

TILTING MODULES AND THE SUBCATEGORIES C_i^M

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ABSTRACT

In this paper we further study the full subcategories C_i^M of the category of finitely generated modules over an artin algebra introduced in [PP], consisting of the modules having an $\text{add}M$ resolution of length i , which remains exact under the functor $\text{Hom}_A(M, -)$. In particular, we characterize tilting modules in terms of these categories and determine when the transpose of a tilting module is a tilting module.

Introduction

Let A be an artin algebra and $\text{mod}A$ be the category of finitely generated right A -modules. Let M be an A -module and denote by $\text{add}M$ the full subcategory of $\text{mod}A$ consisting of the direct sums of direct summands of M . In [PP] we considered for an A -module M and for every $n \geq 0$ the full subcategories C_n^M of $\text{mod}A$ consisting of the modules X such that there is an exact sequence $M_n \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 \rightarrow X \rightarrow 0$ with $M_i \in \text{add}M$, and such that the induced sequence $\text{Hom}_A(M, M_n) \rightarrow \cdots \rightarrow \text{Hom}_A(M, M_1) \rightarrow \text{Hom}_A(M, M_0) \rightarrow \text{Hom}_A(M, X) \rightarrow 0$ is exact, generalizing work of M. Auslander in [A] about the subcategory C_1^M . The results in [PP] refer mainly to C_0^M and C_1^M , and the modules M with the property that $C_0^M = C_1^M$ are studied there. Examples of such modules are semisimple modules, tilting modules, $*$ -modules (as defined in [C]) and the transpose of tilting modules.

In this paper we give some applications of the results in [PP]. On one side, we prove that the transpose $\text{Tr}M$ of a tilting module M is a $*$ -module. Using then a result by D'Este and Happel about $*$ -modules it follows that $\text{Tr}M$ is a tilting module over the algebra $\text{End}({}_B M)/P({}_B M, {}_B M)$, where $B = \text{End}(M_A)$ and $P({}_B M, {}_B M)$ is the set of the endomorphisms of ${}_B M$ which factor through a projective module. As a consequence we obtain conditions for the transpose of a tilting module M to be a tilting module. This is the case, for example, when M is a splitting or a separating tilting module with no nonzero projective summands.

Tilting modules M satisfy $C_0^M = C_1^M$. The converse is not true, even if we assume that $DA \in C_0^M$. An example is provided by the module M direct sum of a complete set of representatives of the isomorphism classes of indecomposable modules over an algebra of finite representation type. However, we characterize tilting modules in terms of the categories C_i^M in the following way. Let $B =$

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$\text{End}(M_A)$. The module M is a tilting module if and only if $DA \in C_0^{M_A}$, $C_0^{M_A} = C_1^{M_A}$ and $C_0^{B^M} = C_1^{B^M}$. We study the relation between the validity of some properties defining a tilting module, and the conditions $C_0^{M_A} = C_1^{M_A}$ and $C_0^{B^M} = C_1^{B^M}$.

Finally, we consider generalized tilting modules and prove that generalized tilting modules M of projective dimension n have the property that $DA \in C_{n-1}^{M_A}$, and the equalities $C_{n-1}^{M_A} = C_n^{M_A}$ and $C_{n-1}^{B^M} = C_n^{B^M}$ hold. However, the converse does not hold.

1. PRELIMINARIES

Throughout this paper A denotes an artin algebra, $\text{mod}A$ the category of finitely generated right A -modules and A^{op} the opposite algebra of A . The word module means finitely generated module and we will write M_A or M to indicate that the A -module M is a right module, and ${}_A M$ to indicate that it is a left module. All subcategories considered are full. We will denote by $D: \text{mod}A \rightarrow \text{mod}A^{op}$ the usual duality for artin algebras. Moreover, $\text{pd}M$ denotes the projective dimension and $\text{id}M$ the injective dimension of the module M . We denote by $\text{Tr}M$ the transpose of M and by $\text{Gen}M$ (respectively, $\text{Cogen}M$) the subcategory of $\text{mod}A$ generated (respectively, cogenerated) by M .

According to [HR, 3] we will say that the module M_A is a tilting module if it satisfies the following conditions:

(T1) $\text{pd} M_A \leq 1$.

(T2) $\text{Ext}_A^1(M_A, M_A) = 0$.

(T3) There exists an exact sequence $0 \rightarrow A \rightarrow M' \rightarrow M'' \rightarrow 0$ with M', M'' in $\text{add}M_A$.

It was shown in [BB] and [HR] that if M is a tilting module and $B = \text{End}(M)$ then: 1) ${}_B M$ is a tilting module and $A \simeq \text{End}({}_B M)$ and 2) the functors $\text{Hom}_A(M, -)$ and $- \otimes_B M$ induce mutually inverse equivalences between the full subcategories $\mathcal{T}(M) = \{X : \text{Ext}_A^1(M, X) = 0\}$ and $\mathcal{Y}(M) = \{Y : \text{Tor}_1^B(Y, M) = 0\}$ while the functors $\text{Ext}_A^1(M, -)$ and $\text{Tor}_1^B(-, M)$ induce mutually inverse equivalences between the full subcategories $\mathcal{F}(M) = \{X : \text{Hom}_A(M, X) = 0\}$ and $\mathcal{X}(M) = \{Y : Y \otimes_B M = 0\}$.

A tilting module M is said to be a separating (respectively, splitting) tilting module if the torsion theory $(\mathcal{T}(M), \mathcal{F}(M))$ splits in $\text{mod}A$ (respectively, the torsion theory $(\mathcal{X}(M), \mathcal{Y}(M))$ splits in $\text{mod}B$).

For a general reference for tilting theory we refer the reader to [As], [R] and [HR].

We recall that the module M is a *-module, as defined in [C], when the functor $\text{Hom}_A(M, -)$ induces an equivalence of categories between $\text{Gen}M$ and $\text{Cogen}D_B M$.

2. THE TRANSPOSE OF A TILTING MODULE

In this section we use results of the subcategories C_i^M to prove that the transpose ${}_A(\text{Tr}M_A)$ of a tilting module M_A is a *-module and give also a necessary and sufficient condition for ${}_A(\text{Tr}M_A)$ to be a tilting module.

Furthermore, we apply this result to obtain that the transpose of a splitting or a separating tilting module without nonzero projective summands is a tilting module.

We start by stating a theorem of D'Este and Happel [DH] which motivated this section.

Theorem 2.1. ([DH]) *Let M_A be an A -module. Then: M_A is a $*$ -module if and only if $M_{\bar{A}}$ is a tilting \bar{A} -module, where $\bar{A} = A/\text{ann}M_A$.*

It is well known that if M_A is a tilting module and $B = \text{End}(M_A)$, then ${}_B M$ is also a tilting module and $\psi : A \rightarrow \text{End}({}_B M)$ defined by $\psi(a)(t) = t.a$, $t \in {}_B M$, $a \in A$ is an isomorphism [HR, 2]. Moreover, we need the following result.

Lemma 2.2. *Let M_A be a tilting A -module and $B = \text{End}(M_A)$. Then ψ induces an isomorphism $\text{ann}(\text{Tr}M_A) \simeq P({}_B M, {}_B M)$, where $P({}_B M, {}_B M)$ is the set of the endomorphisms of ${}_B M$ which factor through a projective module.*

Proof. If M_A is a tilting module then $\psi : A \rightarrow \text{End}({}_B M)$ is an isomorphism. By [PP, 4.1] we know that ${}_A(\text{Tr}M_A) \simeq_A (\text{Tr}{}_B M)$. Then $a \in \text{ann}_A(\text{Tr}M_A)$ if and only if $a \in \text{ann}_A(\text{Tr}{}_B M)$. On the other hand, $a.x = \psi(a).x = \text{Tr}\psi(a)(x) = 0$ for $x \in \text{Tr}{}_B M$. Hence $a \in \text{ann}_A(\text{Tr}M_A)$ if and only if $\psi(a) \in P({}_B M, {}_B M)$. \square

We prove next that the transpose of a tilting module is a $*$ -module, and using D'Este and Happel's result stated in Theorem 2.1 we prove the following theorem.

Theorem 2.3. *Let M_A be a tilting A -module and $B = \text{End}(M_A)$. Then:*

- a) ${}_A(\text{Tr}M_A)$ is a $*$ -module.
- b) $\underline{\text{End}}({}_B M)(\text{Tr}M_A)$ is a tilting $\underline{\text{End}}({}_B M)$ -module.
- c) ${}_A(\text{Tr}M_A)$ is a tilting module if and only if $P({}_B M, {}_B M) = 0$.

Proof. a) It is proven in [PP, 3.8] that a module M_A is a $*$ -module if and only if $C_0^{M_A} = C_1^{M_A}$ and the functor $\text{Hom}_A(M, -)$ is exact on $C_0^{M_A}$.

On the other hand, using that $A \simeq \text{End}({}_B M)$ and ${}_B M$ is a tilting module, we get that $C_0^{\text{Tr}M_A} = C_1^{\text{Tr}M_A}$ and the functor $\text{Hom}_A(\text{Tr}M_A, -)$ is exact on $C_0^{\text{Tr}M_A}$, from [PP, 4.11] and [PP, 4.7] respectively. It follows that ${}_A(\text{Tr}M_A)$ is a $*$ -module.

b) By a) we know that ${}_A(\text{Tr}M_A)$ is a $*$ -module. Then from Theorem 2.1 and Lemma 2.2 we obtain that $\bar{A}(\text{Tr}M_A)$ is a tilting \bar{A} -module, where $\bar{A} = A/\text{ann}(\text{Tr}M_A) \simeq \text{End}({}_B M)/P({}_B M, {}_B M)$.

c) Assume that $P({}_B M, {}_B M) \neq 0$. By Lemma 2.2, $\text{ann}_A(\text{Tr}M_A) \neq 0$. Therefore ${}_A(\text{Tr}M_A)$ is not faithful and consequently ${}_A(\text{Tr}M_A)$ is not a tilting module.

The converse follows directly from b). \square

Lemma 2.4. *Let M_A be a tilting module. Then the number of isomorphism classes of indecomposable projective summands of M_A is equal to the number of isomorphism classes of indecomposable projective summands of ${}_B M$.*

Proof. This result follows from the Connecting Lemma [HR, 2]. In fact, if I, P are respectively the injective envelope and the projective cover of the simple module S , then $\text{Tr}D \text{Hom}_A(M_A, I) \simeq \text{Ext}_A^1(M_A, P)$. Moreover, $P \in \text{add}M_A$ if and only if $D \text{Hom}_A(M_A, I)$ is a projective B^{op} -module. On the other hand, we know that $D {}_B M = \text{Hom}_A(M_A, DA)$, therefore $D \text{Hom}(M_A, I) \in \text{add} {}_B M$.

So to each indecomposable projective summand P of M_A we associate the indecomposable projective summand $\theta(P) = \text{D Hom}_A(M_A, I)$ of ${}_B M$. Since $DA \in \text{Gen} M_A$ and the functor $\text{Hom}_A(M_A, -)|_{\text{Gen} M_A}$ is faithful, the correspondence θ is injective. By [HR, 2][As, 2.3] we know that each indecomposable projective module ${}_B P$ is of the form $\text{D Hom}_A(M_A, I_A)$, with I_A injective, so θ is also surjective. \square

Next, we give equivalent conditions for the transpose of a tilting module M to be a tilting module. Let $(M_A)^* = \text{Hom}_A(M_A, A)$.

Proposition 2.5. *Let M_A be a tilting module and $B = \text{End}(M_A)$. Then the following conditions are equivalent:*

- a) ${}_A(\text{Tr} M_A)$ is a tilting module
- b) $(M_A)^* = 0$
- c) $P(M_A, M_A) = 0$
- d) $(\text{Tr}_B M)_B$ is a tilting module
- e) $({}_B M)^* = 0$
- f) $P({}_B M, {}_B M) = 0$

Proof. To prove that a) implies b) we assume ${}_A(\text{Tr} M_A)$ is a tilting module. Then the module M_A does not have projective summands and $\text{id}(\text{D Tr} M_A) \leq 1$. That is to say, $\text{Tr Tr} M_A \simeq M_A$ and $\text{Hom}_A(\text{Tr Tr} M_A, A) = 0$. Then $(M_A)^* = 0$.

Clearly b) implies c), and it follows directly from Theorem 2.3 that c) implies d). The remaining implications follow from the previous ones using that ${}_B M$ is a tilting module. \square

As a consequence we obtain that the transpose of a separating (or splitting) tilting module with no projective summands is a tilting module. This result can also be obtained from the work of Hoshino [Ho,2].

Corollary 2.6. *Let M_A be a splitting (or separating) tilting module without nonzero projective summands and $B = \text{End}(M_A)$. Then ${}_A(\text{Tr} M_A)$ is a tilting A -module.*

Proof. By the above proposition we only need to prove that $({}_B M)^* = 0$. So we assume that this is not the case and consider a nonzero morphism $f: {}_B M \rightarrow B$. Then $Df: DB \rightarrow D_B M$ is also nonzero. Therefore there exist an indecomposable injective module I_B and a nonzero morphism $h: I_B \rightarrow D_B M$, and since $D_B M \in \mathcal{Y}(M)$ we conclude that $I_B \notin \mathcal{X}(M)$.

We assume first that the tilting module M_A is splitting. Then $I_B \in \mathcal{Y}(M)$. By [As, 2.3] we conclude that there exists an injective module I_A in $\text{mod} A$ such that $I_B \simeq \text{Hom}_A(M, I_A)$, and from the Connecting Lemma we obtain $0 = \text{Tr D} I_B \simeq \text{Tr D Hom}(M, I_A) \simeq \text{Ext}_A^1(M, P_0(I_A/rI_A))$. So the module P_0 belongs to $\mathcal{T}(M)$ and is a projective. Thus $P_0 \in \text{add} M_A$, contradicting our hypothesis.

The proof when M_A is separating is similar. \square

3. MODULES M_A SUCH THAT $C_0^{M_A} = C_1^{M_A}$ AND TILTING MODULES

In this section we consider an A -module M_A , $B = \text{End}(M_A)$ and we show that M_A is a tilting module if and only if $DA \in C_0^{M_A}$, $C_0^{M_A} = C_1^{M_A}$ and $C_0^{B M} = C_1^{B M}$.

We start by studying the relation between the validity of some of the properties (T1), (T2) and (T3) defining a tilting module, and the conditions $C_0^{M_A} = C_1^{M_A}$ and $C_0^{B M} = C_1^{B M}$.

It is well known that $C_0^{M_A} = \text{Gen}M_A$ for any module M_A . By [PP, 3.12], when M_A is a tilting module it satisfies the condition $C_0^{M_A} = C_1^{M_A}$. We will prove that the validity of (T2) and (T3) implies that $C_0^{M_A} = C_1^{M_A}$, and exhibit examples showing that no other combination of two of the properties (T1), (T2) and (T3) implies $C_0^{M_A} = C_1^{M_A}$.

We introduce now the following notation. For a module $M = M_A$, let $M^{\perp n} = \bigcap_{i=1}^n \text{Ker}(\text{Ext}_A^i(M, -))$ and $M^\perp = \bigcap_{i>0} \text{Ker}(\text{Ext}_A^i(M, -))$.

It is well known for a tilting module M that $M^{\perp 1} = \text{Gen}M$. In the next lemma we prove that it is enough to assume that M satisfies (T3) for the inclusion $M^{\perp 1} \subseteq \text{Gen}M$ to hold.

Lemma 3.1. *If M is an A -module and satisfies (T3) then $M^{\perp 1} \subseteq \text{Gen}M$.*

Proof. Since M satisfies (T3) then there is an exact sequence $0 \rightarrow A \xrightarrow{f} M' \rightarrow M'' \rightarrow 0$ with M', M'' in $\text{add}M$.

Assume now that $X \in M^{\perp 1}$. Let f_1, \dots, f_n be generators of the $Z(A)$ -module $\text{Hom}_A(M', X)$ and $\varphi = (f_1, \dots, f_n)^t : (M')^n \rightarrow X$, where $Z(A)$ denotes the center of A . We will prove that φ is an epimorphism, so that $X \in \text{Gen}M$.

By applying the functor $\text{Hom}_A(-, X)$ to the above sequence we get the exact sequence:

$$0 \rightarrow \text{Hom}_A(M'', X) \rightarrow \text{Hom}_A(M', X) \xrightarrow{\text{Hom}(f, X)} \text{Hom}_A(A, X) \rightarrow \text{Ext}_A^1(M'', X) = 0,$$

and the commutative diagram

$$\begin{array}{ccc} \text{Hom}_A(A, M'^n) & \xrightarrow{\text{Hom}(A, \varphi)} & \begin{array}{c} 0 \\ \uparrow \\ \text{Hom}_A(A, X) \end{array} \\ \text{Hom}(f, M'^n) \uparrow & & \uparrow \text{Hom}(f, X) \\ \text{Hom}_A(M', M'^n) & \xrightarrow{\text{Hom}(M', \varphi)} & \text{Hom}_A(M', X) \rightarrow 0 \end{array}$$

shows that $\text{Hom}(A, \varphi)$ is an epimorphism. Therefore so is φ , and $X \in \text{Gen}M$. \square

We will prove that if M satisfies the conditions (T2) and (T3) then $C_0^M = C_1^M$. The converse is not true. In fact, any simple module M satisfies $C_0^M = C_1^M$, and (T3) does not hold for M . Moreover, such a module M can be chosen so that neither (T1) nor (T2) hold, as it is the case when M is the simple module over the algebra $K[X]/(X^2)$.

Proposition 3.2. *Let M be an A -module. If M satisfies (T2) and (T3) then $C_0^M = C_1^M$.*

Proof. Let X in C_0^M and consider an exact sequence $0 \rightarrow K \rightarrow M' \rightarrow X \rightarrow 0$ with $M' \in \text{add}M$ such that the induced sequence $0 \rightarrow \text{Hom}_A(M, K) \rightarrow \text{Hom}_A(M, M') \rightarrow \text{Hom}_A(M, X) \rightarrow 0$ is exact. In order to prove that $C_0^M = C_1^M$ we only need to show that $K \in C_0^M$, by [PP, 3.5].

From the long exact sequence associated to $\text{Hom}_A(M, -)$ and $0 \rightarrow K \rightarrow M' \rightarrow X \rightarrow 0$, and using the fact that $\text{Ext}_A^1(M, M') = 0$ because M satisfies (T2), we get that $\text{Ext}_A^1(M, K) = 0$, so $K \in M^{\perp 1}$. On the other hand, we know from

Lemma 3.1 that $M^{\perp 1} \subseteq \text{Gen}M = C_0^M$, because M satisfies (T3). We conclude that $K \in C_0^M$. \square

The next examples show that neither (T1) and (T2), nor (T1) and (T3) imply $C_0^M = C_1^M$.

Example 3.3. Let A be the hereditary K -algebra given by the quiver:

$$\begin{array}{ccccc} \bullet & \xrightarrow{\alpha} & \bullet & \xrightarrow{\beta} & \bullet \\ 1 & & 2 & & 3 \end{array}$$

The Auslander-Reiten quiver Γ_A is:

$$\begin{array}{ccccccc} & & & P_1 = I_3 & & & \\ & & & \nearrow & & \searrow & \\ & & P_2 & & I_2 & & \\ & \nearrow & \searrow & & \nearrow & \searrow & \\ S_3 = P_3 & \cdot & \cdot & S_2 & \cdot & \cdot & S_1 = I_1 \end{array}$$

We consider the projective module $M = P_2 \oplus P_1$. Clearly M satisfies (T1) and (T2). We observe that $S_2 \in C_0^M$ and $S_2 \notin C_1^M$, so $C_0^M \neq C_1^M$.

Example 3.4. The module $M = P_1 \oplus P_2 \oplus I_2$ over the hereditary algebra of Example 3.3 has projective dimension 1, satisfies (T3) and $C_0^M \neq C_1^M$.

Though, as we observed above, there are modules M_A such that $C_0^M = C_1^M$ which do not satisfy any of the three properties of the definition of a tilting module, we will prove that the condition $C_0^M = C_1^M$ guarantees that the module ${}_B M$ satisfies (T2), where $B = \text{End}(M)$. To prove this result we need the following lemma.

Lemma 3.5. Let M be an A -module such that $C_0^M = C_1^M$. Then $\text{Im}(\text{Hom}_A(M, -)) \subseteq \text{Ker}(\text{Tor}_1^B(-, {}_B M))$.

Proof. It is proven in [PP, 3.7] that $\text{Im}(\text{Hom}_A(M, -)) = \text{Ker}(\text{Tor}_1^B(-, {}_B M))$ when $C_0^M = C_1^M$ under the additional hypothesis that the functor $\text{Hom}_A(M, -)$ is exact in C_0^M . It is not difficult to see that without this additional hypothesis the argument used there applies to prove the desired inclusion. \square

Proposition 3.6. Let M be an A -module and $B = \text{End}(M)$. If $C_0^M = C_1^M$ then ${}_B M$ satisfies (T2).

Proof. We consider the following natural isomorphisms:

$$\text{Ext}_B^1({}_B M, {}_B M) \simeq \text{DTor}_1^B(D_B M, {}_B M) \simeq \text{DTor}_1^B(\text{Hom}_A(M, DA), {}_B M).$$

By Lemma 3.5, $\text{Tor}_1^B(\text{Hom}_A(M, DA), {}_B M) = 0$. Then $\text{Ext}_B^1({}_B M, {}_B M) = 0$. \square

Let ${}_B M_A$ be a B - A -bimodule. We denote by $F = \text{Hom}_A(M, -): \text{mod}A \rightarrow \text{mod}B$ and $G = - \otimes_B M: \text{mod}B \rightarrow \text{mod}A$ the pair of adjoint functors determined by M . Let ϵ_X denote the counit and μ_Y the unit of the adjunction, for $X \in \text{mod}A$ and $Y \in \text{mod}B$. We recall that $F(\epsilon_X)\mu_{FX} = id_{FX}$ and $\epsilon_{GY}G(\mu_Y) = id_{GY}$.

In [PP, 2.2] we proved that if $\epsilon_M: GFM \rightarrow M$ is an isomorphism then $C_1^M \subseteq \text{Im}G \subseteq C_0^M$. In the next lemma we show that if $B = \text{End}(M)$ and $\mu_{DM}: D_B M \rightarrow FGD_B M$ is an isomorphism then $DC_1^B M \subseteq \text{Im}F \subseteq DC_0^B M$.

Lemma 3.7. *Let M be an A -module and $B = \text{End}(M)$. Then:*

- i) $\text{Im}F \subset DC_0^{B^M} = \text{Cogen}D_B M$
- ii) If $\mu_{D_B M} : D_B M \rightarrow FGD_B M$ is an isomorphism then $DC_1^{B^M} \subset \text{Im}F$
- iii) If $\mu_{D_B M}$ is an isomorphism, $C_0^M = C_1^M$ and $C_0^{B^M} = C_1^{B^M}$ then M is a $*$ -module

Proof. i) Let Y_B in $\text{Im}F$ and X' be in $\text{mod}A$ such that $FX' = Y_B$. Since any module X can be immersed in an injective module, there is an exact sequence $0 \rightarrow X' \rightarrow (DA)^n$, with $n \in N$. Applying the functor F we get the exact sequence: $0 \rightarrow FX' \rightarrow (FDA)^n$. Since $D_B M = FDA$ we obtain an exact sequence $0 \rightarrow Y_B \rightarrow (DM)^n$, proving that $FX' = Y_B \in \text{Cogen}D_B M$.

ii) Let ${}_B X \in C_1^{B^M}$, and let ${}_B M_1 \rightarrow_B M_0 \rightarrow_B X \rightarrow 0$ be an exact sequence such that $\text{Hom}_B({}_B M, {}_B M_1) \rightarrow \text{Hom}_B({}_B M, {}_B M_0) \rightarrow \text{Hom}_B({}_B M, {}_B X) \rightarrow 0$ is exact.

By applying the duality we get exact sequences $0 \rightarrow D_B X \rightarrow D_B M_0 \rightarrow D_B M_1$ and $0 \rightarrow D \text{Hom}_B({}_B M, {}_B X) \rightarrow D \text{Hom}_B({}_B M, {}_B M_0) \rightarrow D \text{Hom}_B({}_B M, {}_B M_1)$.

Since there is an isomorphism $D \text{Hom}_B({}_B M, {}_B X) \simeq D_B X \otimes_B M = GD_B X$, natural in ${}_B X$, we get that the sequence

$$0 \rightarrow GD_B X \rightarrow GD_B M_0 \rightarrow GD_B M_1$$

is exact. We apply the functor F and obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & FGD_B X & \rightarrow & FGD_B M_0 & \rightarrow & FGD_B M_1 \\ & & \downarrow \mu_{D_B X} & & \downarrow \mu_{D_B M_0} & & \downarrow \mu_{D_B M_1} \\ 0 & \rightarrow & D_B X & \rightarrow & D_B M_0 & \rightarrow & D_B M_1 \end{array}$$

where $\mu_{D_B M_0}$ and $\mu_{D_B M_1}$ are isomorphisms. So $\mu_{D_B X}$ is an isomorphism and therefore $D_B X \in \text{Im}F$.

iii) Since $C_0^M = C_1^M$, the functor F induces an equivalence of categories between $\text{Gen}M$ and $\text{Im}F$, as follows from [PP, 3.1]. By hypothesis, i) and ii) we know that $DC_1^{B^M} = \text{Im}F = DC_0^{B^M} = \text{Cogen}D_B M$. Then $\text{Im}F = \text{Cogen}D_B M$, so M is a $*$ -module \square

Any tilting module M satisfies that $C_0^M = C_1^M$ and it is well known that $DA \in \text{Gen}M = C_0^M$ [HR, 2]. The converse is not true. We observe then that if M is a tilting module and $B = \text{End}(M)$, then ${}_B M$ is also a tilting module, so that also $C_0^{B^M} = C_1^{B^M}$. Now we prove the main result of this section.

Theorem 3.8. *Let M be an A -module and $B = \text{End}(M)$. Then the following conditions are equivalent:*

- a) M is a tilting module.
- b) $C_0^M = C_1^M$, $C_0^{B^M} = C_1^{B^M}$ and $DA \in C_0^M$.

Proof. We just observed that a) implies b). So we prove that b) implies a). We use the following characterization of tilting modules, given in [PP, 3.12]: the module M is a tilting module if and only if $C_0^M = C_1^M$, the functor F is exact in C_0^M and

$DA \in C_0^M$. We also recall from [PP, 3.8] that the first two properties characterize *-modules.

Thus to prove that b) implies a) we only need to show that a module M satisfying b) is a *-module. This amounts to prove that $\mu_{D_B M}$ is an isomorphism, by Lemma 3.7, since we are assuming that $C_0^M = C_1^M$ and $C_0^{B^M} = C_1^{B^M}$. With this purpose we observe first that ϵ_{DA} is an isomorphism, because $DA \in C_1^M$ [PP, 2.2]. From the equality $1_{FDA} = F(\epsilon_{DA}) \cdot \mu_{FDA}$ we get that μ_{FDA} is also an isomorphism. This proves that $\mu_{D_B M}$ is an isomorphism, since $FDA \simeq D_B M$, ending the proof of the theorem. \square

4. GENERALIZED TILTING MODULES AND THE SUBCATEGORY C_n^M

In this section we study which of the results proven in section 3 for tilting modules can be extended to generalized tilting modules, as defined in [M] and [H, 3]. In particular, we will prove that generalized tilting modules M_A of projective dimension n satisfy that $C_{n-1}^{M_A} = C_n^{M_A}$, $C_{n-1}^{B^M} = C_n^{B^M}$ and $DA \in C_n^{M_A}$, for $B = \text{End}(M_A)$. However, the converse does not hold, so the characterization of tilting modules given in Theorem 3.8 can not be extended to generalized tilting modules.

We recall that a module $M \in \text{mod}A$ is a generalized tilting module if it satisfies the following conditions:

$$(TG1) \text{ pd}M \leq n$$

$$(TG2) \text{ Ext}_A^i(M, M) = 0 \text{ for all } i \geq 1$$

(TG3) There exists an exact sequence $0 \rightarrow A \rightarrow M_0 \rightarrow M_1 \rightarrow \dots \rightarrow M_n \rightarrow 0$ with $M_i \in \text{add}M$.

By [M, 1.16] and [H, 3] we know that if M is a generalized tilting A -module and $B = \text{End}(M_A)$ then ${}_B M$ is a generalized tilting B^{op} -module. We prove some relations between the validity of some of the properties defining a generalized tilting module, the subcategories C_i^M and $M^{\perp i}$. We denote $F = \text{Hom}_A(M, -)$.

Proposition 4.1. *Let M be an A -module and $n, s \in \mathbb{N}$. Then:*

a) *If M satisfies (TG2), $X \in C_{s-1}^M$ and $0 \rightarrow K_{s-1} \rightarrow M_{s-1} \rightarrow \dots \rightarrow M_0 \rightarrow X \rightarrow 0$ is an exact sequence with $M_i \in \text{add}M$ and such that the induced sequence $0 \rightarrow FK_{s-1} \rightarrow FM_{s-1} \rightarrow \dots \rightarrow FM_0 \rightarrow FX \rightarrow 0$ is exact then $K_{s-1} \in M^{\perp s} = \bigcap_{i=1}^s \text{KerExt}_A^i(M, -)$.*

b) *If M satisfies (TG3) for n then $M^{\perp n} \subset C_0^M$.*

c) *If M satisfies (TG2) and (TG3) for n then $C_{n-1}^M = C_n^M$.*

d) *If M satisfies (TG2) and (TG3) for n then $M^{\perp} \subset \bigcap_{i \geq 0} C_i^M$.*

e) *If M satisfies (TG1) and (TG2), then $C_{n-1}^M \subset M^{\perp}$, where $\text{pd}M = n$.*

f) *If M is a generalized tilting A -module, and $\text{pd}M = n$ then $M^{\perp} = C_{n-1}^M = C_n^M$.*

Proof. We start by proving b). Assume M satisfies (TG3) for n . Then there exists an exact sequence $0 \rightarrow A \rightarrow M_0 \rightarrow M_1 \rightarrow \dots \rightarrow M_n \rightarrow 0$ with $M_i \in \text{add}M$.

We denote $K_j = \text{Ker}(M_j \rightarrow M_{j+1})$ for $1 \leq j \leq n-1$. We get the exact sequences:

$$\begin{aligned} 0 \rightarrow A \rightarrow M_0 \rightarrow K_1 \rightarrow 0, \\ 0 \rightarrow K_j \rightarrow M_j \rightarrow K_{j+1} \rightarrow 0, \text{ for } 1 \leq j \leq n-2 \\ 0 \rightarrow K_{n-1} \rightarrow M_{n-1} \rightarrow M_n \rightarrow 0. \end{aligned}$$

Let $X \in M^{\perp n}$. By applying the functor $\text{Hom}_A(-, X)$ to these sequences we obtain the exact sequence:

$0 \rightarrow \text{Hom}_A(K_1, X) \rightarrow \text{Hom}_A(M_0, X) \rightarrow \text{Hom}_A(A, X) \rightarrow \text{Ext}^1(K_1, X) \rightarrow 0$
and the isomorphisms $\text{Ext}_A^i(K_j, X) \simeq \text{Ext}_A^{i+1}(K_{j+1}, X)$ for $1 \leq j \leq n-2$ and $\text{Ext}_A^i(K_{n-1}, X) = 0$ for $1 \leq i \leq n-1$.

In particular $\text{Ext}_A^{n-1}(K_{n-1}, X) = 0$. Then $\text{Ext}_A^1(K_1, X) \simeq \text{Ext}_A^2(K_2, X) \simeq \dots \simeq \text{Ext}_A^{n-1}(K_{n-1}, X) = 0$. Therefore $0 \rightarrow \text{Hom}_A(K_1, X) \rightarrow \text{Hom}_A(M_0, X) \rightarrow \text{Hom}_A(A, X) \rightarrow 0$ is exact. We deduce as in the proof of Lemma 3.1 that there exists an epimorphism $M' \rightarrow X \rightarrow 0$. Then $X \in \text{Gen}M = C_0^M$, proving b)

In order to prove the remaining items we consider M satisfying (TG2), $X \in C_{s-1}^M$ and an exact sequence $0 \rightarrow K_{s-1} \rightarrow M_{s-1} \rightarrow \dots \rightarrow M_0 \rightarrow X \rightarrow 0$, with $M_i \in \text{add}M$ and such that the induced sequence $0 \rightarrow FK_{s-1} \rightarrow FM_{s-1} \rightarrow \dots \rightarrow FM_0 \rightarrow FX \rightarrow 0$ is exact. Then

- i) $\text{Ext}_A^{j+s}(M, K_{s-1}) \simeq \text{Ext}_A^j(M, X)$ for all $j \geq 1$ and
- ii) $\text{Ext}_A^j(M, K_{s-1}) = 0$, for $1 \leq j \leq s$.

In fact, let $0 \rightarrow K_0 \rightarrow M_0 \rightarrow X \rightarrow 0$ be exact. Then $0 \rightarrow FK_0 \rightarrow FM_0 \rightarrow FX \rightarrow 0$ is also exact, and $K_0 \in C_{s-2}^M$. The long exact sequence associated to $0 \rightarrow K_0 \rightarrow M_0 \rightarrow X \rightarrow 0$ yields $\text{Ext}_A^1(M, K_0) = 0$ and $\text{Ext}_A^{j+1}(M, K_0) \simeq \text{Ext}_A^j(M, X)$ for all $j \geq 1$, and i) and ii) follow then by induction on s .

Now, the equalities in ii) mean precisely that that $K_{s-1} \in M^{\perp s}$, proving a). If we also assume that (TG3) holds for n , then from a) and b) we get that $K_{n-1} \in M^{\perp n} \subseteq C_n^M$. Thus $M \in C_n^M$, proving c).

To prove d) we assume, moreover, that $X \in M^{\perp}$. Then from i) we obtain that $\text{Ext}_A^{j+s}(M, K_{s-1}) = 0$ for all $j \geq 1$. Using that $K_{s-1} \in M^{\perp s}$ it follows that $K_{s-1} \in M^{\perp}$. On the other hand, since $M^{\perp} \subseteq C_0^M$, by b), we get that $K_{s-1} \in C_0^M$ and therefore $X \in C_s^M$, ending the proof of d).

e) Let $X \in C_{n-1}$, and assume that the above considered module M satisfying (TG2) satisfies also (TG1) and has projective dimension n . Then $\text{Ext}_A^{j+n}(M, K_{n-1}) = 0$ for all $j \geq 1$ and from i) we get that X is in M^{\perp} , as desired.

Finally, we observe that f) follows directly from c), d) and e). \square

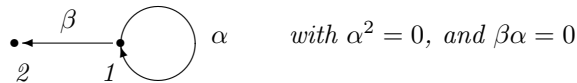
Theorem 4.2. *Let M be a generalized tilting A -module, with $\text{pd}M = n$ and $B = \text{End}(M_A)$. Then $C_{n-1}^{M_A} = C_n^{M_A}$, $C_{n-1}^{B^M} = C_n^{B^M}$ and $DA \in C_n^{M_A}$*

Proof. From Proposition 4.1 f) and the fact that both M_A and ${}_B M$ are generalized tilting modules we have that $C_{n-1}^{M_A} = C_n^{M_A}$ and $C_{n-1}^{B^M} = C_n^{B^M}$.

Since DA is an injective module we know that $\text{Ext}_A^i(M, DA) = 0$, for all $i \geq 1$. Then $DA \in M_A^{\perp}$. By f), $C_n^{M_A} = M_A^{\perp}$, so $DA \in C_n^{M_A}$. \square

The converse of the theorem is not true as we illustrate in the following example.

Example 4.3. *Let A be the K -algebra given by the quiver*



The Auslander-Reiten quiver Γ_A is

$$\begin{array}{ccccc}
 S_1 & & & I_2 & \\
 & \searrow & & \nearrow & \searrow \\
 & & P_1 & & S_1 \\
 & \nearrow & \searrow & \nearrow & \\
 S_2 = P_2 & & & I_1 &
 \end{array}$$

Let $M = I_1 \oplus I_2$. Then $\text{pd}M = \infty$, $DA \in C_1^M = C_2^M$ and $C_1^{BM} = C_2^{BM}$.

In the previous example we saw that the conditions $C_{n-1}^{MA} = C_n^{MA}$, $C_{n-1}^{BM} = C_n^{BM}$ and $DA \in C_n^{MA}$ do not guarantee that M is a generalized tilting module. We will prove that when $DA \in C_n^{MA}$ and $C_{n-1}^{MA} = C_n^{MA}$ the module ${}_B M$ satisfies (TG2).

We start with the following proposition.

Proposition 4.4. *Let M be an A -module and $n \geq 2$. If $X \in C_n^{MA}$ then $\text{Tor}_i^B(FX, M) = 0$, for $i = 1, \dots, n-1$.*

Proof. This result is proven in [PP, 3.7] under the additional hypothesis that $C_0^M = C_1^M$, and the proof there can be easily adapted to the present situation, as we show next.

If $X \in C_n^{MA}$ then there exists an exact sequence $M_n \rightarrow \dots \rightarrow M_0 \rightarrow X \rightarrow 0$ with $M_i \in \text{add}M$, and such that the induced sequence $FM_n \rightarrow \dots \rightarrow FM_0 \rightarrow FX \rightarrow 0$ is exact. Denote $K_j = \text{Ker}(M_j \rightarrow M_{j-1})$ and $K_0 = \text{Ker}(M_0 \rightarrow X)$. We get the exact sequences

- (1) $0 \rightarrow FK_0 \rightarrow FM_0 \rightarrow FX \rightarrow 0$
- (2) $0 \rightarrow FK_j \rightarrow FM_j \rightarrow FK_{j-1} \rightarrow 0$, for $1 \leq j \leq n$

We apply the functor $G = - \otimes_B M$ to the sequence (1) and consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 \text{Tor}_1^B(FM_0, M) & \rightarrow & \text{Tor}_1^B(FX, M) & \rightarrow & GFK_0 & \rightarrow & GFM_0 & \rightarrow & GFX & \rightarrow & 0 \\
 & & & & \downarrow \varepsilon_{K_0} & & \downarrow \varepsilon_{M_0} & & \downarrow \varepsilon_X & & \\
 & & 0 & \rightarrow & K_0 & \rightarrow & M_0 & \rightarrow & X & \rightarrow & 0
 \end{array}$$

where ε_{K_0} , ε_{M_0} and ε_X are isomorphisms by [PP, 2.2] ($K_0 \in C_1^M$ because $n \geq 2$). Since FM_0 is B -projective, then $\text{Tor}_i^B(FM_0, M) = 0$. We obtain $\text{Tor}_1^B(FX, M) = 0$ and $\text{Tor}_i^B(FK_0, M) \simeq \text{Tor}_{i+1}^B(FX, M) = 0$ for $i \geq 1$. The result follows then by induction, using that $K_0 \in C_{n-1}^{MA}$. \square

The previous result holds for all $i \geq 1$ under the additional hypothesis that $C_{n-1}^{MA} = C_n^{MA}$, $n \geq 1$, as we state next.

Corollary 4.5. *Let M be an A -module and $n \geq 1$. If $X \in C_n^{MA}$ and $C_{n-1}^{MA} = C_n^{MA}$ then $\text{Tor}_i^B(FX, {}_B M) = 0$ for all $i \geq 1$.*

Proof. If $C_{n-1}^{MA} = C_n^{MA}$, then $C_{n-1}^{MA} = C_m^{MA}$ for all $m \geq n \geq 1$. The result follows then from Proposition 4.4. \square

Corollary 4.6. *Let M be an A -module and $n \geq 1$. If $DA \in C_n^{MA}$ and $C_{n-1}^{MA} = C_n^{MA}$ then $\text{Ext}_B^i({}_B M, {}_B M) = 0$ for all $i \geq 1$ (${}_B M$ satisfies (TG2)).*

Proof. Since $DA \in C_n^{MA}$ and $C_{n-1}^{MA} = C_n^{MA}$, we know by the above corollary that $\text{Tor}_i^B(FDA, M) = 0$ for all $i \geq 1$. Then $\text{Tor}_i^B(DM, {}_B M) = 0$ because $FDA = DM$. The corollary follows from the isomorphism $\text{Ext}_B^i(M, M) \simeq \text{Tor}_i^B(DM, M)$. \square

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REFERENCES

- [As] I. Assem. *Tilting theory. An introduction.* Topics in Algebra, Banach Centre Publications, vol. 26, PWN, Warsaw, 127-180, (1990).
- [A] M. Auslander *Notes on representation theory.* Brandeis Univ., (1973).
- [ARS] M. Auslander, I. Reiten, and S. O. Smalø. *Representation theory of artin algebras*, vol. 36 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, (1995).
- [BB] S. Brenner and M. C. R. Butler. *Generalization of the Bernstein-Gelfand-Ponomarev reflection functors.* Lecture Notes in Math., 832, Springer-Verlag, Berlin-Heidelberg, New York, 103-169, (1980).
- [C] R. Colpi. *Some remarks on equivalences between categories of modules.* Comm. Algebra 18(6), 1935-1951, (1990).
- [DH] G. D'Este and D. Happel. *Representable equivalences are represented by tilting modules.* Rend. Sem. Mat. Univ. Padova, Vol. 83. (1990).
- [H] D. Happel. *Triangulated categories in the representation theory of finite dimensional algebras.* London Math. Soc. Lecture Note Series. 119, (1988).
- [HR] D. Happel and C.M. Ringel. *Tilted algebras.* Trans. Amer. Math. Soc., 284, 399-443, (1982).
- [Ho] M. Hoshino. *Splitting torsion theories induced by tilting modules.* Comm. Algebra 11, N° 4, 427-441, (1983).
- [M] Y. Miyashita *Tilting modules of finite projective dimension.* Math. Z., 193, 113-146, (1986).
- [PP] M. I. Platzeck and N. I. Pratti. *On a theorem of Auslander and applications.* Comm. in Algebra, 28(6), 2817-2835, (2000)
- [R] C. M. Ringel. *Tame algebras and integral quadratic forms*, Lecture Notes in Math., 1099, Springer-Verlag, Berlin-Heidelberg, (1984).