Algebra univers. **52** (2004) 39–48
0002-5240/04/010039 – 10
DOI 10.1007/s00012-004-1872-2

Algebra Universalis

Topological representation for implication algebras

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Abstract. In this paper we give a description of an implication algebra A as a union of a unique family of filters of a suitable Boolean algebra $\mathbf{Bo}(A)$, called the *Boolean closure* of A. From this representation we obtain a notion of topological implication space and we give a dual equivalence based in the Stone representation for Boolean algebras. As an application we provide the implication space of all free implication algebras.

1. Introduction and preliminaries

The aim of this paper is to give a topological representation for implication algebras, also known as Tarski algebras (see [1, 4, 5]). These algebras have been developed by J. C. Abbott in [2, 3]. They are the $\{\to\}$-subreducts of Boolean algebras (see [2, Theorem 17]), and they are also the algebraic counterpart of the implicational fragment of classical propositional logic [7].

An *implication algebra* is an algebra $\langle A, \to\rangle$ of type $\langle 2\rangle$ that satisfies the equations:

(T1) $(x \to y) \to x = x$,
(T2) $(x \to y) \to y = (y \to x) \to x$,
(T3) $x \to (y \to z) = y \to (x \to z)$.

In any implication algebra A the term $x \to x$ is constant, which we represent by 1. The relation $x \leq y$ if and only if $x \to y = 1$ is a partial order, called *the natural order of A*, with 1 as its greatest element. Relative to this partial order A is a join-semilattice and the join of two elements a and b is given by $a \vee b = (a \to b) \to b$. Besides, for each a in A, $[a) = \{x \in A : a \leq x\}$ is a Boolean algebra in which, for $b, c \geq a$, $b \wedge c = (b \to (c \to a)) \to a$ gives the meet and $b \to a$ is the complement of b. In fact, following [2, Theorems 6 and 7] any implication algebra is a join-semilattice

Presented by B. Jónsson.
Received March 5, 2004; accepted in final form May 3, 2004.
2000 *Mathematics Subject Classification*: 06E15, 03G25, 06F99.
Key words and phrases: Implication algebras, Boolean algebras, implication spaces, Boolean spaces, dual categorical equivalence.
The first two authors were partially supported by the Universidad Nacional del Sur, Bahía Blanca, Argentina. The third author was partially supported by Grants 2001SGR-0017 of D.G.R. of Generalitat de Catalunya and BFM2001-3329 of Ministerio de Ciencia y Tecnología of Spain.

with greatest element, such that for each element a, $[a)$ with the inherited order is a Boolean algebra.

We assume familiarity with the theory of Boolean algebras and implication algebras. The lattice filters of a Boolean algebra B are called simply *filters*. If $X \subseteq B$, we denote by $F(X)$ the filter generated by X, and by $B(X)$ the subalgebra generated by X. $F(X) = \{b \in B : b \geq \bigwedge_{i=1}^{n} x_i,\ x_i \in X,\ n \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of natural numbers, and in particular, if X is increasing, $F(X) = \{b \in B : b = \bigwedge_{i=1}^{n} x_i,\ x_i \in X,\ n \in \mathbb{N}\}$. We recall that maximal proper filters are called *ultrafilters*.

A *generalized Boolean algebra* G is a relatively complemented distributive lattice with upper bound 1. If for any $a, b \in G$ we take $a \to b$ as the complement of a in the interval $[a \wedge b, 1]$, then G is an implication algebra in which the natural order is the lattice order.

The relationship between implication algebras and Boolean and generalized Boolean algebras can be improved as follows.

Lemma 1.1. *For each poset $(A, \leq)$, with $A \neq \emptyset$, the following are equivalent:*

(1) *A is the universe of an implication algebra whose natural order is $\leq$.*
(2) *A is (order isomorphic to) an increasing subset of a Boolean algebra $B(A)$.*
(3) *A is (order isomorphic to) an increasing subset of a generalized Boolean algebra $F(A)$.*

Moreover, if it is the case, A is the universe of an implication subalgebra of $B(A)$ and $F(A)$.

Proof. Since each Boolean algebra is also a generalized Boolean algebra and any increasing subset of a generalized Boolean algebra is closed by $\to$, it suffices to show that (1) implies (2).

Assume that (1). Then there is a Boolean algebra B such that A is an implication subalgebra of B (see [2, Theorem 17]). Let $B(A)$ be the Boolean subalgebra of B generated by A, then it suffices to see that A is increasing in $B(A)$. To show this, take $a \in A$, $b \in B(A)$ such that $a \leq b$, and let us see that $b \in A$. Every $b \in B(A)$ can be written

$$b = \bigwedge_{k=1}^{r} \Big(\big(\bigvee_{i \in I_k} \neg a_i \big) \vee \big(\bigvee_{j \in J_k} c_j \big) \Big),$$

where $I_k \cap J_k = \emptyset$, $\bigcup_{k=1}^{r} (I_k \cup J_k) \neq \emptyset$ is a finite set and $a_i, c_j \in A$.

Let $b_k = (\bigvee_{i \in I_k} \neg a_i) \vee (\bigvee_{j \in J_k} c_j)$, $k = 1, \ldots, r$. Then it is enough to prove that $b_k \in A$ for any k.

If k is such that $J_k \neq \emptyset$, then $\bigvee_{j \in J_k} c_j \in A$. So,

$$b_k = \big(\bigvee_{i \in I_k} \neg a_i \big) \vee \big(\bigvee_{j \in J_k} c_j \big) = \bigvee_{i \in I_k} \big(a_i \to \bigvee_{j \in J_k} c_j\big) \in A.$$

If k is such that $J_k = \emptyset$, then $a \leq \bigvee_{i \in I_k} \neg a_i$, and consequently,

$$b_k = \bigvee_{i \in I_k} \neg a_i = \bigvee_{i \in I_k} \neg a_i \vee a = \bigvee_{i \in I_k} (a_i \to a) \in A. \qquad \square$$

Observe that from Lemma 1.1 it follows that an implication algebra A is a union of filters of $B(A)$.

We denote by $St(B)$ the Stone space of a Boolean algebra B, and by $Clop(X)$ the Boolean algebra of all clopen subsets of a Boolean space X [6]. A dual equivalence between the category of Boolean algebras and homomorphisms and the category of Boolean spaces and continuous maps, is given by

- For a Boolean algebra B, $\begin{cases} s_B \colon B \to Clop(St(B)) \\ \qquad a \mapsto s_B(a) = N_a = \{U \in St(B) : a \in U\} \end{cases}$
- For a Boolean space X, $\begin{cases} t_X \colon X \to St(Clop(X)) \\ \qquad x \mapsto t_X(x) = \{N \in Clop(X) : x \in N\} \end{cases}$

For each Boolean algebra B, the correspondence

$$F \mapsto C_F = \{U \in St(B) : F \subseteq U\}$$

gives a dual order isomorphism from the set $\mathcal{F}$ of all filters of B onto the set $\mathcal{C}$ of all closed sets of $St(B)$, ordered by inclusion. Its inverse is given by

$$C \mapsto F_C = \{a \in B : C \subseteq s_B(a) = N_a\} = \bigcap \{U \in St(B) : U \in C\}.$$

2. Boolean closures of implication algebras

The aim of this section is to give for any implication algebra A the least, up to isomorphism, Boolean algebra in which the filter generated by A is an ultrafilter. First we provide the construction for generalized Boolean algebras.

Lemma 2.1. *Each generalized Boolean algebra G is an ultrafilter of a Boolean algebra $\mathbf{Bo}(G)$.*

Proof. Since G is an implication algebra we can consider $B(G)$ as in Lemma 1.1 and $F(G) = G$. Thus we consider

$$\mathbf{Bo}(G) = \begin{cases} B(G) & \text{if } F(G) \neq B(G) \\ B(G) \times \{0,1\} & \text{if } F(G) = B(G) \end{cases}$$

If $G \neq B(G)$ and $\neg G = \{\neg a : a \in G\}$, then, since $G \cap \neg G = \emptyset$ and $G \cup \neg G = B(G)$, G is an ultrafilter of $B(G) = \mathbf{Bo}(G)$. If $G = B(G)$, identifying G with $G \times \{1\}$, G can be considered as an ultrafilter of $\mathbf{Bo}(G)$. $\square$

A subset C of a Boolean algebra B satisfies the *finite meet property* (fmp for short), provided that 0 cannot be obtained with finite meets of elements of C, that is, the lattice filter generated by C in B is proper.

Theorem 2.2. *Let A be an implication algebra, then there is a Boolean algebra $\mathbf{Bo}(A)$ such that:*

(1) *A is an increasing subset of $\mathbf{Bo}(A)$ and A satisfies the* fmp.

(2) *If $h\colon A \to B$ is an $\{\to\}$-homomorphism from the implication algebra A into a Boolean algebra B, such that $h[A]$ has the* fmp *in B, then there is a Boolean homomorphism $\widehat{h}\colon \mathbf{Bo}(A) \to B$ such that $\widehat{h}\restriction A = h$, i.e., the diagram*

$$\begin{array}{cc} A \subseteq & \mathbf{Bo}(A) \\ {\scriptstyle h}\searrow & \downarrow \widehat{h} \\ & B \end{array}$$

commutes.

Moreover, the proper filter $F(A)$ generated by A in $\mathbf{Bo}(A)$ is an ultrafilter.

Proof. Let $F(A)$ be the filter generated by A in $B(A)$ and let $\mathbf{Bo}(A) = \mathbf{Bo}(F(A))$. Then by Lemma 2.1 we have (1).

Now let B be a Boolean algebra and $h\colon A \to B$ satisfying the hypothesis of (2). In order to define $\widehat{h}$ consider $b \in \mathbf{Bo}(A)$. If $b \in F(A)$, then there are $b_1, \ldots, b_n \in A$, such that $b = \bigwedge_{i=1}^{n} b_i$ and we take $\widehat{h}(b) = \bigwedge_{i=1}^{n} h(b_i)$. If $b \notin F(A)$, then we take $\widehat{h}(b) = \neg\widehat{h}(\neg b)$. Observe that $h[A]$ has the fmp and hence $F(h[A]) \cap \neg F(h[A]) = \emptyset$.

Since for all $a, b \in F(A)$, $a = \bigwedge_{j=1}^{m} a_j$ and $b = \bigwedge_{i=1}^{n} b_i$ where $a_j, b_i \in A$, we have

$$a \to b = \Big(\bigwedge_{j=1}^{m} a_j\Big) \to \Big(\bigwedge_{i=1}^{n} b_i\Big) = \bigwedge_{i=1}^{n} \Big(\bigvee_{j=1}^{m} (a_j \to b_i)\Big)$$

and for all j, i, $a_j \to b_i \in A$, then it is straightforward to see that $\widehat{h}$ is well defined, preserves $\to$ and $\widehat{h}(0) = 0$. Thus $\widehat{h}$ is an homomorphism of Boolean algebras and by definition $\widehat{h}\restriction A = h$. □

As a consequence of the above results we obtain.

Corollary 2.3. *Let $h\colon A_1 \to A_2$ be a homomorphism of implication algebras. Then there is a Boolean homomorphism $\widehat{h}\colon \mathbf{Bo}(A_1) \to \mathbf{Bo}(A_2)$ such that $\widehat{h}\restriction A_1 = h$, and $\widehat{h}^{-1}[F(A_2)] = F(A_1)$. In particular, if h is an isomorphism, then $\widehat{h}$ is also an isomorphism.*

Proof. To prove the first part it is enough to consider in Theorem 2.2, $A = A_1$, and $B = \mathbf{Bo}(A_2)$. Since $F(A_1) \subseteq \widehat{h}^{-1}[F(A_2)]$, the equality follows from the maximality of $F(A_1)$ and $\widehat{h}^{-1}[F(A_2)]$. Moreover, if h is isomorphism, then $\widehat{h^{-1}} = \widehat{h}^{-1}$ and $\widehat{h}$ is also isomorphism. □

Remark 2.4. Observe that if A is an implication algebra and is an increasing subset of a generalized Boolean algebra G and A generates G, then $\mathbf{Bo}(G) = \mathbf{Bo}(A)$.

Given an implication algebra we refer to $F(A)$ and $\mathbf{Bo}(A) = \mathbf{Bo}(F(A))$ *as the generalized Boolean closure* and *Boolean closure* of A, respectively.

Two implication algebras may have the same Boolean closure, but they can be distinguished by means of the filters contained in them. Indeed, if $\mathcal{M}(A)$ is the family of all maximal elements in the set of all filters of $\mathbf{Bo}(A)$ contained in the implication algebra A, ordered by inclusion, then:

(a) $A = \bigcup_{F \in \mathcal{M}(A)} F$,
(b) $\mathcal{M}(A)$ is an antichain, relative to the inclusion,
(c) if M is a filter of $\mathbf{Bo}(A)$ contained in A, then $M \subseteq F$ for some $F \in \mathcal{M}(A)$.

Moreover, these properties characterize $\mathcal{M}(A)$, in the sense that if A is an implication algebra and $\mathcal{G}$ is an antichain of filters of $\mathbf{Bo}(A)$ contained in A and satisfying (a), (b) and (c), then $\mathcal{G} = \mathcal{M}(A)$. Notice that the case $\mathcal{M}(A) = \{A\}$ is not excluded.

3. Implication spaces

By an *implication space* we mean a triple $(X, u, \mathcal{C})$ such that

(i) X is a Boolean space,
(ii) u is a fixed element of X,
(iii) $\mathcal{C}$ is an antichain, with respect to inclusion, of closed sets of X such that $\bigcap \mathcal{C} = \{u\}$,
(iv) if C is a closed subset of X such that for every clopen N of X, $C \subseteq N$ implies $D \subseteq N$ for some $D \in \mathcal{C}$, then there exists $D' \in \mathcal{C}$ such that $D' \subseteq C$.

Let A be an implication algebra. If $St(\mathbf{Bo}(A))$ is the Boolean space of its Boolean closure, then, by Theorem 2.2, $F(A) \in St(\mathbf{Bo}(A))$. By taking $\mathcal{C}(A) = \{C_F : F \in \mathcal{M}(A)\}$ the family of closed sets in $St(\mathbf{Bo}(A))$ corresponding with the set $\mathcal{M}(A)$, it is straightforward to see that

$$\mathbb{X}(A) = (St(\mathbf{Bo}(A)), F(A), \mathcal{C}(A))$$

is an implication space. Condition (iv) is a consequence of condition (c).

Theorem 3.1. *Let* $f : A_1 \to A_2$ *be a homomorphism of implication algebras, let* $\widehat{f}\colon \mathbf{Bo}(A_1) \to \mathbf{Bo}(A_2)$ *be as in Corollary* 2.3*, and consider*

$$St(\widehat{f})\colon\ St(\mathbf{Bo}(A_2)) \to St(\mathbf{Bo}(A_1))$$
$$U \mapsto \widehat{f}^{-1}(U).$$

Then we have:

(1) $St(\widehat{f})$ *is continuous;*

(2) *$St(\widehat{f})(F(A_2)) = F(A_1)$;*
(3) *for $F \in \mathcal{M}(A_1)$, there is $H \in \mathcal{M}(A_2)$ such that $C_H \subseteq St(\widehat{f})^{-1}[C_F]$.*

Proof. (1) follows from the Boolean duality, and (2) from the fact that, by definition, $F(A_1) \subseteq \widehat{f}^{-1}([F(A_2)]) \in St(\mathbf{Bo}(A_1))$.
(3) Let $F \in \mathcal{M}(A_1)$, then

$$U \in St(\widehat{f})^{-1}(C_F) \text{ iff } St(\widehat{f})(U) \in C_F \text{ iff } F \subseteq \widehat{f}^{-1}(U) \text{ iff } \widehat{f}[F] \subseteq U.$$

Hence $St(\widehat{f})^{-1}(C_F) = C_{F(f[F])}$. Now, since $f[F] \subseteq A_2$ is closed under finite meets, the increasing set $\{a \in A_2 : a \geq b, \text{ for some } b \in f[F]\}$ is a filter of $\mathbf{Bo}(A_2)$ contained in A_2 and containing $f[F]$. This shows that $F(f[F]) \subseteq A_2$, and $f[F]$ is contained in some $H \in \mathcal{M}(A_2)$, or equivalently $C_H \subseteq St(\widehat{f})^{-1}(C_F)$. □

The above results motivate the next definition. Let $(X_1, u_1, \mathcal{C}_1)$ and $(X_2, u_2, \mathcal{C}_2)$ be implication spaces. We say that a map $f\colon X_1 \to X_2$ is *i-continuous* provided that:

(1) f is continuous;
(2) $f(u_1) = u_2$;
(3) for all $C \in \mathcal{C}_2$, there is $D \in \mathcal{C}_1$ such that $D \subseteq f^{-1}[C]$.

Moreover, if f is a homeomorphism and its inverse is also i-continuous, then we refer to f as an *i-homeomorphism.* Observe that in this case, the maximality of $D \in \mathcal{C}_1$ implies that $f[D] \in \mathcal{C}_2$ and in the same way for all $C \in \mathcal{C}_2$, $f^{-1}[C] \in \mathcal{C}_1$. Hence we have.

Corollary 3.2. *Let f be an i-continuous map from the implication space $(X_1, u_1, \mathcal{C}_1)$ into the implication space $(X_2, u_2, \mathcal{C}_2)$ such that f is a homeomorphism. Then f is an i-homeomorphism if and only if for all $D \in \mathcal{C}_1$, $f[D] \in \mathcal{C}_2$.*

It follows from the above that

Corollary 3.3. *If $f : A_1 \to A_2$ is a homomorphism of implication algebras, then $\mathbb{X}(f) = St(\widehat{f})$, defined as in Theorem 3.1, is an i-continuous map from $\mathbb{X}(A_2)$ into $\mathbb{X}(A_1)$. Moreover, if f is an isomorphism, then $\mathbb{X}(f)$ is an i-homeomorphism.*

Let $(X, u, \mathcal{C})$ be an implication space and consider

$$\mathbb{I}(X) = \{N \in Clop(X) : C \subseteq N \text{ for some } C \in \mathcal{C}\}.$$

Since $\mathbb{I}(X)$ is an increasing subset in the Boolean algebra $Clop(X)$, $\mathbb{I}(X)$ is an implication algebra. Moreover, if $F_C = \{N \in Clop(X) : C \subseteq N\}$, then $\mathbb{I}(X) = \bigcup_{C \in \mathcal{C}} F_C$. Then we have:

Lemma 3.4. *For each implication space $(X, u, \mathcal{C})$, the following properties hold for the implication algebra $\mathbb{I}(X)$:*

(1) $F(\mathbb{I}(X)) = \{N \in \mathit{Clop}(X) : u \in N\}$;
(2) $\mathbf{Bo}(\mathbb{I}(X)) = \mathit{Clop}(X)$;
(3) $\mathcal{M}(\mathbb{I}(X)) = \{F_C : C \in \mathcal{C}\}$.

Proof. Observe that for any closed subset C of X, and for any $x \in X$, we have $x \in C$ if and only if $F_C \subseteq t_X(x)$. Hence $\bigcup_{C\in\mathcal{C}} F_C \subseteq t_X(x)$ if and only if $x \in \bigcap_{C\in\mathcal{C}} C$ if and only if $x = u$. Thus $F(\mathbb{I}(X)) = F(\bigcup_{C\in\mathcal{C}} F_C) = t_X(u) = \{N \in \mathit{Clop}(X) : u \in N\}$, and we have (1). (2) is an immediate consequence of (1). Finally, condition (iv) in the definition of implication spaces implies that each filter F of $\mathit{Clop}(X)$ such that $F \subseteq \mathbb{I}(X)$ is contained in some F_C. Hence (3) follows. □

Lemma 3.5. *Let $h\colon X_1 \to X_2$ be an i-continuous map from the implication space $(X_1, u_1, \mathcal{C}_1)$ into the implication space $(X_2, u_2, \mathcal{C}_2)$, then*

$$\begin{aligned}\mathbb{I}(h) = \mathit{Clop}(h) \upharpoonright \mathbb{I}(X_2)\colon \mathbb{I}(X_2) &\to \mathbb{I}(X_1)\\ N &\mapsto \mathit{Clop}(h)(N) = h^{-1}[N]\end{aligned}$$

defines a homomorphism of implication algebras. Moreover, if h is an i-homeomorphism, then $\mathbb{I}(h)$ is an isomorphism.

Proof. Since $\mathit{Clop}(h)$ gives a Boolean homomorphism, it is enough to see that $\mathit{Clop}(h)[\mathbb{I}(X_2)] \subseteq \mathbb{I}(X_1)$. Indeed, if $N \in \mathbb{I}(X_2)$, then there exists $C \in \mathcal{C}_2$ such that $C \subseteq N$. Thus $h^{-1}[C] \subseteq h^{-1}[N]$, and since h is i-continuous, there is $D \in \mathcal{C}_1$ such that $D \subseteq h^{-1}[C]$, hence $D \subseteq h^{-1}[N] = \mathbb{I}(h)(N)$, and so $\mathbb{I}(h)(N) \in \mathbb{I}(X_1)$.
If h is i-homeomorphism, then it follows from Corollary 3.2 that $\mathbb{I}(h)$ is an isomorphism. □

4. The natural dual equivalence

The results obtained in the preceding section suggest us the dual equivalence for implication algebras and implication spaces.

We consider $\mathfrak{I}$ the category whose objects are implication algebras and its arrows are implication homomorphisms, and $\mathfrak{X}$ the category with implication spaces as objects and i-continuous maps as arrows. Then from Corollary 3.1 and the fact that St is a functor and hence compatible with composition, we obtain:

Theorem 4.1. *The correspondence $\mathbb{X}\colon \mathfrak{I} \rightsquigarrow \mathfrak{X}$ defined by*

$$\begin{array}{ccc} A_1 & \overset{\mathbb{X}}{\rightsquigarrow} & \mathbb{X}(A_1)\\ f\downarrow & & \uparrow \mathbb{X}(f)=St(\widehat{f})\\ A_2 & \overset{\mathbb{X}}{\rightsquigarrow} & \mathbb{X}(A_2)\end{array}$$

defines a contravariant functor from $\mathfrak{I}$ into $\mathfrak{X}$.

Since $\mathbb{I}$ is compatible with composition and transforms i-homeomorphisms into isomorphisms, we have the following result:

Theorem 4.2. *The correspondence* $\mathbb{I}\colon \mathfrak{X} \rightsquigarrow \mathfrak{I}$ *defined by*

$$\begin{array}{ccc} (X_1, u_1, \mathcal{C}_1) & \overset{\mathbb{I}}{\rightsquigarrow} & \mathbb{I}(X_1) \\ {\scriptstyle h}\downarrow & & \uparrow {\scriptstyle \mathbb{I}(h)} \\ (X_2, u_2, \mathcal{C}_2) & \overset{\mathbb{I}}{\rightsquigarrow} & \mathbb{I}(X_2) \end{array}$$

gives a contravariant functor from $\mathfrak{X}$ *into* $\mathfrak{I}$.

Now we can state the duality theorem.

Theorem 4.3. *The functors* $\mathbb{I}$, $\mathbb{X}$ *define a dual equivalence between the categories* $\mathfrak{I}$ *and* $\mathfrak{X}$. *More explicitly,*

(1) *If for each implication algebra* A *we consider*

$$\sigma_A\colon A \to \mathbb{IX}(A)$$
$$a \mapsto \sigma_A(a) = N_a = \{U \in St(\mathbf{Bo}(A)) : a \in U\},$$

then σ *defines a natural transformation from the functor* $\mathbb{IX}$ *into the identity functor* $\mathrm{Id}_{\mathfrak{I}}$.

(2) *If for each implication space* $(X, u, \mathcal{C})$ *we consider*

$$\tau_X : X \to \mathbb{XI}(X)$$
$$x \mapsto \tau_X(x) = U_x = \{N \in St(Clop(X)) : x \in N\},$$

then τ *defines a natural transformation from the functor* $\mathbb{XI}$ *into the identity functor* $\mathrm{Id}_{\mathfrak{X}}$.

Proof. (1) Observe that if $s_{\mathbf{Bo}(A)}$ is the natural isomorphism from the Boolean algebra $\mathbf{Bo}(A)$ onto $Clop(St(\mathbf{Bo}(A)))$, then $\sigma_A = s_{\mathbf{Bo}(A)}\restriction A$. Hence in order to see that σ_A is an isomorphism it suffices to see that $s_{\mathbf{Bo}(A)}[A] = \mathbb{IX}(A)$. If $a \in A$, then there exists $F \in \mathcal{M}(A)$ such that $a \in F$. So $C_F \subseteq N_a$, that is, $\sigma_A(a) = N_a \in \mathbb{I}(\mathbb{X}(A))$. If $N \in \mathbb{I}(\mathbb{X}(A))$, there exists $a \in \mathbf{Bo}(A)$ such that $N = N_a \in \mathbb{I}(\mathbb{X}(A))$. Thus there exists $F \in \mathcal{M}(A)$ such that $C_F \subseteq N_a$, that is, every ultrafilter containing F also contains a. So $a \in F \subseteq A$.

Observe that if A and A' are implication algebras, then it follows from Corollary 2.3, Lemma 3.5, definition of σ and the fact that s is a natural transformation from *Clop St* into the identity, that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\sigma_A} & \mathbb{IX}(A) \\ {\scriptstyle f}\downarrow & & \downarrow {\scriptstyle \mathbb{IX}(f)} \\ A' & \overset{\sigma_{A'}}{\to} & \mathbb{IX}(A') \end{array}$$

commutes. Thus σ gives a natural transformation.

(2) Observe that if $t_{\mathbf{Bo}(A)}$ is the natural isomorphism from the Boolean space X onto $St(Clop(X))$, then by definition, $\tau_X = t_X$. By (1) of Lemma 3.4, $\tau_X(u) = F(\mathbb{I}(X))$, and by (3) of the Lemma 3.4, we have that for a closed set C,

$$C \in \mathcal{C} \text{ iff } F_{\tau_X}[C] \in \mathcal{M}(\mathbb{I}(X)).$$

Then τ_X is i-homeomorphism.

Finally it is easy to see that if $(X, u, \mathcal{C})$ and $(X', u', \mathcal{C})$ are implication spaces, then the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\tau_X} & \mathbb{X}\mathbb{I}(X) \\ h\downarrow & & \downarrow \mathbb{X}\mathbb{I}(h) \\ X' & \stackrel{\tau_{X'}}{\rightarrow} & \mathbb{X}\mathbb{I}(X') \end{array}$$

commutes. So τ gives a natural transformation. □

In the following section we give an example of an application of this duality.

5. The implication space of a free implication algebra

We recall that the $|Y|$-free implication algebra $\mathcal{F}_{\mathfrak{I}}(Y)$ is the increasing subset of the $|Y|$-free Boolean algebra $\mathcal{F}_{\mathfrak{B}}(Y)$ generated by a set Y. In other words, $\mathcal{F}_{\mathfrak{I}}(Y) = \{x \in \mathcal{F}_{\mathfrak{B}}(Y) : y \leq x \text{ for some } y \in Y\} = \bigcup_{y\in Y} F_y$, where $F_y = F(\{y\})$ is the principal filter generated by y in $\mathcal{F}_{\mathfrak{B}}(Y)$ (see [4] and the references given there).

Now since any infinite Y generates an ultrafilter $F(Y)$ in $\mathcal{F}_{\mathfrak{B}}(Y)$, then $\mathbf{Bo}(\mathcal{F}_{\mathfrak{I}}(Y)) = \mathcal{F}_{\mathfrak{B}}(Y)$.

Lemma 5.1. *For every set Y, $\mathcal{M}(\mathcal{F}_{\mathfrak{I}}(Y)) = \{F_y : y \in Y\}$.*

Proof. We need to see that: (1) $F_y \in \mathcal{M}(F_{\mathfrak{I}}(Y))$, and (2) if F is a filter of $F_{\mathfrak{B}}(Y)$ such that $F \subseteq F_{\mathfrak{I}}(Y) = \bigcup_{y\in Y} F_y$, then $F \subseteq F_y$ for some $y \in Y$.

For (1), suppose that there exists a proper filter F such that $F \subseteq \mathcal{F}_{\mathfrak{I}}(Y)$ and $F_y \subseteq F$. If $x \in F \setminus F_y$, then $x \wedge y \in F$ and there exists $y' \in Y$ with $y' \leq x \wedge y < y$. Consider the extension $\overline{f} : F_{\mathfrak{B}}(Y) \to \mathbf{2} = \{0, 1\}$ of $f : Y \to \mathbf{2}$ defined by $f(y) = 0$ and $f(y') = 1$ for every $y' \in Y \setminus \{y\}$. Then $\overline{f}(y') = 1 < 0 = \overline{f}(y)$, a contradiction. Hence $F \setminus F_y = \emptyset$ and $F = F_y$.

In order to prove (2), we recall that if Y is infinite, then for any infinite subset Y_0 of Y, the unique upper bound of Y_0 in $\mathcal{F}_{\mathfrak{B}}(Y)$ is 1. Hence, for any $x \in \mathcal{F}_{\mathfrak{I}}(Y)$, $x \neq 1$ implies that $P(x) = \{y \in Y : y \leq x\}$ is finite.

Now, let $F \subseteq \bigcup_{y\in Y} F_y$ be a filter and let $n = \min\{|P(x)| : x \in F\}$. If $t \in F$ is such that $|P(t)| = n$, then for each $z \in F$ we have $z \wedge t \in F$ and $P(z \wedge t) \subseteq P(t)$, so $P(z \wedge t) = P(t)$. Hence $\bigvee_{y\in P(t)} y \leq z$, that is, $y \leq z$ for every $y \in P(t)$. Consequently, $z \in F_y$ for every $y \in P(t)$, that is, $F \subseteq F_y$ for every $y \leq t$. This closes the proof. □

Hence, for all $y \in Y$, $C_{F_y} = N_y$ which is clopen. On the other hand, the Boolean space associated to the $|Y|$-free Boolean algebra is the *Cantor space* $\mathbf{2}^Y$ endowed with the product topology by considering the discrete topology in $\mathbf{2} = \{0, 1\}$. It is clear that it can be identify with $\mathcal{P}(Y)$, the family of all subsets of Y. If for all $y \in Y$, we take $C_y = \{Z \subseteq Y : y \in Z\}$, then the implication space of the $|Y|$-free implication algebra is

$$\mathbb{I}(\mathcal{F}_{\mathfrak{I}}(Y)) = (\mathbf{2}^Y, Y, \{C_y : y \in Y\}).$$

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