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Embedding of the vertices of the Auslander–Reiten quiver of an iterated tilted algebra of Dynkin type Δ in $\mathbb{Z}\Delta$

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Abstract

Let Δ be a Dynkin diagram and k an algebraically closed field. Let A be an iterated tilted finite-dimensional k-algebra of type Δ and denote by $\hat{A}$ its repetitive algebra. We approach the problem of finding a combinatorial algorithm giving the embedding of the vertices of the Auslander–Reiten quiver Γ_A of A in the Auslander–Reiten quiver $\Gamma(\underline{\mathrm{mod}}(\hat{A})) \simeq \mathbb{Z}\Delta$ of the stable category $\underline{\mathrm{mod}}(\hat{A})$. Let T be a trivial extension of finite representation type and Cartan class Δ. Assume that we know the vertices of $\mathbb{Z}\Delta$ corresponding to the radicals of the indecomposable projective T-modules. We determine the embedding of Γ_A in $\mathbb{Z}\Delta$ for any algebra A such that $T(A) \simeq T$.

Introduction

The algebras to be considered in this paper are basic finite-dimensional algebras over an algebraically closed field k. Any such algebra A can be written as a bound quiver algebra kQ_A/I, where I is an admissible ideal of the path algebra kQ_A and Q_A is the quiver associated to A.

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For a quiver Q, let Q_0 denote the set of vertices of Q and Q_1 the set of arrows of Q. An arrow α of Q_1 starts at the vertex $o(\alpha)$ and ends at $e(\alpha)$.

Let k be an algebraically closed field, Δ a Dynkin diagram and let A be an iterated tilted algebra of type Δ [1]. Let $T(A) = A \ltimes D_A(A)$ be the trivial extension of A by its minimal injective cogenerator $D_A(A) = \mathrm{Hom}_k(A, k)$. The algebra $T(A)$ is known to be of finite representation type [3] and there exists an embedding of $\mathrm{mod}\, A$ in the stable category $\underline{\mathrm{mod}}\, T(A)$. Then the set of vertices $(\Gamma_A)_0$ of the AR-quiver Γ_A of A can be embedded in the stable part ${}_S\Gamma_{T(A)}$ of the AR-quiver $\Gamma_{T(A)}$ of $T(A)$. Moreover, $T(A)$ admits universal Galois covering $\hat{A} \to T(A)$, where $\hat{A}$ is the repetitive algebra of A, ${}_S\Gamma_{\hat{A}} \simeq \mathbb{Z}\Delta$ and thus Γ_A can be embedded in $\mathbb{Z}\Delta$ [1,7,8,11]. This is, the vertices of the AR-quiver Γ_A of any iterated tilted algebra A of type Δ can be embedded in $\mathbb{Z}\Delta$, and in such way that knowing which vertices of $\mathbb{Z}\Delta$ correspond to A-modules we can obtain the arrows of Γ_A in a canonical way, so that we get the AR-quiver Γ_A of A. Taking this into account and for simplicity we will just say that the AR-quiver Γ_A embeds in $\mathbb{Z}\Delta$ to mean that there is an injective map $\varphi : (\Gamma_A)_0 \to (\mathbb{Z}\Delta)_0$. Our main objective is to describe this embedding explicitly. We recall that the trivial extensions of finite representation type and Cartan class Δ are precisely the trivial extensions of iterated tilted algebras of Dynkin type Δ [3]. We divided the problem in two parts.

Let T be a trivial extension of finite representation type and Cartan class Δ.

(1) Assume that we know the vertices of $\mathbb{Z}\Delta$ corresponding to the radicals of the indecomposable projective T-modules. Determine the embedding of Γ_A in $\mathbb{Z}\Delta$ for any algebra A such that $T(A) \simeq T$.
(2) Describe an algorithm to determine which subsets of vertices in $\mathbb{Z}\Delta$ represent the radicals of the indecomposable projective modules over the trivial extension T.

In this paper we solve the first part. The second is studied in the first author's Ph.D. thesis [15] where an algorithm is given for $\Delta = \mathbf{A}_n$ and $\Delta = \mathbf{D}_n$, and will be published in a forthcoming paper.

We describe the embedding more explicitly. Let A be an iterated tilted algebra of type Δ and let $T(A) = A \ltimes D_A(A)$ be the trivial extension of A by $D_A(A) = \mathrm{Hom}_k(A, k)$. The canonical epimorphism $p : T(A) \to A$ given by $p(a, \varphi) = a$ induces a full and faithful functor

$$F_p : \mathrm{mod}\, A \hookrightarrow \mathrm{mod}\, T(A),$$

which identifies $\mathrm{mod}\, A$ with the full subcategory of $\mathrm{mod}\, T(A)$ whose objects are the $T(A)$-modules annihilated by $D_A(A)$. Moreover, the composition of F_p with the canonical functor $\theta : \mathrm{mod}\, T(A) \to \underline{\mathrm{mod}}\, T(A)$ is also a full and faithful functor

$$\theta F_p : \mathrm{mod}\, A \hookrightarrow \underline{\mathrm{mod}}\, T(A).$$

Therefore the AR-quiver Γ_A of A can be embedded in the AR-quiver $\Gamma_{T(A)}$ of $T(A)$ and in the stable AR-quiver ${}_S\Gamma_{T(A)}$ making the following diagram commutative

$$\begin{array}{ccc} & & \Gamma_{T(A)} \\ & \nearrow & \uparrow \\ \Gamma_A & \hookrightarrow & {}_S\Gamma_{T(A)} \end{array}$$

It is known (see 2.6 in [8]) that there exists a translation quiver morphism $\pi : {}_S\Gamma_{\hat{A}} \to {}_S\Gamma_{T(A)}$, which is the universal covering of ${}_S\Gamma_{T(A)}$, and that ${}_S\Gamma_{\hat{A}} \simeq \mathbb{Z}\Delta$

$$\begin{array}{ccc} & & {}_S\Gamma_{\hat{A}} = \mathbb{Z}\Delta \\ & & \downarrow \pi \\ \Gamma_A & \hookrightarrow & {}_S\Gamma_{T(A)} \end{array}$$

Then we can consider a connected lifting ${}_S\Gamma_{T(A)}[0]$ of the quiver ${}_S\Gamma_{T(A)}$ to $\mathbb{Z}\Delta$ (see Section 3). Since the quiver Γ_A is embedded in ${}_S\Gamma_{T(A)}$ the above lifting induces a subquiver $\Gamma_A[0]$ of ${}_S\Gamma_{T(A)}[0]$ in such way that the following diagram is commutative

$$\begin{array}{ccc} \Gamma_A[0] & \hookrightarrow & {}_S\Gamma_{\hat{A}} = \mathbb{Z}\Delta \\ \downarrow \pi & & \downarrow \pi \\ \Gamma_A & \hookrightarrow & {}_S\Gamma_{T(A)} \end{array}$$

We get an embedding of Γ_A in $\mathbb{Z}\Delta$ and we are looking for the vertices of $\mathbb{Z}\Delta$ corresponding to indecomposable A-modules under such embedding.

We start by studying the embedding $\Gamma_A \hookrightarrow \Gamma_{T(A)}$ induced by the canonical epimorphism $p : T(A) \to A$. Thus, we have to determine which vertices of $\Gamma_{T(A)}$ correspond to indecomposable A-modules. We know that $A \simeq T(A)/D_A(A)$, and that a $T(A)$-module M is an A-module if and only if $D_A(A)M = 0$. Therefore we have to know what the condition $D_A(A)M = 0$ means in the Auslander–Reiten quiver $\Gamma_{T(A)}$. Let $A = kQ_A/I$, in [9,10] the quiver of $Q_{T(A)}$ is obtained from Q_A by adding some arrows. Moreover, the ideal $D_A(A)$ of $T(A)$ is generated precisely by these added arrows [9]. On the other hand, given a trivial extension T of finite representation type a method is given in [9] to obtain the iterated tilted algebras B such that $T(B) \simeq T$. In fact, such algebras are obtained by deleting exactly one arrow in each nonzero oriented cycle of Q_T and considering the induced relations. Thus B is the factor of T by an ideal generated by arrows.

First we will study when an ideal generated by arrows annihilates a module M. In Section 2 we give a characterization of modules M over a quotient k-algebra $\Lambda/\mathcal{J}$ where $\mathcal{J}$ is an ideal of Λ generated by arrows of Q_Λ. In particular, when Λ is $T(A)$ and $\mathcal{J} = D(A)$ we describe the vertices of $\Gamma_{T(A)}$ corresponding to $A \simeq T(A)/\mathcal{J}$-modules. More precisely, suppose that $\mathcal{J}$ is generated by some arrows $\alpha_1, \alpha_2, \ldots, \alpha_t$ of $Q_{T(A)}$. We consider the subquiver $\mathcal{P}_{\alpha_1,\alpha_2,\ldots,\alpha_t}$ of $\Gamma_{T(A)}$ induced by the nonzero paths in $\Gamma_{T(A)}$ starting at the projective $P_{o(\alpha_i)}$ and ending at the projective $P_{e(\alpha_i)}$ for some $i = 1, 2, \ldots, t$. We prove that the vertices of Γ_A are exactly the vertices of $\Gamma_{T(A)}$ which are not contained in $\mathcal{P}_{\alpha_1,\alpha_2,\ldots,\alpha_t}$. A similar description is given in Section 3 for the embedding of Γ_A in $\Gamma_{\hat{A}}$. To do that, we define an appropriate lifting of $\Gamma_{T(A)}$ to $\Gamma_{\hat{A}}$, and we study how nonzero paths between projective modules in $\Gamma_{T(A)}$ lift to $\Gamma_{\hat{A}}$. In this way we obtain the embedding $\Gamma_A \hookrightarrow \Gamma_{\hat{A}}$, and then the desired embedding $\Gamma_A \hookrightarrow \mathbb{Z}\Delta \simeq {}_S\Gamma_{\hat{A}}$.

1. Preliminaries

Let Q be a quiver, which may be infinite. A *path* γ in the quiver Q is either an oriented sequence of arrows $\alpha_n \cdots \alpha_1$ with $e(\alpha_t) = o(\alpha_{t+1})$ for $1 \leqslant t < n$, or the symbol e_i for $i \in Q_0$. The *length* $\ell(\gamma)$ of γ is n in the first case, and $\ell(e_i) = 0$. We call the paths e_i trivial paths and we define $o(e_i) = e(e_i)$. Let I be an ideal of the path algebra kQ. We consider $\Lambda = kQ/I$ as a k-category whose objects are the vertices Q_0 of Q and the morphism space $\Lambda(i, j)$ from i to j is $\overline{e_j}\Lambda\overline{e_i}$, where $\overline{e_i} = e_i + I$ (see [5]).

Let A be a k-algebra. For a given vertex j of Q_A we denote by S_j the simple A-module corresponding to j, by P_j the projective cover of S_j, and by I_j the injective envelope of S_j. We will use freely properties of the module category $\operatorname{mod} A$ of finitely generated left A-modules, the stable category $\underline{\operatorname{mod}}\, A$ module projectives, the Auslander–Reiten quiver Γ_A and the Auslander–Reiten translations $\tau = DTr$ and $\tau^{-1} = TrD$, as can be found in [4]. We denote by $\operatorname{ind} A$ (respectively by $\underline{\operatorname{ind}}\, A$) the full subcategory of $\operatorname{mod} A$ ($\underline{\operatorname{mod}}\, A$) formed by chosen representatives of the indecomposable modules. Moreover, we will frequently identify the objects of $\operatorname{ind} A$ with the vertices of the AR-quiver Γ_A representing such objects.

We will freely use the notions of locally finite k-category, translation quiver, covering functor, well behaved functor and related notions. We refer the reader to [4,5,11,17,18] for definitions and basic properties of these objects.

Let Δ be an oriented tree. Following Chr. Riedtmann [17] (see also [4]) we will consider the translation quiver $\mathbb{Z}\Delta$, defined as follows:

$$(\mathbb{Z}\Delta)_0 = \mathbb{Z} \times \Delta_0, \quad (\mathbb{Z}\Delta)_1 = \{-1, 1\} \times \mathbb{Z} \times \Delta_1.$$

For an arrow $x \xrightarrow{\alpha} y$ of Δ we define the arrows $(-1, n, \alpha)$ and $(1, n, \alpha)$ as

$$(n-1, y) \xrightarrow{(-1,n,\alpha)} (n, x) \quad \text{and} \quad (n, x) \xrightarrow{(1,n,\alpha)} (n, y).$$

Finally, the translation τ is $\tau(n, y) = (n-1, y)$.

2. Modules over quotients of quasi-schurian weakly symmetric algebras

We start this section by giving a characterization of modules M over a quotient k-algebra $\Lambda/\mathcal{J}$ where $\mathcal{J}$ is an ideal of Λ generated by arrows of Q_Λ. Then we go on to study the case when Λ is quasi-schurian and weakly symmetric. Finally, we give an application to trivial extensions of finite representation type.

We recall from [14] that an algebra Λ is *quasi-schurian* if it satisfies:

(a) $\dim_k \operatorname{Hom}_\Lambda(P, Q) \leqslant 1$ if P and Q are non isomorphic indecomposable projective Λ-modules and
(b) $\dim_k \operatorname{End}_\Lambda(P) = 2$ for any indecomposable projective Λ-module P.

Let $A = kQ_A/I$ be a *schurian* (that is, $\dim_k \operatorname{Hom}_A(P_i, P_j) \leqslant 1$ for any vertices i and j of Q_A) and *triangular* (that is, Q_A has non oriented cycles) k-algebra, with I admissible ideal. Then the trivial extension $T(A)$ of A is a quasi-schurian algebra.

As a consequence we get that the trivial extensions of finite representation type are quasi-schurian. This follows from the fact, proved by K. Yamagata in [20], that the trivial extension of a non triangular algebra is of infinite representation type.

Since we want to describe the Λ-modules M annihilated by a finite number of arrows of Q_Λ, we start by studying when $\bar{\alpha} M = 0$ for a given arrow α.

Lemma 2.1. *Let $\Lambda = kQ_\Lambda/I$ be a k-algebra with I an admissible ideal. Let $\alpha : i \to j$ be an arrow in Q_Λ and $M \in \operatorname{mod} \Lambda$.*

The following conditions are equivalent:

(a) $\bar{\alpha} M \neq 0$.
(b) $\operatorname{Hom}_\Lambda(\rho_\alpha, M) : \operatorname{Hom}_\Lambda(P_i, M) \to \operatorname{Hom}_\Lambda(P_j, M)$ *is nonzero, where $\rho_\alpha : P_j \to P_i$ is the right multiplication by $\bar{\alpha}$.*

Proof. The proof is straightforward. □

Lemma 2.2. *Let $\Lambda = kQ_\Lambda/I$ be a k-algebra with I an admissible ideal. Let $\alpha : i \to j$ be an arrow in Q_Λ and $M \in \operatorname{mod} \Lambda$. Then*

(a) *If $\bar{\alpha} M \neq 0$ then there are morphisms $f : P_i \to M$, $g : M \to I_j$ such that $gf \neq 0$.*
(b) *Assume that $\operatorname{Hom}_\Lambda(\rho_\alpha, I_j) : \operatorname{Hom}_\Lambda(P_i, I_j) \to \operatorname{Hom}_\Lambda(P_j, I_j)$ is a monomorphism, where $\rho_\alpha : P_j \to P_i$ is the right multiplication by $\bar{\alpha}$. If there are morphisms $f : P_i \to M$, $g : M \to I_j$ with $gf \neq 0$, then $\bar{\alpha} M \neq 0$.*

Proof. (a) From Lemma 2.1 we know that there is a nonzero morphism $f : P_i \to M$ such that $f\rho_\alpha : P_j \to M$ is nonzero. Then there is $g : M \to I_j$ such that $gf\rho_\alpha \neq 0$, and consequently $gf \neq 0$.

(b) Assume that $\operatorname{Hom}_\Lambda(\rho_\alpha, I_j)$ is a monomorphism and let $f : P_i \to M$, $g : M \to I_j$ such that $gf \neq 0$. Then $0 \neq \operatorname{Hom}_\Lambda(\rho_\alpha, I_j)(gf) = (gf)\rho_\alpha = g(f\rho_\alpha)$, proving that $f\rho_\alpha \neq 0$. Thus $\operatorname{Hom}_\Lambda(\rho_\alpha, M)(f) \neq 0$ and by Lemma 2.1 we get that $\bar{\alpha} M \neq 0$. □

In case Λ is a quasi-schurian weakly symmetric algebra we obtain the following theorem.

Theorem 2.3. *Let $\Lambda = kQ_\Lambda/I$ be a quasi-schurian and weakly-symmetric k-algebra with I an admissible ideal. Let $\alpha : i \to j$ be an arrow in Q_Λ. Then the following conditions are equivalent for an indecomposable Λ-module M:*

(a) $\bar{\alpha} M \neq 0$.
(b) *There are morphisms $P_i \xrightarrow{f} M$, $M \xrightarrow{g} P_j$ with $gf \neq 0$.*

Proof. (a) ⇒ (b) Since Λ is weakly-symmetric then $P_j = I_j$ for any vertex $j \in Q_\Lambda$. So Lemma 2.2(a) proves the result in this case.

(b) ⇒ (a) Assume that $i \neq j$. Using Lemma 2.2(b) we only need to prove that

$$\mathrm{Hom}_\Lambda(\rho_\alpha, P_j) : \mathrm{Hom}_\Lambda(P_i, P_j) \to \mathrm{Hom}_\Lambda(P_j, P_j)$$

is nonzero. Since Λ is quasi-schurian and weakly-symmetric it is not hard to prove that there exists a path δ starting at j, ending at i and such that $\delta\alpha$ is nonzero (see in [14, 2.2 and 3]). In particular, from [14, Theorem 3, IV] we obtain that $\alpha\delta$ is nonzero. Thus $\mathrm{Hom}_\Lambda(\rho_\alpha, P_j)$ is nonzero.

If $i = j$ then α is a loop. Now, the only (up to isomorphisms) indecomposable quasi-schurian and weakly-symmetric k-algebra with loops is $\Lambda \simeq k[x]/\langle x^2\rangle$ (see [14, Lemma 14]). Assume that $e(\alpha) = o(\alpha) = 1$. Then the projective P_1 and the simple S_1 are the unique (up to isomorphism) indecomposable Λ-modules.

Suppose that $M = P_1$. Then $\bar{\alpha} P_1 \neq 0$ and the morphisms $f = \rho_\alpha$ and $g = 1_{P_1}$ satisfy (b).

Let $M = S_1$, then $\bar{\alpha} S_1 = 0$. On the other hand, since $\mathrm{rad}^2(P_1, P_1) = 0$ we get that $gf = 0$ for any $f : P_1 \to S_1$ and $g : S_1 \to P_1$. □

Corollary 2.4. *Let $\Lambda = kQ_\Lambda/I$ be a quasi-schurian and weakly-symmetric k-algebra with I an admissible ideal. Let $\alpha_i : a_i \to b_i$ be arrows in Q_Λ for $i = 1, 2, \ldots, t$. Then the following conditions are equivalent for an indecomposable Λ-module M.*

(a) *M is a $\Lambda/\langle\bar{\alpha}_1, \ldots, \bar{\alpha}_t\rangle$-module.*
(b) *If $f : P_{a_i} \to M$, $g : M \to P_{b_i}$ are morphisms in* $\mathrm{mod}\,\Lambda$*, then their composition gf is zero for all $i = 1, 2, \ldots, t$.*

Proof. Follows easily from the preceding theorem. □

We are now in a position to characterize the modules M over Λ which are in $\mathrm{mod}\,\Lambda/\mathcal{J}$ in terms of certain chains of irreducible morphism, in case Λ is quasi-schurian, weakly-symmetric and of finite representation type, and $\mathcal{J}$ is an ideal of Λ generated by arrows of Q_Λ.

Corollary 2.5. *Let $\Lambda = kQ_\Lambda/I$ be a quasi-schurian and weakly-symmetric k-algebra of finite representation type, with I an admissible ideal. Let $\alpha_i : a_i \to b_i$ be arrows in Q_Λ for $i = 1, 2, \ldots, t$. Then the following conditions are equivalent for an indecomposable Λ-module M:*

(a) *M is a $\Lambda/\langle\bar{\alpha}_1,\dots,\bar{\alpha}_t\rangle$-module.*
(b) *Any chain of irreducible maps in* ind Λ

$$X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_j = M \xrightarrow{f_{j+1}} X_{j+1} \to \cdots \xrightarrow{f_r} X_r$$

with $X_0 = P_{a_i}$, $X_r = P_{b_i}$ has zero composition for all $i = 1, 2, \dots, t$.

Proof. Follows from the above corollary using that if Λ is of finite representation type, then each nonzero morphism between indecomposable modules can be written as a sum of compositions of irreducible morphisms between indecomposable modules [4]. □

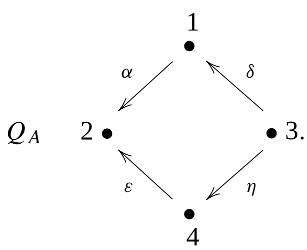
Let Λ be a k-algebra as in the preceding corollary, and let $A = \Lambda/\mathcal{J}$ where $\mathcal{J}$ is the ideal of Λ generated by some arrows $\alpha_1, \alpha_2, \dots, \alpha_t$ of Q_Λ. We denote by $\mathcal{P}_{\alpha_1,\alpha_2,\dots,\alpha_t}$ the subquiver of Γ_Λ induced by the nonzero paths in $k(\Gamma_\Lambda)$ starting at the projective $P_{o(\alpha_i)}$ and ending at the projective $P_{e(\alpha_i)}$ for some $i = 1, 2, \dots, t$. Then by Corollary 2.5 we have that the vertices of Γ_A can be identified with the vertices of Γ_Λ which are not in $\mathcal{P}_{\alpha_1,\alpha_2,\dots,\alpha_t}$. That is, $(\Gamma_A)_0 = (\Gamma_\Lambda)_0 \backslash (\mathcal{P}_{\alpha_1,\alpha_2,\dots,\alpha_t})_0$.

Let $A = kQ_A/I$ be an iterated tilted k-algebra of Dynkin type, with I an admissible ideal and let $T(A)$ be the trivial extension of A. Then $\Lambda = T(A)$ satisfies the hypothesis of Corollary 2.5. This is the case because the trivial extension of an iterated tilted algebra of Dynkin type is of finite representation type (see [3]) and, as we have seen at the beginning of this section, $T(A)$ is quasi-schurian.

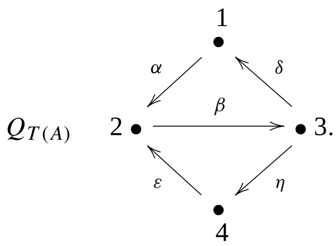
Remark 2.6. Let $T = kQ_T/I_T$ be a trivial extension of finite representation type and let A be an iterated tilted k-algebra of Dynkin type such that $T \simeq T(A)$. As we observed in the introduction, A is obtained by deleting exactly one arrow in each nonzero cycle of Q_T, and considering the induced relations. So we have that $A = T/\langle\bar{\alpha}_1,\dots,\bar{\alpha}_t\rangle$ where $\alpha_1, \alpha_2, \dots, \alpha_t$ are arrows in Q_T. Suppose that we know which vertices of the AR-quiver Γ_T correspond to the projective T-modules P_j associated with each vertex j of Q_T. As we observed above, the vertices of Γ_A can be identified with the vertices of Γ_T which are not in $\mathcal{P}_{\alpha_1,\alpha_2,\dots,\alpha_t}$.

Therefore the embedding $\Gamma_A \hookrightarrow \Gamma_T$ is determined by the position in Γ_T of the vertices corresponding to the projective T-modules P_j for $j \in (Q_T)_0$.

Example. Let A be the iterated tilted algebra of type $\mathbf{D}_4$ with ordinary quiver Q_A, and with relation $0 = \alpha\delta - \varepsilon\eta$, where



By [10] the ordinary quiver $Q_{T(A)}$ of the trivial extension $T(A)$ of A is



and the ideal I such that $T(A) = kQ_{T(A)}/I$ is generated by the relations: $\alpha\delta - \varepsilon\eta$, $\delta\beta\varepsilon$, $\eta\beta\alpha$, $\beta\alpha\delta\beta$, $\alpha\delta\beta\alpha$, $\varepsilon\eta\beta\varepsilon$. In this case we have $A = T(A)/\langle\overline{\beta}\rangle$. Hence we have to look for the nonzero paths in $\Gamma_{T(A)}$ from $P_{o(\beta)} = P_2$ to $P_{e(\beta)} = P_3$. The shaded region of Fig. 1 corresponds to $\mathcal{P}_\beta$.

Then we delete from the quiver $\Gamma_{T(A)}$ the modules which are in $\mathcal{P}_\beta$. In Fig. 2 we indicate with $\square$ the vertices of $\Gamma_{T(A)}$ corresponding to A-modules.

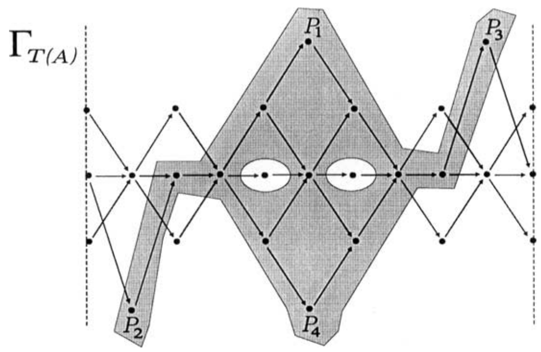


Fig. 1.

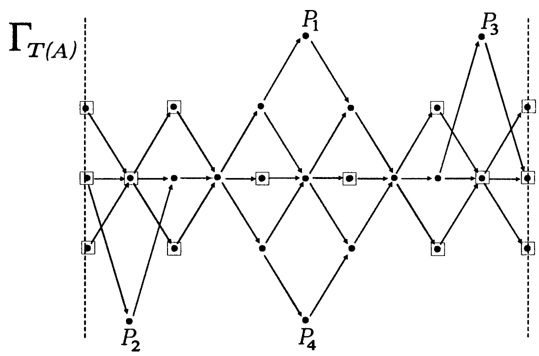


Fig. 2.

$_S\Gamma_{T(A)}$

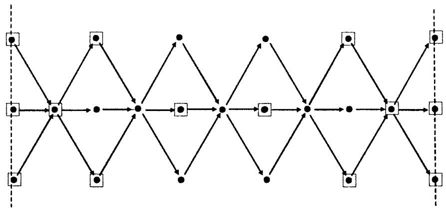


Fig. 3.

Then the embedding $\Gamma_A \hookrightarrow {}_S\Gamma_{T(A)}$ is described in Fig. 3, where we indicate with $\square$ the vertices of ${}_S\Gamma_{T(A)}$ corresponding to A-modules.

The other iterated tilted algebras B such that $T(B) \simeq T(A)$ are of the form $T(A)/\langle\bar{\alpha}, \bar{\varepsilon}\rangle$, $T(A)/\langle\bar{\alpha}, \bar{\eta}\rangle$, $T(A)/\langle\bar{\delta}, \bar{\varepsilon}\rangle$, and $T(A)/\langle\bar{\delta}, \bar{\eta}\rangle$. The embedding of Γ_B in ${}_S\Gamma_{T(A)}$ for these algebras B is obtained in the same way.

The embedding $\Gamma_A \hookrightarrow {}_S\Gamma_{T(A)}$ is reduced to the embedding $\Gamma_A \hookrightarrow \Gamma_{T(A)}$, since the stable part ${}_S\Gamma_{T(A)}$ of $\Gamma_{T(A)}$ is obtained from $\Gamma_{T(A)}$ by deleting the vertices of $\Gamma_{T(A)}$ associated to projective modules. In general, we have information about the stable quiver ${}_S\Gamma_{T(A)}$. Indeed, suppose that the trivial extension $\Lambda = T(A)$ of A is of Cartan class Δ, where Δ is a Dynkin diagram. Then ${}_S\Gamma_\Lambda \xrightarrow{\sim} \mathbb{Z}\Delta/\Pi({}_S\Gamma_\Lambda, x)$ where $\Pi({}_S\Gamma_\Lambda, x)$ is the fundamental group associated to the universal covering $\pi : \mathbb{Z}\Delta \to {}_S\Gamma_\Lambda$ of the stable translation quiver ${}_S\Gamma_\Lambda$ (see [17]). Moreover, the group $\Pi({}_S\Gamma_\Lambda, x)$ is generated by τ^{m_Δ}, where m_Δ is the Loewy length of the mesh category $k(\mathbb{Z}\Delta)$ [2,6]. We recall that the values of m_Δ are: $m_{\mathbf{A}_n} = n$, $m_{\mathbf{D}_n} = 2n-3$, $m_{\mathbf{E}_6} = 11$, $m_{\mathbf{E}_7} = 17$, $m_{\mathbf{E}_8} = 29$.

In this way we have information about the structure of the stable quiver ${}_S\Gamma_\Lambda$. Our problem now is to recover the structure of Γ_Λ from the knowledge we have about ${}_S\Gamma_\Lambda$. To do that, we need to know which vertices of ${}_S\Gamma_\Lambda$ correspond to the radicals of the projective modules P_i for $i \in (Q_\Lambda)_0$, since $0 \to rP_i \to P_i \amalg rP_i/\operatorname{soc} P_i \to P_i/\operatorname{soc} P_i \to 0$ is an AR-sequence for each vertex i of Q_Λ. We denote by $\mathcal{C}_\Lambda$ the set of vertices of ${}_S\Gamma_\Lambda$ representing the radicals of the projective Λ-modules. It is well known that $\mathcal{C}_\Lambda$ is a configuration of ${}_S\Gamma_\Lambda$, as defined by Chr. Riedtmann in [18]. This is, the elements of $\mathcal{C}_\Lambda$ satisfy the following definition.

Definition 2.7. [18]. Let Γ be a stable translation quiver and $k(\Gamma)$ the mesh-category associated to Γ. A *configuration* $\mathcal{C}$ of Γ is a set of vertices of Γ which satisfies the following conditions:

(a) For any vertex $x \in \Gamma_0$ there exists a vertex $y \in \mathcal{C}$ such that $k(\Gamma)(x, y) \neq 0$,
(b) $k(\Gamma)(x, y) = 0$ if x and y are different elements of $\mathcal{C}$,
(c) $k(\Gamma)(x, x) = k$ for all $x \in \mathcal{C}$.

Let Δ be a Dynkin diagram, Λ a trivial extension of Cartan class Δ, and $\pi : \mathbb{Z}\Delta \to {}_S\Gamma_\Lambda$ the universal covering of ${}_S\Gamma_\Lambda$. Since $\mathcal{C}_\Lambda$ is a configuration of ${}_S\Gamma_\Lambda$, we obtain from [18] that $\widetilde{\mathcal{C}}_\Lambda = \pi^{-1}(\mathcal{C}_\Lambda)$ is a configuration of $\mathbb{Z}\Delta$. We will say that $\widetilde{\mathcal{C}}_\Lambda$ is the configuration of $\mathbb{Z}\Delta$ associated to Λ.

3. The lifting process

Throughout this section Δ denotes a Dynkin diagram. Let A be an iterated tilted k-algebra of type Δ and let $T(A)$ be the trivial extension of A. In the preceding section we described an embedding of Γ_A into ${}_S\Gamma_{T(A)}$ which we will lift to an embedding of Γ_A in $\mathbb{Z}\Delta = {}_S\Gamma_{\hat{A}}$. Our purpose now is describing directly this embedding in terms of a section in $\mathbb{Z}\Delta$ and some nonzero paths in $\Gamma_{\hat{A}}$ between projective $\hat{A}$-modules. A similar description was done in the preceding section for the embedding of Γ_A into $\Gamma_{T(A)}$. So, we will define a connected lifting ${}_S\Gamma_{T(A)}[0]$ of ${}_S\Gamma_{T(A)}$ to $\mathbb{Z}\Delta$ and extend it to a connected lifting $\Gamma_{T(A)}[0]$ of $\Gamma_{T(A)}$ to $\Gamma_{\hat{A}}$. Afterwards we will study how nonzero paths in $\Gamma_{T(A)}$ between projective modules lift to $\Gamma_{\hat{A}}$. Since there are infinitely many $\Gamma_{\hat{A}}$-projectives and we want to circumscribe to a small part of $\mathbb{Z}\Delta$, we need to study how long the nonzero paths between the projective modules in $\Gamma_{\hat{A}}$ are. So we start with some preliminaries.

Following [6,12] we denote the *Nakayama-permutation* on $\mathbb{Z}\Delta$ by ν_Δ. This is the bijection $\nu_\Delta : (\mathbb{Z}\Delta)_0 \to (\mathbb{Z}\Delta)_0$ which satisfies the following condition: *for each vertex x of $\mathbb{Z}\Delta$ there exists a path $w : x \to \nu_\Delta(x)$ whose image $\overline{w}$ in the mesh-category $k(\mathbb{Z}\Delta)$ is not zero, and w has longest length among all nonzero paths starting at x*. The Loewy length m_Δ of the mesh-category $k(\mathbb{Z}\Delta)$ is the smallest integer m such that $\bar{v} = 0$ in $k(\mathbb{Z}\Delta)$ for all paths v in $\mathbb{Z}\Delta$ whose length is greater than or equal to m. Thus $m_\Delta - 1$ is the common length of all nonzero paths from x to $\nu_\Delta(x)$. Moreover, we have that $\tau^{-m_\Delta} = \nu_\Delta^2 \tau^{-1}$.

Let (Γ, τ) be a connected stable translation quiver. Following P. Gabriel in [12] we will call *slice* of Γ to a full connected subquiver whose vertices are determined by choosing a unique element in each τ-orbit of Γ_0. Then for each vertex $x \in \Gamma$ there is a well-determined slice admitting x as its unique source. We call it *slice starting at x* and denote it by $\mathcal{S}_{x\to}$. Likewise, the *slice ending at x* admits x as its unique sink and is denoted by $\mathcal{S}_{\to x}$.

Let $f : (\mathbb{Z}\Delta)_0 \to \mathbb{Z}$. We recall that f is *additive* if it satisfies the equation

$$f(x) + f\big(\tau(x)\big) = \sum_{z \in x^-} f(z)$$

for each vertex x. It is well known that the additive function f_x, which has value 1 on $\mathcal{S}_{x\to}$, determines the support of the functor $k(\mathbb{Z}\Delta)(x, -)$. In fact, $\dim_k k(\mathbb{Z}\Delta)(x, y) = f_x(y)$.

Proposition 3.1. *Let x be a vertex of $\mathbb{Z}\Delta$. Then*

(a) $\operatorname{Supp} k(\mathbb{Z}\Delta)(x, -) = \operatorname{Supp} k(\mathbb{Z}\Delta)(-, \nu_\Delta(x))$,
(b) $\operatorname{Supp} k(\mathbb{Z}\Delta)(x, -) \cap \operatorname{Supp} k(\mathbb{Z}\Delta)(-, \nu_\Delta^2(x)) = \{\nu_\Delta(x)\}$.

Proof. (a) The proof given by Chr. Riedtmann for the $\mathbf{D}_n$ case in [19, page 312] can be adapted to the other Dynkin diagrams.

(b) Follows from (a) and the fact that $\mathbb{Z}\Delta$ has no oriented cycles. □

Let x be a vertex of $\mathbb{Z}\Delta$. Using (a) of the preceding proposition we obtain that the support of the functor $k(\mathbb{Z}\Delta)(x, -)$ is contained in the set of vertices of $\mathbb{Z}\Delta$ laying on or between the sections $\mathcal{S}_{x\to}$ and $\mathcal{S}_{\to\nu_\Delta(x)}$. Though this inclusion is not in general an equality it is so in the case $\Delta = \mathbf{A}_n$.

Remark 3.2. Let Λ be a trivial extension of Cartan class Δ, and let $F : k(\mathbb{Z}\Delta) \to \underline{\mathrm{ind}}\,\Lambda$ be a well-behaved functor induced by the universal covering $\pi : \mathbb{Z}\Delta \to {}_S\Gamma_\Lambda$. Since F is a covering functor, then it induces a k-vector space isomorphism

$$\coprod_{y\in\pi^{-1}(Y)} k(\mathbb{Z}\Delta)(x,y) \xrightarrow{\sim} \underline{\mathrm{Hom}}_\Lambda\big(\pi(x),Y\big).$$

Since Δ is of Dynkin type we can say more: if $\underline{\mathrm{Hom}}_\Lambda(\pi(x),Y) \neq 0$, then the left side has a unique nonzero summand. Dually, if $\underline{\mathrm{Hom}}_\Lambda(X,\pi(y)) \neq 0$ there exists a unique $x \in \pi^{-1}(X)$ such that $k(\mathbb{Z}\Delta)(x,y) \neq 0$.

In fact, we assume that $k(\mathbb{Z}\Delta)(x,y_i) \neq 0$ for $i = 1,2$ and $\pi(y_1) = \pi(y_2)$. Suppose that $y_1 \neq y_2$. Then $y_1 = \tau^{jm_\Delta} y_2$ for some integer j, which we may assume positive. Let $\delta : y_1 \to y_2$ and $\gamma : x \to y_1$ be paths in $\mathbb{Z}\Delta$. Therefore we have a path $\delta\gamma : x \to y_2$ with length $\ell(\delta\gamma) \geqslant \ell(\delta) = 2jm_\Delta$. Since paths between vertices of $\mathbb{Z}\Delta$ have the same length, we obtain that any path starting at x and ending at y_2 has length at least $2jm_\Delta$. This is a contradiction because the longest length of a nonzero path in $k(\mathbb{Z}\Delta)$ is $m_\Delta - 1$. This proves the first statement of the remark. The second statement follows by duality.

As a consequence of the above remark we can see that the information we have about the support of the functor $k(\mathbb{Z}\Delta)(x,-)$ in $\mathbb{Z}\Delta$ can be carried out through the universal covering $\pi : \mathbb{Z}\Delta \to {}_S\Gamma_\Lambda$ to determine the support of $\underline{\mathrm{Hom}}_\Lambda(\pi(x),-)$ in ${}_S\Gamma_\Lambda$.

Proposition 3.3. *Let Λ be a trivial extension of Cartan class Δ. Then the universal covering $\pi : \mathbb{Z}\Delta \to {}_S\Gamma_\Lambda$ induces the following bijections*:

(i) $\mathrm{Supp}\, k(\mathbb{Z}\Delta)(x,-) \xrightarrow{\sim} \mathrm{Supp}\, \underline{\mathrm{Hom}}_\Lambda(\pi(x),-)$.
(ii) $\mathrm{Supp}\, k(\mathbb{Z}\Delta)(-,x) \xrightarrow{\sim} \mathrm{Supp}\, \underline{\mathrm{Hom}}_\Lambda(-,\pi(x))$.

The next result is an interesting application of the preceding corollary.

Corollary 3.4. *Let Λ be a trivial extension of Cartan class Δ with Δ a Dynkin diagram. Then for all $X, Y \in \underline{\mathrm{ind}}\,\Lambda$ we have*

$$\dim_k \underline{\mathrm{Hom}}_\Lambda(X,Y) \leqslant \begin{cases} 1 & \text{if } \Delta = \mathbf{A}_n, \\ 2 & \text{if } \Delta = \mathbf{D}_n, \\ 3 & \text{if } \Delta = \mathbf{E}_p \text{ and } p = 6,7, \\ 6 & \text{if } \Delta = \mathbf{E}_8. \end{cases}$$

Proof. Let $\pi : \mathbb{Z}\Delta \to {}_S\Gamma_\Lambda$ be the universal covering of ${}_S\Gamma_\Lambda$. To describe $\underline{\mathrm{Hom}}_\Lambda(X,Y)$ we consider a fixed $x \in \pi^{-1}(X)$. We know by Remark 3.2 that there exists a unique $y \in \pi^{-1}(Y)$ such that $\underline{\mathrm{Hom}}_\Lambda(X,Y)$ is isomorphic to $k(\mathbb{Z}\Delta)(x,y)$. On the other hand,

$\dim_k k(\mathbb{Z}\Delta)(x, y) = f_x(y)$ where f_x is the additive function starting at x. We use the work of Gabriel [12, p. 53] where he computes the values of this function for some vertices x of $\mathbb{Z}\Delta$, to get the bounds for $\dim_k \underline{\mathrm{Hom}}_\Lambda(X, Y) = f_x(y)$ above stated. □

When A is an iterated tilted algebra of Cartan class Δ, there is an embedding $\mathrm{ind}\, A \hookrightarrow \underline{\mathrm{ind}}\, T(A)$. Thus, the bounds given in the preceding corollary are also bounds for $\dim_k \underline{\mathrm{Hom}}_A(X, Y)$ if $X, Y \in \mathrm{ind}\, A$.

For a fixed vertex x of $\mathbb{Z}\Delta$ we define the partition $\{\mathbb{P}_x[j]\colon\ j \in \mathbb{Z}\}$ of $\mathbb{Z}\Delta$, where $\mathbb{P}_x[0]$ is the full subquiver of $\mathbb{Z}\Delta$ with vertices lying on or between the slices $\mathcal{S}_{x\to}$ and $\tau^{-m_\Delta+1}\mathcal{S}_{x\to}$, and $\mathbb{P}_x[j] = \tau^{-jm_\Delta}\mathbb{P}_x[0]$ for any $j \in \mathbb{Z}$. Let z be a vertex of $\mathbb{P}_x[0]$, for any integer j we denote by $z[j]$ the vertex $\tau^{-jm_\Delta}z$ of $\mathbb{P}_x[j]$.

Let Λ be a trivial extension of Cartan class Δ, and let $\pi : \mathbb{Z}\Delta \to {}_S\Gamma_\Lambda$ be the universal covering of ${}_S\Gamma_\Lambda$. Let $M \in \underline{\mathrm{ind}}\, \Lambda$ and let $M[0]$ be a fixed element of the fibre $\pi^{-1}(M)$. Then $\pi|_{\mathbb{P}_{M[0]}} : \mathbb{P}_{M[0]} \to {}_S\Gamma_\Lambda$ is a quiver morphism, which is a bijection on the vertices of $\mathbb{P}_{M[0]}$, since the quiver ${}_S\Gamma_\Lambda$ is isomorphic to the cylinder $\mathbb{Z}\Delta/\langle\tau^{m_\Delta}\rangle$. The inverse $\varphi_M : ({}_S\Gamma_\Lambda)_0 \to (\mathbb{Z}\Delta)_0$ of this bijection defines an embedding of ${}_S\Gamma_\Lambda$ into $\mathbb{Z}\Delta$. Moreover, the map $\pi|_{\mathbb{P}_{M[0]}}$ is injective on the arrows of $\mathbb{P}_{M[0]}$ but not surjective. Indeed, the arrows $X \to Y$ of ${}_S\Gamma_\Lambda$ with $X \in \mathcal{S}_{\tau M\to}$ and $Y \in \mathcal{S}_{M\to}$ are not in the image of $\pi|_{\mathbb{P}_{M[0]}}$ (see Fig. 4).

Definition 3.5. Let Λ be a trivial extension of Cartan class Δ and let $M \in \underline{\mathrm{ind}}\, \Lambda$. We say that the quiver ${}_S\Gamma_\Lambda[0] = \mathbb{P}_{M[0]}$ *is a lifting of* ${}_S\Gamma_\Lambda$ *to* $\mathbb{Z}\Delta$ *at* M. Moreover, if we do not want to state precisely the lifting vertex we will say that ${}_S\Gamma_\Lambda[0]$ is a lifting of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\Delta$.

For an algebra A such that $\Lambda \simeq T(A)$ we denote by $\Gamma_A[0]$ the embedding of Γ_A in $\mathbb{Z}\Delta$ obtained as the composition of the embeddings $\Gamma_A \hookrightarrow {}_S\Gamma_{T(A)}$ (given in the preceding section) and $\varphi_M : {}_S\Gamma_\Lambda \hookrightarrow \mathbb{Z}\Delta$.

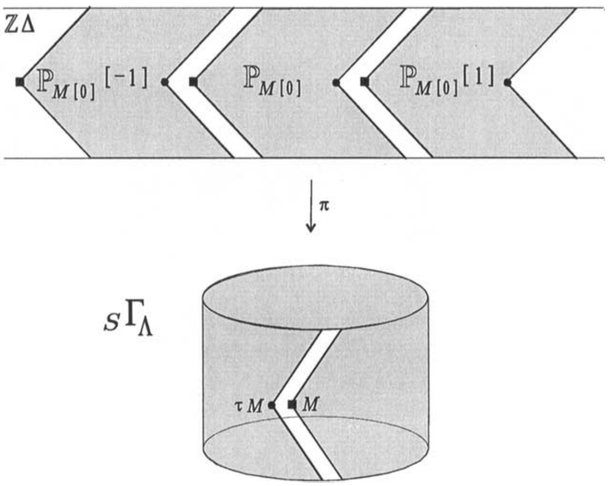


Fig. 4.

Remark 3.6. Let ${}_S\Gamma_\Lambda[0]$ be a lifting of ${}_S\Gamma_\Lambda$ to $\mathbb{Z}\Delta$ at M, and let $\alpha : X \to Y$ be an arrow of ${}_S\Gamma_\Lambda$. For any $j \in \mathbb{Z}$, there exists a unique arrow $\alpha_j : X[j] \to Y_j$ in $\mathbb{Z}\Delta$ such that $\pi(\alpha_j) = \alpha$, where $\pi : \mathbb{Z}\Delta \to {}_S\Gamma_\Lambda$ is the universal covering of ${}_S\Gamma_\Lambda$. Moreover, we have that Y_j is either equal to $Y[j]$ or to $Y[j+1]$. The latter case occurs when $Y \in \mathcal{S}_{M\to}$.

Let A be an iterated tilted algebra of Cartan class Δ, with Δ a Dynkin diagram. Let $\pi : \mathbb{Z}\Delta \to {}_S\Gamma_{T(A)}$ be the universal covering of ${}_S\Gamma_{T(A)}$, $\mathcal{C}_{T(A)} = \{rP_i \colon i \in (Q_{T(A)})_0\}$ and let $\widetilde{\mathcal{C}}_{T(A)} = \pi^{-1}(\mathcal{C}_{T(A)})$ be the configuration of $\mathbb{Z}\Delta$ associated to $T(A)$. From this data Chr. Riedtmann constructed in [18] the universal covering of $\Gamma_{T(A)}$ by adding to $\mathbb{Z}\Delta$ the "projective vertices", exactly one for each vertex of the configuration $\widetilde{\mathcal{C}}_{T(A)}$, and appropriate arrows. This can be described as follows. Let ${}_S\Gamma_{T(A)}[0]$ be a lifting of ${}_S\Gamma_{T(A)}$ to $\mathbb{Z}\Delta$. Then $\{rP_i[j] \colon j \in \mathbb{Z}\} = \pi^{-1}(rP_i)$ for any vertex i of $Q_{T(A)}$. We denote by $\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}}$ the translation quiver obtained from $\mathbb{Z}\Delta$ by adding a new vertex $\mathbf{P}_i[j]$ and arrows $rP_i[j] \to \mathbf{P}_i[j]$, $\mathbf{P}_i[j] \to \tau^{-1}rP_i[j]$ for each $rP_i[j] \in \widetilde{\mathcal{C}}_{T(A)}$. The translation of $\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}}$ coincides with the translation of $\mathbb{Z}\Delta$ on the common vertices and is not defined on the remaining ones.

The action of $\Pi({}_S\Gamma_{T(A)}, x) = \langle \tau^{m_\Delta} \rangle$ on $\mathbb{Z}\Delta$ can be extended to $\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}}$ by defining $\tau^{m_\Delta}(\mathbf{P}_i[j]) = \mathbf{P}_i[j-1]$. Moreover, the covering $\pi : \mathbb{Z}\Delta \to {}_S\Gamma_{T(A)}$ admits an extension $\tilde{\pi} : \mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}} \to \Gamma_{T(A)}$ by defining $\tilde{\pi}(\mathbf{P}_i[j]) = P_i$ for any i and j. It is not difficult to see that $\tilde{\pi} : \mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}} \to \Gamma_{T(A)}$ is the universal covering of $\Gamma_{T(A)}$ and that it induces an isomorphism $\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}}/\langle \tau^{m_\Delta} \rangle \xrightarrow{\sim} \Gamma_{T(A)}$.

For any $M \in \underline{\mathrm{ind}}\, T(A)$ the embedding $\varphi_M : {}_S\Gamma_{T(A)} \hookrightarrow \mathbb{Z}\Delta$ can be extended to an embedding $\tilde{\varphi}_M : \Gamma_{T(A)} \hookrightarrow \mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}}$ by defining $\tilde{\varphi}_M(P_j) = \mathbf{P}_j[0]$ for any vertex j of $Q_{T(A)}$. We denote by $\Gamma_{T(A)}[0]$ the full subquiver of $\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}}$ with vertices $\tilde{\varphi}_M((\Gamma_{T(A)})_0)$. Then $\tilde{\pi}|_{\Gamma_{T(A)}[0]} : \Gamma_{T(A)}[0] \to \Gamma_{T(A)}$ is a quiver morphism, which is a bijection with inverse $\tilde{\varphi}_M$ on the vertices of $\Gamma_{T(A)}[0]$. In this way, we have that the lifting ${}_S\Gamma_{T(A)}[0]$ of ${}_S\Gamma_{T(A)}$ to $\mathbb{Z}\Delta$ extends directly to a lifting $\Gamma_{T(A)}[0]$ of $\Gamma_{T(A)}$ to $\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}}$.

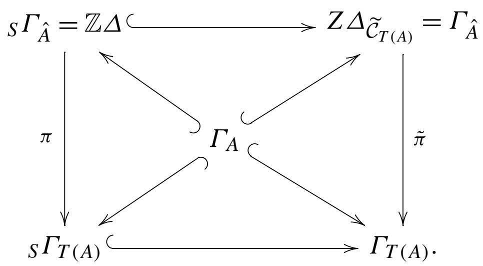
Given a set X of vertices of $\Gamma_{T(A)}[0]$ we denote by $X[j]$ the shifted set $\tau^{-jm_\Delta}X$.

Proposition 3.7. *With the above notation we have that $\Gamma_{\hat{A}} \simeq \mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}}$ and the protective vertices $\mathbf{P}_i[j]$ of $\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}}$ represent the projective $\hat{A}$-modules. Moreover, there is a commutative diagram*

$$\begin{array}{ccc} {}_S\Gamma_{\hat{A}} = \mathbb{Z}\Delta & \hookrightarrow & Z\Delta_{\widetilde{\mathcal{C}}_{T(A)}} = \Gamma_{\hat{A}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tilde{\pi}} \\ {}_S\Gamma_{T(A)} & \hookrightarrow & \Gamma_{T(A)}. \end{array}$$

Proof. Let $F : k(\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}}) \to \mathrm{ind}\, T(A)$ be a well-behaved functor induced by the universal covering $\tilde{\pi} : \mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}} \to \Gamma_{T(A)}$. Let $\tilde{A}$ be the full subcategory of $k(\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}})$ whose objects are the projective vertices of $\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}}$. Then the restriction of the functor F to $\tilde{A}$ induces a covering functor $F' : \tilde{A} \to T(A)$ (see [11, 2]). This functor is the universal covering since $T(A)$ is standard [13, 3]. On the other hand, it is proven in [16] that the Galois covering $\hat{A} \to T(A)$ is universal. So $\tilde{A} \simeq \hat{A}$ proving the result. □

Remark 3.8. For any $M \in \underline{\text{ind}}\, T(A)$ the embeddings $\varphi_M : {}_S\Gamma_{T(A)} \hookrightarrow \mathbb{Z}\Delta$ and $\tilde{\varphi}_M : \Gamma_{T(A)} \hookrightarrow \mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}}$ induce embeddings of Γ_A in ${}_S\Gamma_{\hat{A}}$ and $\Gamma_{\hat{A}}$, respectively, making the following diagram commutative



Moreover, we have that $\Gamma_A[j] \hookrightarrow {}_S\Gamma_{T(A)}[j] \hookrightarrow \Gamma_{T(A)}[j]$ for any $j \in \mathbb{Z}$.

We know that $A = T(A)/\langle\bar{\alpha}_1, \ldots, \bar{\alpha}_t\rangle$, where $\alpha_1, \alpha_2, \ldots, \alpha_t$ are arrows of $Q_{T(A)}$. In Section 2 we have seen that $(\Gamma_A)_0 = (\Gamma_{T(A)})_0 \backslash (\mathcal{P}_{\alpha_1,\alpha_2,\ldots,\alpha_t})_0$, where $\mathcal{P}_{\alpha_1,\alpha_2,\ldots,\alpha_t}$ is the full subquiver of $\Gamma_{T(A)}$ induced by the nonzero paths in $k(\Gamma_{T(A)})$ starting at the projective $P_{o(\alpha_i)}$ and ending at the projective $P_{e(\alpha_i)}$ for some $i = 1, 2, \ldots, t$. Thus, to obtain the embedding $\Gamma_A \hookrightarrow \Gamma_{\hat{A}}$ and then the desired embedding $\Gamma_A \hookrightarrow \mathbb{Z}\Delta \simeq {}_S\Gamma_{\hat{A}}$ we have to lift $\mathcal{P}_{\alpha_1,\alpha_2,\ldots,\alpha_t}$ through the universal covering $\tilde{\pi} : \mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}} \to \Gamma_{T(A)}$.

As we recalled at the beginning of this section, the length of any nonzero path in $k(\mathbb{Z}\Delta)$ is at most $m_\Delta - 1$. Though in $\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}}$ there are longer paths which are nonzero in $k(\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}})$, we have that the length of these paths is bounded by $2m_\Delta$, as follows from the following known result.

Lemma 3.9 [6, 1.2]. *Any nonzero path $v : x \to y$ in $k(\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}})$ can be extended to a nonzero path $\mathbf{P}_i[j] \xrightarrow{u} x \xrightarrow{v} y \xrightarrow{w} \mathbf{P}_i[j+1] = \tau^{-m_\Delta}\mathbf{P}_i[j]$ for some $i \in (Q_{T(A)})_0$ and $j \in \mathbb{Z}$. In particular, the nonzero path $v : x \to y$ has length $\ell(v) \leqslant 2m_\Delta$.*

Remark 3.10. Let Λ be a trivial extension of Cartan class Δ, with Δ a Dynkin diagram. Let $F : k(\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_\Lambda}) \to \text{ind}\,\Lambda$ be a well-behaved functor induced by the universal covering $\tilde{\pi} : \mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_\Lambda} \to \Gamma_\Lambda$. We consider now the isomorphism

$$\coprod_{y \in \tilde{\pi}^{-1}(Y)} k\big(\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_\Lambda}\big)(x, y) \xrightarrow{\sim} \text{Hom}_\Lambda\big(\tilde{\pi}(x), Y\big) \qquad (*)$$

induced by the covering functor $F : k(\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_\Lambda}) \to \text{ind}\,\Lambda$. In analogy with the result stated in Remark 3.2 for the stable case, we obtain that if $\text{Hom}_\Lambda(\tilde{\pi}(x), Y) \neq 0$ then the left side of $(*)$ has a unique nonzero summand, unless $\tilde{\pi}(x) \simeq Y$. Though this is not true when $\tilde{\pi}(x) \simeq Y$; in this case the left side of $(*)$ has at most two nonzero summands.

In fact, the last claim follows directly from Lemma 3.9. To prove the first, let $y \in \tilde{\pi}^{-1}(Y)$ be such that $k(\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_\Lambda})(x, y) \neq 0$. Using Lemma 3.9 we only need to prove that $k(\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_\Lambda})(x, \tau^{jm_\Delta} y) = 0$ for $j = \pm 1$. Since any path $w : y \to \tau^{-m_\Delta} y$ has length $2m_\Delta$ and we have a path $v : x \to y$ with $x \neq y$, we conclude that any path $u : x \to \tau^{-m_\Delta} y$ has length $\ell(u) \geqslant 2m_\Delta + 1$. Thus by Lemma 3.9 we obtain that $k(\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_\Lambda})(x, \tau^{-m_\Delta} y) = 0$. Likewise, we get that also $k(\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_\Lambda})(x, \tau^{m_\Delta} y) = 0$, proving the result.

We are now in a position to prove the main result of this section.

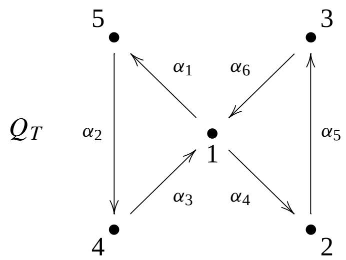
Theorem 3.11. *Let A be an iterated tilted algebra of Dynkin type Δ, and let $A = T(A)/\langle \bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n \rangle$, where $\alpha_1, \alpha_2, \ldots, \alpha_t$ are arrows of $Q_{T(A)}$. Let ${}_S\Gamma_{T(A)}[0]$ be a lifting of ${}_S\Gamma_{T(A)}$ to $\mathbb{Z}\Delta$. For any integer j we denote by $\mathcal{P}_{\alpha_1,\alpha_2,\ldots,\alpha_t}[j]$ the full subquiver of $\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}}$ induced by the nonzero paths in $k(\mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}})$ starting at $\mathbf{P}_{o(\alpha_i)}[j]$ and ending either at $\mathbf{P}_{e(\alpha_i)}[j]$ or at $\mathbf{P}_{e(\alpha_i)}[j+1]$ for some $i = 1, 2, \ldots, t$. Then the vertices of $\Gamma_A[0]$ are the vertices of ${}_S\Gamma_{T(A)}[0]$ which are not in $\mathcal{P}_{\alpha_1,\alpha_2,\ldots,\alpha_n}[-1] \cup \mathcal{P}_{\alpha_1,\alpha_2,\ldots,\alpha_n}[0]$.*

Proof. Let $\tilde{\pi} : \mathbb{Z}\Delta_{\widetilde{\mathcal{C}}_{T(A)}} \to \Gamma_{T(A)}$ be the universal covering of $\Gamma_{T(A)}$. By Remarks 2.6 and 3.8 we know that $\Gamma_A[0] = {}_S\Gamma_{T(A)}[0] \backslash \tilde{\pi}^{-1}(\mathcal{P}_{\alpha_1,\alpha_2,\ldots,\alpha_t})$. On the other hand, $\mathcal{P}_{\alpha_1,\alpha_2,\ldots,\alpha_t}[j] \cap {}_S\Gamma_{T(A)}[0] = \emptyset$ for $j \geqslant 1$ and $j \leqslant -2$. Then the desired result follows from the equality

$$\tilde{\pi}^{-1}(\mathcal{P}_{\alpha_1,\alpha_2,\ldots,\alpha_t}) = \bigcup_{j \in \mathbb{Z}} \mathcal{P}_{\alpha_1,\alpha_2,\ldots,\alpha_t}[j],$$

which is a consequence of Lemma 3.9 and Remark 3.10. □

Example. Let T be the trivial extension of Cartan class $\mathbf{A}_5$ with ordinary quiver Q_T and with the relations $\alpha_4\alpha_3 = 0, \alpha_1\alpha_6 = 0$, $\alpha_3\alpha_2\alpha_1 - \alpha_6\alpha_5\alpha_4 = 0$, $\alpha_2\alpha_1\alpha_3\alpha_2 = 0$, $\alpha_5\alpha_4\alpha_6\alpha_5 = 0$.



Let $A = T/\langle \overline{\alpha_2}, \overline{\alpha_5} \rangle$ and $B = T/\langle \overline{\alpha_3}, \overline{\alpha_4} \rangle$. Hence $T(A) = T = T(B)$ and the embeddings $\Gamma_A[j] \hookrightarrow \Gamma_{\hat{A}}$, $\Gamma_B[j] \hookrightarrow \Gamma_{\hat{B}}$ for each integer j are as follows:

(1) The shaded regions in Fig. 5 correspond to $\mathcal{P}_{\alpha_2,\alpha_5}[j]$ for $j \in \mathbb{Z}$. Hence, the vertices of $\Gamma_{\hat{A}}$ which are not in these shaded regions correspond to A-modules.

(2) The shaded regions in Fig. 6 correspond to $\mathcal{P}_{\alpha_3,\alpha_4}[j]$ for $j \in \mathbb{Z}$. Consequently, the vertices of $\Gamma_{\hat{B}}$ which are not in these regions correspond to B-modules.

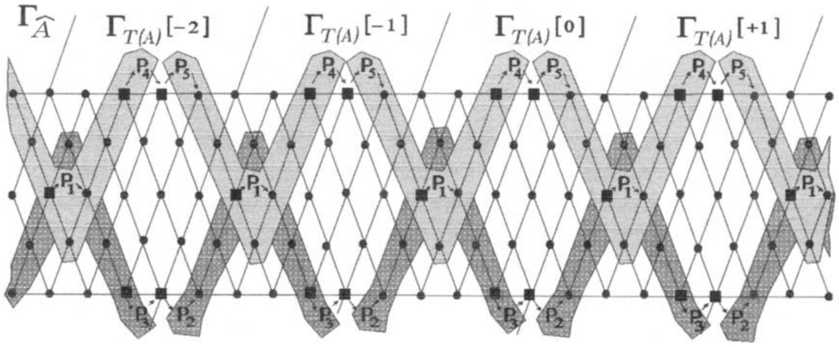


Fig. 5.

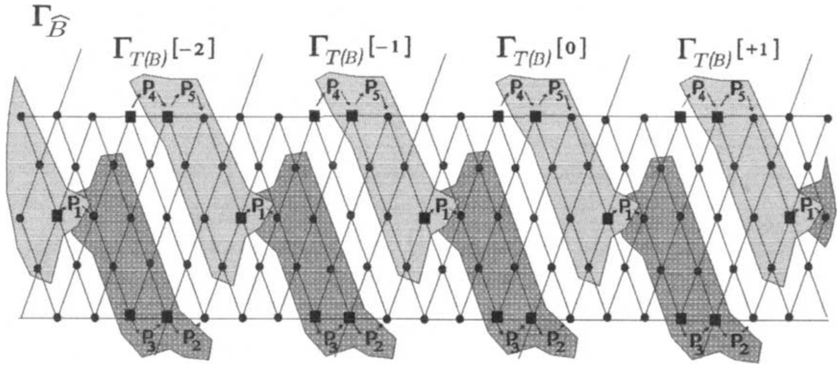
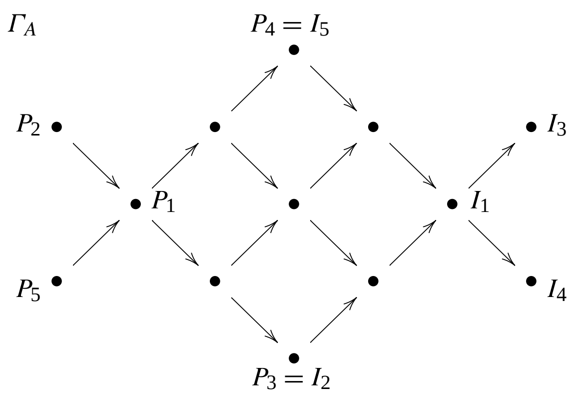
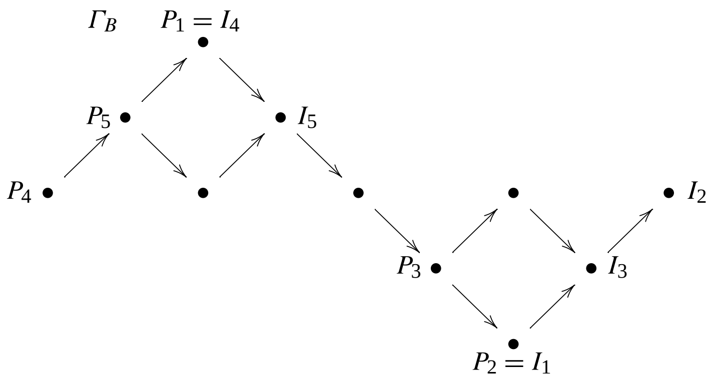


Fig. 6.

Finally, we can describe Γ_A and Γ_B from this information. Indeed, the vertices of Γ_A can be represented by the vertices of ${}_S\Gamma_{T(A)}[0]$, which are not in the shaded regions. The arrows of Γ_A are obtained by studying the paths in ${}_S\Gamma_{T(A)}[-1] \cup {}_S\Gamma_{T(A)}[0] \cup {}_S\Gamma_{T(A)}[1]$, as follows from Remarks 3.2 and 3.6. Then we get the AR-quivers Γ_A and Γ_B





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