitant Laurent expansion derived from (3.2) is

$$\mathcal{Z} = \frac{4}{9\pi}(4\pi^2\beta G^2 m^3 M^2)^{\frac{3}{2}}\left[\boldsymbol{C} + 8\ln 2 + \frac{10}{3} - \ln\left(\frac{4\pi^2\beta G^2 m^3 M^2}{3}\right)\right]. \tag{6.1}$$

Positivity of $\mathcal{Z}$ leads us again to a T-lower bound:

$$T > \frac{e^{-(\boldsymbol{C}+8\ln 2+\frac{10}{3})}}{3k_B}4\pi^2 G^2 m^3 M^2. \tag{6.2}$$

For $\langle \mathcal{U} \rangle$ we deduce, from (3.4),

$$\mathcal{Z}\langle \mathcal{U} \rangle = \frac{1}{2\pi}(4\pi^2\beta G^2 m^3 M^2)^{\frac{3}{2}}\left[8 + 3\ln 3 - 3\ln\left(\pi^2\beta G^2 m^3 M^2\right) - \ln 16 - 5\boldsymbol{C}\right], \tag{6.3}$$

and

$$\langle \mathcal{U} \rangle = \frac{9}{8\beta}\frac{8 + 3\ln 3 - 3\ln\left(4\pi^2\beta G^2 m^3 M^2\right) - \ln 16 - 5\boldsymbol{C}}{\boldsymbol{C} + 8\ln 2 + \frac{10}{3} + \ln 3 - \ln\left(4\pi^2\beta G^2 m^3 M^2\right)}. \tag{6.4}$$

7. Specific heats

We deal now with a specific heat constructed via $\mathcal{C} = \frac{\partial \langle \mathcal{U} \rangle}{\partial T}$. Thus for $= \frac{3}{2}$ we obtain

$$\mathcal{C} = \frac{8k[\ln(\pi^2 G^2 m^3 M^2) - 4 - \ln(kT) - 2\boldsymbol{C}]}{\frac{22}{3} + \ln(kT) - \ln(16\pi^2 G^2 m^3 M^2)} - \frac{8k[3\ln(\pi^2 G^2 m^3 M^2) - 3 - \ln(kT) - 2\boldsymbol{C}]}{\left[\frac{22}{3} + \ln(kT) - \ln(16\pi^2 G^2 m^3 M^2)\right]^2} \tag{7.1}$$

For $q = \frac{1}{3}$ one has

$$\mathcal{C} = \frac{9k[11 + 3\ln 3 - \ln 16 + 3\ln(kT) - 3\ln(4\pi^2 G^2 m^3 M^2) - 5\boldsymbol{C}]}{8\left[\boldsymbol{C} + 8\ln 2 + 3\ln 3 + \frac{10}{3} + \ln(kT) - \ln(\pi^2 G^2 m^3 M^2)\right]} - \frac{9k[8 + 3\ln 3 - \ln 16 + 3\ln(kT) - 3\ln(4\pi^2 G^2 m^3 M^2) - 5\boldsymbol{C}]}{8\left[\boldsymbol{C} + 8\ln 2 + 3\ln 3 + \frac{10}{3} + \ln(kT) - \ln(\pi^2 G^2 m^3 M^2)\right]^2} \tag{7.2}$$

Figs. 1–2 depict specific heats corresponding to Eqs. (7.1)–(7.2). We call $E = G^2 m^3 M^2$ with $m \ll M$. We express quantities in $k_B T/E$-units. Specific heats are negative, as befits gravitation. Indeed, such an occurrence has been associated to self-gravitational systems [81]. In turn, Verlinde has associated this type of systems to an entropic force [82]. It is natural to conjecture then that such a force may appear at the energy-associated poles. Notice also that temperature ranges are restricted. There is an T−lower bound.

8. Discussion

In this work we have appealed to an elementary regularization procedure to study the poles in the partition function and the mean energy that appear, for specific, discrete q-values, in Tsallis' statistics of Newton's two-body problem. We studied the thermodynamic behavior at the poles and found interesting peculiarities. The analysis was made in one, two, three, and 3 dimensions. Amongst the pole-traits we emphasize:

- The poles appear, both in the partition function and the mean energy, for $q \neq 1$
- These poles are an artifact of having $q \neq 1$.
- We have proved that there is a lower bound to the temperature at the poles.
- Negative specific heats, characteristic trait of self-gravitating systems, are encountered.

The poles arise only because $q \neq 1$. They are a property of the entropic quantifier, not of the Hamiltonian. Indeed, only for $q \neq 1$ a Gamma function appears in the partition function. It is this Gamma function that displays poles.

Future research should be concerned with cases where it is already known in advance that $q \neq 1$. For these cases, the traits here discovered may acquire some degree of physical "reality".

In this effort we limit ourselves to the two body problem, as the divergences produced by gravitation emerge already at the two-body level. The reader is reminded that N-body gravitation is a frontier research topic of Celestial Mechanics.
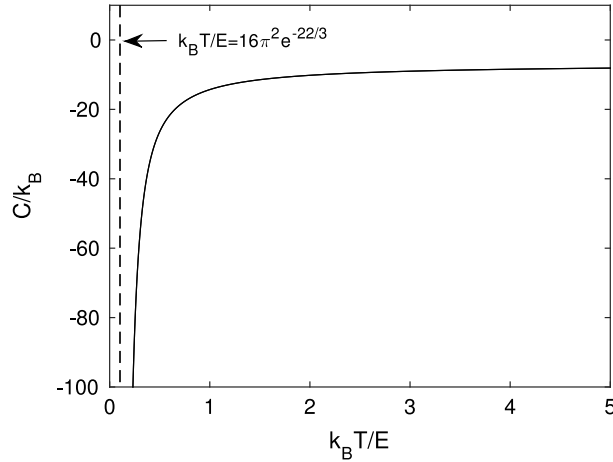
Fig. 1. Specific heat versus $k_B T/E$ for $q = 3/2$. It is well known that gravitational effects make specifics heats to be negative [81]. This is clearly appreciated in this graph and in the following one.
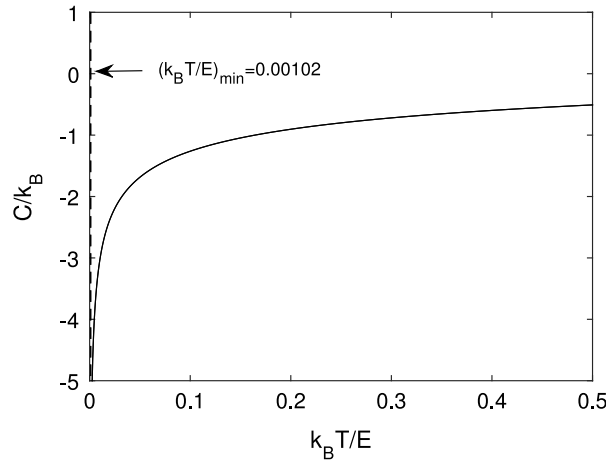
Fig. 2. Specific heat versus $k_B T/E$ for $q = 1/3$. It is well known that gravitational effects make specifics heats to be negative [81]. This is clearly appreciated here.

The $q = 1$ case cannot be analyzed with the present formulation, that is *not valid* for it. The $q = 1$ scenario is discussed in the paper "Gravitational partition function for the Boltzmann–Gibbs classical distribution" by D.J. Zamora, M.C. Rocca, A. Plastino and G.L. Ferri. (see the paper in Researchgate)

The importance of the present communication resides in that fact of having disclosed Tsallis' entropy traits that could not have been suspected before.

Appendix. A simple example of Dimensional Regularization

This Appendix illustrates Dimensional Regularization (DR) with a simple example. The justification of the DR procedure is given in [19], being a generalization of the treatment advanced in [15].

Here we discuss the partition function $\mathcal{Z}$ of the 3D Harmonic Oscillator (HO) for $q = \frac{2}{3}$. We first consider $\mathcal{Z}$ in en ν dimensions

$$\mathcal{Z} = \int [1 + (1 - q)\beta(P^2 + Q^2)]^{\frac{1}{q-1}} d^\nu p \, d^\nu q, \tag{A.1}$$

where $P^2 = p_1^2 + p_2^2 + \cdots p_\nu^2$ and $Q^2 = q_1^2 + q_2^2 + \cdots q_\nu^2$. For the integral we use hyper-spherical coordinates. One finds

$$\mathcal{Z} = \frac{2\pi^\nu}{\Gamma(\nu)} \int_0^\infty S^{2\nu-1} [1 + (1 - q)\beta S^2]^{\frac{1}{q-1}} dS, \tag{A.2}$$

with $S^2 = P^2 + Q^2$. Effecting the change $S^2 = x$ we obtain

$$\mathcal{Z} = \frac{\pi^v}{\Gamma(v)} \int_0^\infty x^{v-1}[1 + (1-q)\beta x]^{\frac{1}{q-1}} dx, \tag{A.3}$$

or

$$\mathcal{Z} = \frac{\pi^v}{\Gamma(v)} \int_0^\infty \frac{x^{v-1}}{[1 + (1-q)\beta x]^{\frac{1}{1-q}}} dx. \tag{A.4}$$

To evaluate this integral we look it up in [80] and find

$$\frac{\pi^v}{\Gamma(v)} \int_0^\infty \frac{x^{\mu-1}}{(1+\gamma x)^v} dx = \gamma^{-\mu} B(\mu, v - \mu), \tag{A.5}$$

where $B(\mu, v - \mu)$ is Euler's Beta function. Comparing (A.4) with (A.5) one encounters $\mu = v$, $v = \frac{1}{1-q}$, $\gamma = (1-q)\beta$, and then

$$\mathcal{Z} = \frac{\pi^v}{\Gamma(v)} [\beta(1-q)]^{-v} B\left(v, \frac{1}{1-q} - v\right). \tag{A.6}$$

We see that for $q = \frac{2}{3}$ and $v = 3$ (A.6) diverges since

$$B\left(v, \frac{1}{1-q} - v\right) = \frac{\Gamma(v)\Gamma\left(\frac{1}{1-q} - v\right)}{\Gamma\left(\frac{1}{1-q}\right)}, \tag{A.7}$$

with $\Gamma(z)$ being Euler's Gamma function, that exhibits poles at $z = 0, -1, -2, -3, \ldots$. From (A.7) it follows that

$$\mathcal{Z} = \frac{\pi^v}{\Gamma(v)} [\beta(1-q)]^{-v} \frac{\Gamma(v)\Gamma\left(\frac{1}{1-q} - v\right)}{\Gamma\left(\frac{1}{1-q}\right)}, \tag{A.8}$$

or

$$\mathcal{Z} = \left[\frac{\pi}{\beta(1-q)}\right]^v \frac{\Gamma\left(\frac{1}{1-q} - v\right)}{\Gamma\left(\frac{1}{1-q}\right)}. \tag{A.9}$$

Setting $q = \frac{2}{3}$ in (A.9) one finds

$$\mathcal{Z} = \left(\frac{3\pi}{\beta}\right)^v \frac{\Gamma(3-v)}{\Gamma(3)}. \tag{A.10}$$

Since $\Gamma(3) = 2$, this leads to

$$\mathcal{Z} = \frac{1}{2}\left(\frac{3\pi}{\beta}\right)^v \Gamma(3-v). \tag{A.11}$$

Note that for $v = 3$, $\mathcal{Z}$ indeed diverges. Bollini–Giambiagi's DR approach consists in performing the Laurent-expansion of $\mathcal{Z}$ around $v = 3$ and select afterwards, as the physical result for $\mathcal{Z}$, the $v - 3$-independent term in the expansion. The justification for such a procedure is clearly explained in [19]. In order to proceed with the Laurent expansion we first define

$$f(v) = \left(\frac{3\pi}{\beta}\right)^v, \tag{A.12}$$

whose Taylor's expansion is

$$f(v) = \left(\frac{3\pi}{\beta}\right) \sum_{n=0}^\infty \ln^n\left(\frac{3\pi}{\beta}\right) \frac{(v-3)^n}{n!}. \tag{A.13}$$

The Gamma function Laurent expansion is

$$\Gamma(3-v) = \frac{1}{3-v} + \mathbf{C} + \sum_{m=1}^\infty c_m(3-v)^m, \tag{A.14}$$

where $\mathbf{C}$ is Euler's constant. Multiplying the two series we have

$$f(v)\Gamma(3-v) = \left(\frac{3\pi}{\beta}\right)^3 \frac{1}{3-v} + \left(\frac{3\pi}{\beta}\right)^3 \mathbf{C} - \left(\frac{3\pi}{\beta}\right)^3 \ln\left(\frac{3\pi}{\beta}\right) + \sum_{m=1}^\infty a_m(3-v)^m. \tag{A.15}$$

Accordingly, $\mathcal{Z}$ becomes

$$\mathcal{Z} = \frac{1}{2}\left(\frac{3\pi}{\beta}\right)^3 \left[\boldsymbol{C} - \ln\left(\frac{3\pi}{\beta}\right)\right],$$ (A.16)

or

$$\mathcal{Z} = \frac{1}{2}(3\pi k_B T)^3 \left[\boldsymbol{C} - \ln(3\pi k_B T)\right].$$ (A.17)

Since one demands $\mathcal{Z} > 0$, T obeys

$$0 < T < \frac{e^{\boldsymbol{C}}}{3\pi k_B},$$ (A.18)

entailing an upper bound for T, typical of Tsallis' formalism (see [83]).

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