Arch. Math. Logic
DOI 10.1007/s00153-008-0067-5

Mathematical Logic

Free Łukasiewicz implication algebras

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Received: 6 December 2006 / Revised: 8 March 2007

Abstract Łukasiewicz implication algebras are the $\{\rightarrow, 1\}$-subreducts of MV-algebras. They are the algebraic counterpart of Super-Łukasiewicz Implicational Logics investigated in Komori (Nogoya Math J 72:127–133, 1978). In this paper we give a description of free Łukasiewicz implication algebras in the context of McNaughton functions. More precisely, we show that the $|X|$-free Łukasiewicz implication algebra is isomorphic to $\bigcup_{x\in X}[x_\theta)$ for a certain congruence θ over the $|X|$-free MV-algebra. As corollary we describe the free algebras in all subvarieties of Łukasiewicz implication algebras.

Keywords Łukasiewicz implication algebras · Free algebras · MV-algebras · Wajsberg algebras · McNaughton functions

Mathematics Subject Classification (2000) 06F99 · 08A30 · 08B20

1 Introduction and preliminaries

Łukasiewicz implication algebras are the algebraic counterpart of the implicational fragment of Super-Łukasiewicz Logic [9,10]. In fact they are the class of all $\{\rightarrow, 1\}$-subreducts of the MV-algebras (MV-algebras are term-wise equivalent to Wajsberg algebras and bounded commutative BCK-algebras [5,8,11]). They are also called C-algebras in [9,10] and Łukasiewicz residuation algebras by Berman and Blok in [2].

The support of Universidad Nacional del Sur and CONICET is gratefully acknowledged.

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A ***Łukasiewicz implication algebra*** is an algebra $\boldsymbol{A} = \langle A, \to, 1\rangle$ of type $\langle 2, 0\rangle$ that satisfies the equations:

(ł1) $1 \to x \approx x$,
(ł2) $(x \to y) \to ((y \to z) \to (x \to z)) \approx 1$,
(ł3) $(x \to y) \to y \approx (y \to x) \to x$,
(ł4) $(x \to y) \to (y \to x) \approx y \to x$.

We will denote by $\mathcal{L}$ the variety of all Łukasiewicz implication algebras. The following properties are satisfied in $\mathcal{L}$:

(ł5) $x \to x \approx 1$,
(ł6) $x \to 1 \approx 1$,
(ł7) if $x \to y \approx y \to x \approx 1$, then $x \approx y$,
(ł8) $x \to (y \to x) \approx 1$,
(ł9) $x \to (y \to z) \approx y \to (x \to z)$.

If $\boldsymbol{A} \in \mathcal{L}$ then the relation $a \leq b$ if and only if $a \to b = 1$ is a partial order on A, called the *natural order of* $\boldsymbol{A}$, with 1 as its greatest element. The join operation $x \vee y$ is given by the term $(x \to y) \to y$ and if $c \in A$, then the polynomial $p(x, y, c) := ((x \to c) \vee (y \to c)) \to c$ is such that $p(a, b, c) = a \wedge b = \inf\{a, b\}$ for $a, b \geq c$. The lattice operation satisfies the following properties:

(ł10) $(x \vee y) \to z \approx (x \to z) \wedge (y \to z)$,
(ł11) $z \to (x \vee y) \approx (z \to x) \vee (z \vee y)$,

and if for $a, b \in A$, $a \wedge b$ exists then for any $c \in A$,

(ł12) $(a \wedge b) \to c \approx (a \to c) \vee (b \to c)$,
(ł13) $c \to (a \wedge b) \approx (c \to a) \wedge (c \to b)$.

For properties and definitions of MV-algebras see [6]. An ***MV*-*algebra*** (term equivalent to Wajsberg algebra [6, Theorem 4.2.5] and [8]) is an algebra $\boldsymbol{A} = \langle A, \oplus, \neg, 0\rangle$, of type $\langle 2, 1, 0\rangle$ that satisfies the equations:

(MV1) $x \oplus (y \oplus z) \approx (x \oplus y) \oplus z$,
(MV2) $x \oplus y \approx y \oplus x$,
(MV3) $x \oplus 0 \approx x$,
(MV4) $\neg\neg x \approx x$,
(MV5) $x \oplus \neg 0 \approx \neg 0$,
(MV6) $\neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x$.

We will denote by $\mathcal{MV}$, the variety of all MV-algebras. For $\boldsymbol{A} \in \mathcal{MV}$ we can define the terms

(MV7) $1 := \neg 0$,
(MV8) $x \to y := \neg x \oplus y$,
(MV9) $x \odot y := \neg(\neg x \oplus \neg y)$.

It is known that if $\boldsymbol{A} \in \mathcal{MV}$, the reduct $\boldsymbol{A}^{\to} = \langle A, \to, 1\rangle$ of $\boldsymbol{A}$ is a Łukasiewicz implication algebra. For basic concepts and properties of universal algebra we refer the reader to [4].

Łukasiewicz implication algebras (and MV-algebras) are congruence 1-regular. For each congruence relation θ on an algebra $\boldsymbol{A} \in \mathcal{L}$ (or $\mathcal{MV}$), $1/\theta$ is an implicative filter, i.e., contains 1 and if $a, a \to b \in 1/\theta$, then $b \in 1/\theta$ (modus ponens); in particular, $1/\theta$ is upwardly-closed in the natural order. Conversely, for any implicative filter F of $\boldsymbol{A}$ the relation

$$\theta_F = \{\langle a, b\rangle \in A^2 : a \to b, b \to a \in F\}$$

is a congruence on $\boldsymbol{A}$ such that $F = 1/\theta_F$. In fact, the correspondence $\theta \mapsto 1/\theta$ gives an order isomorphism from the family of all congruence relations on $\boldsymbol{A}$ onto the family of all implicative filters of $\boldsymbol{A}$, ordered by inclusion. Since any implicative filter F contains 1 and is closed by $\to$, then it is the universe of a subalgebra $\boldsymbol{F}$ of $\boldsymbol{A}$. The bounded distributive lattice of the congruence relations on $\boldsymbol{A}$ is algebraic.

The subdirectly irreducible algebras in $\mathcal{L}$ are linearly ordered relative to the natural order, or $\mathcal{L}$-*chains* (commutative BCK-chains). Finite $\mathcal{L}$-chains are the $\{\to, 1\}$-reducts $\boldsymbol{Ł}_n^{\to}$ of the finite $\mathcal{MV}$-chains $\boldsymbol{Ł}_n$. The algebra $\boldsymbol{Ł}_n^{\to}$ has as universe the set of rationals $Ł_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$, and for each $a, b \in Ł_n$, $a \to b = \min(1, 1 - a + b)$. Another important $\mathcal{L}$-chain is the $\{\to, 1\}$-reduct of the Chang's algebra $\boldsymbol{C}_\omega$ [5, p. 474]:

$$\boldsymbol{C}_\omega^{\to} = \langle\{(0, y) : y \in \mathbb{N}\} \cup \{(1, -y) : y \in \mathbb{N}\}, \to, (1, 0)\rangle,$$

where $\mathbb{N}$ is the set of non-negative integers and

$$(x, y) \to (z, u) = \begin{cases} (1, 0) & \text{if } z > x, \\ (1, \min(0, u - y)) & \text{if } z = x, \\ (1 - x + z, u - y) & \text{otherwise.} \end{cases}$$

The set $Ł_\omega^{\to} = \{(1, -y) : y \in \mathbb{N}\}$ is the unique maximal (proper) implicative filter of $\boldsymbol{C}_\omega^{\to}$ with $\boldsymbol{C}_\omega^{\to}/\theta_{Ł_\omega^{\to}} \cong \boldsymbol{Ł}_1^{\to}$. Its associated subalgebra $\boldsymbol{Ł}_\omega^{\to}$ is not finitely generated, and any infinite subalgebra of $\boldsymbol{Ł}_\omega^{\to}$ is isomorphic to a copy of it. Moreover, every non-trivial finite subalgebra of $\boldsymbol{Ł}_\omega^{\to}$ is isomorphic to $\boldsymbol{Ł}_n^{\to}$, for some $n > 0$. In addition, $\boldsymbol{C}_\omega^{\to}$ and all $\boldsymbol{Ł}_n^{\to}$ are two-generated and every subalgebra of $\boldsymbol{Ł}_\omega^{\to}$ finitely generated is isomorphic to $\boldsymbol{Ł}_n^{\to}$, for some $n > 0$. In particular, $\boldsymbol{Ł}_n^{\to}$ is a subalgebra of $\boldsymbol{Ł}_m^{\to}$ for all $n \leq m$, and every infinite $\mathcal{L}$-chain contains a copy of $\boldsymbol{Ł}_n^{\to}$ for all $n \geq 0$ [10].

The lattice of all subvarieties of $\mathcal{L}$ was described in [10], and it is a $\omega + 1$-chain:

$$V(\boldsymbol{Ł}_0^{\to}) \subsetneq V(\boldsymbol{Ł}_1^{\to}) \subsetneq \dots V(\boldsymbol{Ł}_n^{\to}) \subsetneq \dots V(\boldsymbol{Ł}_\omega^{\to}) = V(\boldsymbol{C}_\omega^{\to}) = \mathcal{L},$$

where $V(\boldsymbol{A})$ denotes the variety generated by an algebra $\boldsymbol{A}$. Observe that $V(\boldsymbol{Ł}_0^{\to})$ is the trivial variety and $V(\boldsymbol{Ł}_1^{\to})$ is the variety of all implication algebras. In order to describe equationally the varieties $V(\boldsymbol{Ł}_n^{\to})$, let us write $x \to^0 y := y$ and for $n \geq 0$, $x \to^{n+1} y := x \to (x \to^n y)$. For any $k \in \omega$, we consider the equation

$$\varepsilon_k : x \to^k y \approx x \to^{k+1} y,$$

then we have:

Theorem 1 *$V(\mathbf{Ł}_k^{\rightarrow})$ is the variety of implication Łukasiewicz algebras satisfying the equation ε_k.*

Let $\mathbf{[0,1]}^{\rightarrow} = \langle [0,1], \rightarrow, 1\rangle$ the $\{\rightarrow, 1\}$-reduct of the MV-algebra $\mathbf{[0,1]} = \langle [0,1], \oplus, \neg, 0, 1\rangle$, where $a \oplus b = min\{1, x+y\}$ and $\neg a = 1-a$, for all $a, b \in [0,1]$. For each k, $\mathbf{Ł}_k^{\rightarrow}$ is a subalgebra of $\mathbf{[0,1]}^{\rightarrow}$, therefore $\mathcal{L} = V(\mathbf{[0,1]}^{\rightarrow})$.

2 Free algebras in $\mathcal{L}$

The goal of this section is to provide a description of the free algebras in $\mathcal{L}$. For this purpose, we would first need to refer briefly to the free algebras in $\mathcal{MV}$ and some of their properties.

A *McNaughton function over the n-cube* [1,6,11,13] is a continuous function $f : [0,1]^n \rightarrow [0,1]$ for which the following holds: there exist finitely many affine linear polynomials $f_1, \ldots, f_k$, each f_i of the form $f_i = a_i^0 x_0 + a_i^1 x_1 + \cdots + a_i^{n-1} x_{n-1} + a_i^n$, with $a_i^0, \ldots, a_i^n$ integers, such that, for each $v \in [0,1]^n$, there exists $i \in \{1, \ldots, k\}$ with $f(v) = f_i(v)$.

If κ is an infinite cardinal, a *McNaughton function over the κ-cube*, is a continuous function $f : [0,1]^\kappa \rightarrow [0,1]$ which depends on finitely many variables $x_{i_1}, \ldots, x_{i_n}$ and such that $f(x_{i_1}, \ldots, x_{i_n})$ is a McNaughton function over the n-cube. It is well known [6] that the free MV-algebra over κ-generators $\boldsymbol{F}_\kappa(\mathcal{MV})$ is the algebra of all McNaughton functions over the κ-cube, and the generators are the projection functions $x_i : [0,1]^\kappa \rightarrow [0,1]$. We can limit ourselves to the case of κ finite. This is not restrictive, since every element of $\boldsymbol{F}_\kappa(\mathcal{MV})$ is generated by finitely many projections, and is therefore essentially an element of $\boldsymbol{F}_n(\mathcal{MV})$, for an appropriate choice of indices. Then in what follows, we will consider $\kappa = n$.

For a set $G \subseteq [0,1]^n$, let

$$F_G = \{f \in \boldsymbol{F}_n(\mathcal{MV}) : f(v) = 1 \text{ for all } v \in G\}.$$

Clearly F_G is an implicative filter of $\boldsymbol{F}_n(\mathcal{MV})$. We denote [1]

$$\boldsymbol{F}_n(\mathcal{MV}) \restriction G = \boldsymbol{F}_n(\mathcal{MV})/F_G,$$

and $|f|_G$ the congruence class of f in the quotient $\boldsymbol{F}_n(\mathcal{MV})/F_G$. Observe that two elements $f_1, f_2 \in \boldsymbol{F}_n(\mathcal{MV})$ have the property $|f_1|_G = |f_2|_G$ if and only if $f_1(G) = f_2(G)$.

A *rational point* of the n-cube is a point $v \in [0,1]^n$ such that $x_i(v)$ is a rational number for every $i \in \{1, \ldots, n\}$. If v is a rational point then there exists a uniquely determined sequence $\{a_i : 0 \leq i \leq n\}$ of positive integers such that:

- $a_0 > 0$,
- $x_i(v) = \frac{a_i}{a_0}$ for every $0 \leq i \leq n$,
- the greatest common divisor of the a_i's is 1.

The numbers a_i's are named *homogeneous coordinates* of v, and a_0 is the *denominator* of v, $den(v)$. Observe that if v is a rational point,

$$\boldsymbol{F}_n(\mathcal{MV}) \upharpoonright \{v\} \cong \boldsymbol{Ł}_{den(v)},$$

and the homomorphism $k_v : \boldsymbol{F}_n(\mathcal{MV}) \to \boldsymbol{Ł}_{den(v)}$, is the extension of the function defined $k_v(x_i) = x_i(v) = \frac{a_i}{a_0}$ on the projections x_i, for $1 \leq i \leq n$, and has kernel $F_{\{v\}}$. Moreover the application $v \to F_{\{v\}}$, between rational points over $[0, 1]^n$ and maximal filters of $\boldsymbol{F}_n(\mathcal{MV})$, such that the quotient is isomorphic to $\boldsymbol{Ł}_s$ for some s, is a bijection.

For a given class $\mathcal{K}$ of $\{\to, 1\}$-algebras, we say that two terms s, t are $\mathcal{K}$-*equivalent* if the equation $s \approx t$ holds in $\mathcal{K}$. Since the class $\mathcal{L}$ is a variety of BCK-algebras, it follows from [3, Fact 0] that any $\{\to, 1\}$-term $s(x_1, \ldots, x_n)$ is $\mathcal{L}$-equivalent to a $\{\to, 1\}$-term

$$s'(x_1, \ldots, x_n) = s_1 \to (s_2 \to (\ldots \to (s_r \to x_i)\ldots)),$$

where $x_i \in \{x_1, \ldots, x_n\}$ and s_i, $1 \leq i \leq r$, are terms in the variables $x_1, \ldots, x_n$ in which 1 does not appear. Thus, for any $\{\to, 1\}$-term s there is a variable x, which appears in s, such that the equation $x \to s \approx 1$ holds in $\mathcal{L}$. Therefore, for every subvariety V of $\mathcal{L}$, every element of $\boldsymbol{F}_n(V)$ is greater than or equal to some generator. Hence:

Lemma 2 *If V is a subvariety of $\mathcal{L}$, then $\boldsymbol{F}_n(V) = \bigcup_{x \in X}[x)$, where X is the set of generators of $\boldsymbol{F}_n(V)$ and $[x) = \{y \in \boldsymbol{F}_n(V) : x \leq y\}$.*

Remark 3 As $\mathcal{L} = V(\mathbf{[0, 1]}^{\to})$, is a fact, from standard Universal Algebra, that $\boldsymbol{F}_n(\mathcal{L})$ is a subalgebra of the $\mathcal{L}$-algebra whose elements are the functions from $(\mathbf{[0, 1]}^{\to})^n$ to $\mathbf{[0, 1]}^{\to}$, under pointwise operations. For $i \in \{1, \ldots, n\}$, the ith free generator of $\boldsymbol{F}_n(\mathcal{L})$ is the ith projection $x_i : (\mathbf{[0, 1]}^{\to})^n \to \mathbf{[0, 1]}^{\to}$. By the previous lemma, if $f \in \boldsymbol{F}_n(\mathcal{L})$ there is $i \in \{1, \ldots, n\}$ such that $x_i \leq f$. As $\mathcal{L}$ is the class of all $\{\to, 1\}$-subreducts of all $\mathcal{MV}$-algebras, $\boldsymbol{F}_n(\mathcal{L})$, is a subreduct of $\boldsymbol{F}_n(\mathcal{MV})$, moreover is isomorphic to the implication subalgebra of $\boldsymbol{F}_n(\mathcal{MV})$ generated by $x_1, \ldots, x_n$. Then we can consider $\boldsymbol{F}_n(\mathcal{L}) \subseteq \boldsymbol{F}_n(\mathcal{MV})$. Moreover $\boldsymbol{F}_n(\mathcal{L}) \subseteq \bigcup_{i=1}^n [x_i)$ with $[x_i) = \{f \in \boldsymbol{F}_n(\mathcal{MV}) : x_i \leq f\}$.

For the n-cube $[0, 1]^n$, we have exactly $2n$, $(n-1)$-faces (faces of dimension $n-1$), they are for $i \in \{1, \ldots, n\}$:

$$C_i^0 = \{(v_1, \ldots, v_n) \in [0, 1]^n : v_i = 0\},$$

and

$$C_i^1 = \{(v_1, \ldots, v_n) \in [0, 1]^n : v_i = 1\},$$

i.e., the faces C_i^0 are the $(n-1)$-faces that contain the origin $\mathbf{0} = (0, \ldots, 0)$, and the faces C_i^1 are the $(n-1)$-faces that contain the vertex $\mathbf{1} = (1, \ldots, 1)$.
Now we are ready to prove the main theorem of the paper.

Theorem 4 *Let $\boldsymbol{F}_n(\mathcal{L})$ be the free Łukasiewicz implication algebra over n generators. For $i \in \{1, \ldots, n\}$, let $O = \bigcup_{i=1}^n C_i^0$ and $|x_i|_O$ be the congruence class of the projections x_i in $\boldsymbol{F}_n(\mathcal{MV}) \restriction O$. Then*

$$\boldsymbol{F}_n(\mathcal{L}) \cong \bigcup_{i=1}^{n} [|x_i|_O),$$

where $[|x_i|_O) = \{|f|_O \in \boldsymbol{F}_n(\mathcal{MV}) \restriction O : |x_i|_O \leq |f|_O\}$.

For the proof of this theorem we reproduce the construction of the elements in the free MV-algebra given in [1, Theorem 3.1] and [12].

(1) For each vertex $v = (v_1, \ldots, v_n)$ of the n-cube, let $h_v \in \boldsymbol{F}_n(\mathcal{MV})$ be defined as follows:

$$h_v = \begin{cases} x_1 \vee \ldots \vee x_n, & \text{if } v = \mathbf{0}, \\ \neg x_1 \vee \ldots \vee \neg x_n, & \text{if } v = \mathbf{1}, \\ \bigvee\{x_i \to x_j : v_i = 1, \text{ and } v_j = 0\} & \text{otherwise.} \end{cases}$$

Let $S_0 = \{h_v : v \text{ a vertex of the } n\text{-cube}\}$.

(2) Given a finite subset S of $\boldsymbol{F}_n(\mathcal{MV})$, a *starring* of S consists in choosing two elements $h \neq k$ in S, and in forming the new set

$$S' = (S \backslash \{h, k\}) \cup \{h \vee k, h \to k, k \to h\}.$$

(3) For every $f \in \boldsymbol{F}_n(\mathcal{MV})$, there exists a finite sequence of starrings $S_0, \ldots, S_l$, leading from S_0 to the set S_l having the property that

$$f = k_1 \odot \ldots \odot k_m,$$

for some $k_1, \ldots, k_m \in S_l$.

For $f \in \boldsymbol{F}_n(\mathcal{MV})$, we say that f is $\{\to\}$-term if there exists a term containing only $\to$ corresponding to f (if 1 appears it can be replaced by $x \to x$), i.e., f is the interpretation of a $\{\to\}$-term $t_f(y_1, \ldots, y_n)$ in $\boldsymbol{F}_n(\mathcal{MV})$, when we replace the variables $y_1, \ldots, y_n$ by the free generators $x_1, \ldots, x_n$.

Proof of the Theorem 4 First observe that

$$|x_1 \wedge \ldots \wedge x_n|_O = |0|_O.$$

This is immediate from the fact that $x_i(C_i^0) = 0$, then $(x_1 \wedge \ldots \wedge x_n)(v) = 0$ for every $v \in O$. Thus for $f \in \boldsymbol{F}_n(\mathcal{MV})$, by **(Ł13)**,

$$|\neg f|_O = |f|_O \to |0|_O = \bigwedge_{i=1}^{n}(|f|_O \to |x_i|_O).$$

Let $f_1, f_2 \in \boldsymbol{F}_n(\mathcal{MV})$ and $g = f_1 \odot f_2$. Then $g = \neg(f_1 \to \neg f_2)$ and therefore,

$$|g|_O = |\neg(f_1 \to \neg f_2)|_O = \bigwedge_{i=1}^{n}\left(\left|f_1 \to \left(\bigwedge_{j=1}^{n}(f_2 \to x_j)\right)\right|_O \to |x_i|_O\right)$$
$$= \bigwedge_{i=1}^{n}\bigvee_{j=1}^{n}(|f_1 \to (f_2 \to x_j)|_O \to |x_i|_O).$$

Thus, as $\vee$ is an $\{\to\}$-term (in the sense that $x \vee y = (x \to y) \to y$), $|f_1 \odot f_2|_O$ is equivalent to an infimum of $\{\to\}$-terms.

For this simple observation we have that

$$|h_v|_O = \begin{cases} |x_1 \vee \ldots \vee x_n|_O, \text{ is a } \{\to\}\text{-term} & \text{if } v = \mathbf{0}, \\ |\neg x_1 \vee \ldots \vee \neg x_n|_O = |1|_O, \text{ is a } \{\to\}\text{-term} & \text{if } v = \mathbf{1}, \\ \left|\bigvee\{x_i \to x_j : v_i = 1, \text{ and } v_j = 0\}\right|_O, \text{ is a } \{\to\}\text{-term} & \text{otherwise.} \end{cases}$$

By the construction of the starrings given in (2), the elements of a starring S of S_0 are in the same congruence class of a $\{\to\}$-term. Since $|f_1 \odot f_2|_O$ is equivalent to an infimum of $\{\to\}$-terms, by item (3), every $f \in \boldsymbol{F}_n(\mathcal{MV})$ is in the same class of an infimum of $\{\to\}$-terms. Then for every $f \in \boldsymbol{F}_n(\mathcal{MV})$

$$|f|_O = \bigwedge_{i=1}^{l}|f_i^{\to}(x_1, \ldots, x_n)|_O,$$

where $f_i^{\to}(x_1, \ldots, x_n)$ are $\{\to\}$-terms.

Suppose that $|x_i|_O \leq |f|_O$, then

$$|f|_O = |x_i|_O \vee |f|_O = \left(\left(\bigwedge_{i=1}^{l}|f_i^{\to}(x_1, \ldots, x_n)|_O\right) \to |x_i|_O\right) \to |x_i|_O$$
$$= \left(\bigvee_{i=1}^{l}(|f_i^{\to}(x_1, \ldots, x_n)|_O \to |x_i|_O)\right) \to |x_i|_O,$$

thus $|f|_O$ is a $\{\to\}$-term and $\bigcup_{i=1}^{n}[|x_i|_O)$ is an implication algebra generated by $|X|_O = \{|x_1|_O, \ldots, |x_n|_O\}$. Then $\bigcup_{i=1}^{n}[|x_i|_O)$ is a homomorphic image of $\boldsymbol{F}_n(\mathcal{L})$.

Let $f_1, f_2 \in \boldsymbol{F}_n(\mathcal{L})$, $f_1 \neq f_2$. Since $\mathcal{L} = V(\{\boldsymbol{Ł}_s\}_{s \geq 1})$, there is $s \geq 1$ and an epimorphism $k : \boldsymbol{F}_n(\mathcal{L}) \to \boldsymbol{Ł}_s$ such that $k(f_1) \neq k(f_2)$. Observe that k is determined by the image of the generators $k(x_i)$ and, as k is onto, there is an $i_0 \in \{1, \ldots, n\}$ such that $\boldsymbol{Ł}_s = [k(x_{i_0}))$. Hence, there exists $i_0 \in \{1, \ldots, n\}$ such that $k(x_{i_0}) = 0$.

On the other hand, if $X = \{x_1, \ldots, x_n\}$, we have that $X \subseteq \boldsymbol{F}_n(\mathcal{L}) \subseteq \boldsymbol{F}_n(\mathcal{MV})$. Then k can be extended to an onto homomorphism $k_v : \boldsymbol{F}_n(\mathcal{MV}) \to \boldsymbol{Ł}_s$ $(s = den(v))$ such that $k_v \upharpoonright \boldsymbol{F}_n(\mathcal{L}) = k$. Hence $x_{i_0}(v) = k_v(x_{i_0}) = k(x_{i_0}) = 0$ and therefore $v \in O$. Since $v \in O$ and $f_1(v) = k_v(f_1) = k(f_1) \neq k(f_2) = k_v(f_2) = f_2(v)$, we have $|f_1|_O \neq |f_2|_O$, and the theorem is proved. □

As example we give a representation for $\boldsymbol{F}_2(\mathcal{L})$. Consider the MV-algebra,

$$\boldsymbol{M}_2 = \boldsymbol{F}_1(\mathcal{MV}) \times \boldsymbol{F}_1(\mathcal{MV}).$$

Let $(f_1, f_2) \in \boldsymbol{M}_2$, we say that the pair (f_1, f_2) is *compatible* if $f_1(0) = f_2(0)$. Let $\boldsymbol{M}_2^c$ be the $\mathcal{MV}$-subalgebra of $\boldsymbol{M}_2$ of compatible pairs. Let $x_1 = (x, 0)$ and $x_2 = (0, x)$ where x is the free generator of $\boldsymbol{F}_1(\mathcal{MV})$. For $i = 1, 2$ let $[x_i) = \{(f_1, f_2) \in \boldsymbol{M}_2^c : x_i \leq (f_1, f_2)\}$. By the previous theorem we have that

$$\boldsymbol{F}_2(\mathcal{L}) \cong [x_1) \cup [x_2).$$

Let $\mathcal{MV}_k$ be the variety of k-potent $\mathcal{MV}$-algebras, i.e., the variety of $\mathcal{MV}$-algebras generated by the algebras $\boldsymbol{Ł}_s$ with $s \leq k$ (the subvariety that satisfies ε_k). In [13], the free algebra $\boldsymbol{F}_n(\mathcal{MV}_k)$ is described. More precisely

$$\boldsymbol{F}_n(\mathcal{MV}_k) \cong \prod\{\boldsymbol{F}_n(\mathcal{MV}) \upharpoonright \{v\} : v \text{ rational point}, den(v) \leq k\}.$$

As immediate consequence of this and Theorem 4 we have:

Corollary 5 *Let $\mathcal{L}_k = V(\boldsymbol{Ł}_k)$ be the k-potent subvariety of $\mathcal{L}$. Then*

$$\boldsymbol{F}_n(\mathcal{L}_k) \cong \bigcup_{i=1}^{n} [|x_i|_O),$$

Where $[x_i)_O = \{|f|_O \in \boldsymbol{F}_n(\mathcal{MV}_k) \upharpoonright O : |x_i|_O \leq |f|_O\}$, with x_i the generators of $\boldsymbol{F}_n(\mathcal{MV}_k)$.

This is a new representation of $\boldsymbol{F}_n(\mathcal{L}_k)$ different of that given in [2].
Examples:

- For all $k \geq 1$,

$$\boldsymbol{F}_1(\mathcal{L}) \cong \boldsymbol{F}_1(\mathcal{L}_k) \cong \boldsymbol{F}_1(\mathcal{MV}) \upharpoonright \{0\} \cong \mathbf{2},$$

  where **2** is the two-element implication algebra.
- $\boldsymbol{F}_n(\mathcal{L}_1) \cong \bigcup_{x \in X} [x)$, with X the set of free generators of $\boldsymbol{F}_n(\mathcal{B})$ and $\mathcal{B}$ the variety of Boolean algebras.

- $\boldsymbol{F}_2(\mathcal{L}_2)$ is described in [7]. If $g_1 = (0, 1, 0, \frac{1}{2}, 0)$, and $g_2 = (0, 0, 1, 0, \frac{1}{2})$ are elements in $\boldsymbol{F}_2(\mathcal{MV}_2) \upharpoonright O \cong \boldsymbol{Ł}_1^3 \times \boldsymbol{Ł}_2^2$, then

$$\boldsymbol{F}_2(\mathcal{L}_2) \cong [g_1) \cup [g_2).$$

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