

Injective Envelopes and Local Multiplier Algebras of Some Spatial Continuous Trace C*-algebras*

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Abstract

A precise description of the injective envelope of a spatial continuous trace C*-algebra A over a Stonean space Δ is given. The description is based on the notion of a weakly continuous Hilbert bundle, which we show herein to be a Kaplansky–Hilbert module over the abelian AW*-algebra $C(\Delta)$. We then use the description of the injective envelope of A to study the first- and second-order local multiplier algebras of A . In particular, we show that the second-order local multiplier algebra of A is precisely the injective envelope of A .

Introduction

A commonly used technique in the theory of operators algebras is to study a given C*-algebra A by one or more of its enveloping algebras. Well known examples of such enveloping algebras are the enveloping von Neumann algebra A^{**} and the multiplier algebra $M(A)$. In this paper we consider two others: the local multiplier algebra $M_{\text{loc}}(A)$ and the injective envelope $I(A)$, both

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of which have received considerable study and application in recent years (see, for example, [1, 6, 7, 9, 11, 19, 21, 22]).

The C^* -algebras $M_{\text{loc}}(A)$ and $I(A)$ are difficult to determine precisely, even for fairly rudimentary types of C^* -algebras A . For instance, if we denote by $C_0(T)$ an abelian C^* -algebra and by $K(H)$ the ideal of compact operators over H , their local multiplier algebra and injective envelope have been readily computed; but the injective envelope of $C_0(T) \otimes K(H)$ is much more difficult to describe: see [15] for an abstract description and [3, 4] for a somewhat more concrete one.

Our first goal in the present paper is to make a further contribution to the issue of the determination of $I(A)$ and $M_{\text{loc}}(A)$ from A by considering continuous trace C^* -algebras studied by Fell [10] that arise from continuous Hilbert bundles. The class of such algebras contains in particular all C^* -algebras of the form $C_0(T) \otimes K(H)$, which were studied in [4]. Because the centres of $I(A)$ and $M_{\text{loc}}(A)$ are AW^* -algebras, and thus have Stonean maximal ideal spaces, we restrict ourselves in this paper to locally compact Hausdorff spaces T that are Stonean. In so doing, we establish an important first step toward a complete analysis, in the case of non-Stonean T , of the C^* -algebras $I(A)$, $M_{\text{loc}}(A)$, and $M_{\text{loc}}(M_{\text{loc}}(A))$ for spatial continuous trace C^* -algebras A with spectrum T . As the passage from general T to Stonean T involves a number of technicalities, the application of the main results herein to the case of arbitrary locally compact Hausdorff spaces T will be deferred to a subsequent article.

Our second goal is to study and use the notion of a weakly continuous Hilbert bundle Ω_{wk} relative to a continuous Hilbert bundle Ω over a locally compact Hausdorff space T . Particular cases of this notion have been previously considered in [15, 23]. It is natural to consider Ω as a C^* -module over the abelian C^* -algebra $C_0(T)$; if, moreover, T is a Stonean space Δ , we then show Ω_{wk} carries the structure of a faithful AW^* -module over $C(\Delta)$. In this latter situation, such C^* -modules are called Kaplansky–Hilbert modules. We study the C^* -modules Ω and Ω_{wk} , as well as certain C^* -algebras of endomorphisms of these modules, using the beautiful machinery Kaplansky developed in his seminal work from the early 1950s [16]. In particular, we prove that the C^* -algebra $B(\Omega_{\text{wk}})$ of bounded adjointable endomorphisms of Ω_{wk} is the injective envelope and second-order local multiplier algebra of the C^* -algebra $K(\Omega)$ of “compact” endomorphisms of Ω .

Assuming that $T = \Delta$, a Stonean space, and in postponing the precise definitions until the following section, we summarise in this paragraph the main results of the paper. In Section 2, we show that Ω_{wk} is a Kaplansky–Hilbert module that contains Ω as a C^* -submodule such that $\Omega^\perp = \{0\}$. In

Section 3, we prove that $B(\Omega_{\text{wk}})$ is the injective envelope of both $K(\Omega)$ and the Fell continuous trace C^* -algebra A induced by the bundle Ω . Section 4 deals with local multipliers, and we show that $B(\Omega_{\text{wk}})$ is the second-order local multiplier algebra of both $K(\Omega)$ and Fell algebra A . We also prove that the equality $M_{\text{loc}}(M_{\text{loc}}(A)) = I(A)$ holds for certain type I non-separable C^* -algebras, generalising a result of Somerset [21]. Finally, in Section 5 we find that a direct-sum decomposition of Ω_{wk} leads to a corresponding decomposition of (the generally non-AW^{*}) algebra $M_{\text{loc}}(A)$ but not to a decomposition of A .

1 Preliminaries

If T is a locally compact Hausdorff space and $\{H_t\}_{t \in T}$ is family of Hilbert spaces, a vector field on T with fibres H_t is a function $\nu : T \rightarrow \bigsqcup_t H_t$ in which $\nu(t) \in H_t$, for every $t \in T$. Such a vector field ν is said to be bounded if the function $t \mapsto \|\nu(t)\|$ is bounded. From this point on, the notation $T \rightarrow \bigsqcup_t H_t$ will be taken to also imply that, for all t , the point t is mapped into the corresponding fibre H_t .

Definition 1.1. *A continuous Hilbert bundle [8] is a triple $(T, \{H_t\}_{t \in T}, \Omega)$, where Ω is a set of vector fields on T with fibres H_t such that:*

- (I) Ω is a $C(T)$ -module with the action $(f \cdot \omega)(t) = f(t)\omega(t)$;
- (II) for each $t_0 \in T$, $\{\omega(t_0) : \omega \in \Omega\} = H_{t_0}$;
- (III) the map $t \mapsto \|\omega(t)\|$ is continuous, for all $\omega \in \Omega$;
- (IV) Ω is closed under local uniform approximation—that is, if $\xi : T \rightarrow \bigsqcup_t H_t$ is any vector field such that for every $t_0 \in T$ and $\varepsilon > 0$ there is an open set $U \subset T$ containing t_0 and a $\omega \in \Omega$ with $\|\omega(t) - \xi(t)\| < \varepsilon$ for all $t \in U$, then necessarily $\xi \in \Omega$.

Dixmier and Douady [8] show that (I), (II), and (IV) can be replaced by other axioms, such as those given by Fell [10], without altering the structure that arises. For example, in the presence of the other axioms, (II) is equivalent to “ $\{\omega(t_0) : \omega \in \Omega\}$ is dense in H_{t_0} , for each $t_0 \in T$ ”; in the presence of (IV), axiom (I) can be replaced by “ Ω is a complex vector space”.

We turn next to the notion of a weakly continuous Hilbert bundle. If $(T, \{H_t\}_{t \in T}, \Omega)$ is a continuous Hilbert bundle then, by the polarisation identity, the function $t \mapsto \langle \omega_1(t), \omega_2(t) \rangle$ is continuous for all $\omega_1, \omega_2 \in \Omega$. In defining $\langle \omega_1, \omega_2 \rangle$ to be the map $T \rightarrow \mathbb{C}$ given by $t \mapsto \langle \omega_1(t), \omega_2(t) \rangle$, one

obtains a $C(T)$ -valued inner product on Ω which gives Ω the structure of an inner product module over $C(T)$.

Definition 1.2. A vector field $\nu : T \rightarrow \bigsqcup_t H_t$ is said to be weakly continuous with respect to the continuous Hilbert bundle $(T, \{H_t\}_{t \in T}, \Omega)$ if the function

$$t \longmapsto \langle \nu(t), \omega(t) \rangle$$

is continuous for all $\omega \in \Omega$. The set of all bounded weakly continuous vector fields with respect to a given Ω will be denoted by Ω_{wk} , that is

$$\Omega_{\text{wk}} = \left\{ \nu : T \rightarrow \bigsqcup_t H_t : \sup_t \|\nu(t)\| < \infty \text{ and } \nu \text{ is weakly continuous} \right\}.$$

We will call the quadruple $(T, \{H_t\}_{t \in T}, \Omega, \Omega_{\text{wk}})$ a weakly continuous Hilbert bundle over T .

We remark that when T is compact, Ω_{wk} is a $C(T)$ -module under the pointwise module action, and also $\Omega \subset \Omega_{\text{wk}}$ (because then every continuous field on T is bounded). However, the function $t \mapsto \langle \nu_1(t), \nu_2(t) \rangle$ is generally not continuous for arbitrary $\nu_1, \nu_2 \in \Omega_{\text{wk}}$. Thus, although Ω_{wk} is, algebraically, a module over $C_b(T)$, it is not in general an inner product module over $C_b(T)$. Nevertheless, if T has the right topology—namely that of a Stonean space—then we show (Theorem 2.6) that it is possible to endow a weakly continuous Hilbert bundle with the structure of a C^* -module over the C^* -algebra of continuous complex-valued functions on T .

The continuous trace C^* -algebras we consider herein were first studied by Fell [10]. We now recall their definition.

Assume that $\{A_t\}_{t \in T}$ is a family of C^* -algebras indexed by the locally compact Hausdorff topological space T . An operator field is a map $a : T \rightarrow \bigsqcup_t A_t$ such that $a(t) \in A_t$, for each $t \in T$.

Definition 1.3. Let $(T, \{H_t\}_{t \in T}, \Omega)$ be a continuous Hilbert bundle. An operator field $a : T \rightarrow \bigsqcup_{t \in T} K(H_t)$ is:

i. almost finite-dimensional (with respect to Ω) if for each $t_0 \in T$ and $\varepsilon > 0$ there exist an open set $U \subset T$ containing t_0 and $\omega_1, \dots, \omega_n \in \Omega$ such that

- (a) $\omega_1(t), \dots, \omega_n(t)$ are linearly independent for every $t \in U$, and
- (b) $\|p_t a(t) p_t - a(t)\| < \varepsilon$ for all $t \in U$, where $p_t \in B(H_t)$ is the projection with range $\text{Span} \{\omega_j(t) : 1 \leq j \leq n\}$;

ii. weakly continuous (with respect to Ω) if the complex-valued function

$$t \longmapsto \langle a(t)\omega_1(t), \omega_2(t) \rangle$$

is continuous for every $\omega_1, \omega_2 \in \Omega$.

Definition 1.4. ([10]) Let $(T, \{H_t\}_{t \in T}, \Omega)$ be a continuous Hilbert bundle. The Fell algebra of the Hilbert bundle $(T, \{H_t\}_{t \in T}, \Omega)$, denoted by $A = A(T, \{H_t\}_{t \in T}, \Omega)$, is the set of all weakly continuous, almost finite-dimensional operator fields $a : T \rightarrow \bigsqcup_{t \in T} K(H_t)$ for which $t \mapsto \|a(t)\|$ is continuous and vanishes at infinity, endowed with pointwise operations and norm

$$\|a\| = \max_{t \in T} \|a(t)\|, \quad a \in A.$$

We shall make repeated use of the following fact about the Fell algebras of Hilbert bundles: if $A = A(T, \{H_t\}_{t \in T}, \Omega)$, for some continuous Hilbert bundle $(T, \{H_t\}_{t \in T}, \Omega)$, then A is a continuous trace C^* -algebra with spectrum $\hat{A} \simeq T$ [10, Theorems 4.4, 4.5].

2 An AW^* -module Structure for Ω_{wk}

Assume henceforth that $T = \Delta$ is a Stonean space; that is, Δ is Hausdorff, compact, and extremely disconnected. The abelian C^* -algebra $C(\Delta)$ is an AW^* -algebra and so one may ask whether the C^* -modules Ω and Ω_{wk} are AW^* -modules in the sense of Kaplansky [16]. We shall show that this is indeed true for the module Ω_{wk} . As a consequence of this last fact we shall get that the C^* -algebra $B(\Omega_{\text{wk}})$ of bounded adjointable endomorphisms of Ω_{wk} is an AW^* -algebra of type I.

The following lemmas are needed to describe the $C(\Delta)$ -Hilbert module structure of Ω_{wk} .

Lemma 2.1. Let $f : \Delta \rightarrow \mathbb{R}$ be a lower semicontinuous function such that there exist $g \in C(\Delta)$ and a meagre set $M \subset \Delta$ with $f(s) = g(s)$ for all $s \in \Delta \setminus M$. Then

$$\sup_{s \in \Delta} g(s) = \sup_{s \in \Delta \setminus M} f(s) = \sup_{s \in \Delta} f(s).$$

Proof. Let $\rho = \sup_{s \in \Delta \setminus M} f(s) = \sup_{s \in \Delta \setminus M} g(s) \leq \sup_{s \in \Delta} g(s)$; then $f(s) \leq \rho$ for all $s \in \Delta \setminus M$. Because Δ is a Baire space, $\overline{\Delta \setminus M} = \Delta$; thus, by the lower semi-continuity, $f(s) \leq \rho$ for every $s \in \Delta$. The same argument yields that $g(s) \leq \rho$ for all $s \in \Delta$. \square

Lemma 2.2. *Assume that $(\Delta, \{H_s\}_{s \in \Delta}, \Omega)$ is a continuous Hilbert bundle and $\nu \in \Omega_{\text{wk}}$. Then*

- i. the function $s \mapsto \|\nu(s)\|^2$ is lower semicontinuous;*
- ii. there is a meagre subset $M \subset \Delta$ and a continuous function $h : \Delta \rightarrow \mathbb{R}_+$ such that*

- (a) $h(s) = \|\nu(s)\|^2$ for all $s \in \Delta \setminus M$, and*
- (b) $\|h\| = \sup_{s \in \Delta \setminus M} \|\nu(s)\|^2 = \sup_{s \in \Delta} \|\nu(s)\|^2$.*

Proof. Let $r \in \mathbb{R}$ be fixed and consider $U_r = \{s \in \Delta : r < \|\nu(s)\|^2\}$. We aim to show that U_r is open. Choose $s_0 \in U_r$. Thus, $r < \|\nu(s_0)\|^2$. By Parseval's formula, there are orthonormal vectors $\xi_1, \dots, \xi_n \in H_{s_0}$ such that $r < \sum_{j=1}^n |\langle \nu(s_0), \xi_j \rangle|^2 \leq \|\nu(s_0)\|^2$. Choose any $\mu_1, \dots, \mu_n \in \Omega$ such that $\mu_j(s_0) = \xi_j$, for each j . Because ξ_1, \dots, ξ_n are orthogonal, $\mu_1(s), \dots, \mu_n(s)$ are linearly independent in an open neighbourhood of s_0 . Hence, by [10, Lemma 4.2], there is an open set V containing s_0 and vector fields $\omega_1, \dots, \omega_n \in \Omega$ such that $\omega_1(s), \dots, \omega_n(s)$ are orthonormal for all $s \in V$, and $\omega_j(s_0) = \xi_j$ for each j . The function

$$g(s) = \sum_{j=1}^n |\langle \nu(s), \omega_j(s) \rangle|^2$$

on Δ is continuous and satisfies $g(s) \leq \|\nu(s)\|^2$, for every $s \in V$, and $r < g(s_0)$. Therefore, by the continuity of g , there is an open set $W \subset V$ containing s_0 such that $r < g(s) \leq \|\nu(s)\|^2$ for all $s \in W$. This proves that U_r contains an open set around each of its points. That is, U_r is open.

Because every bounded nonnegative lower semicontinuous function on a Stonean space Δ agrees with a nonnegative continuous function off a meagre set [24, Proposition III.1.7], the function $h \in C(\Delta)$ as in (ii) exists and satisfies $h(s) = \|\nu(s)\|^2$ for $s \in \Delta \setminus M$.

The last statement follows from Lemma 2.1. □

Let $(\Delta, \{H_t\}_{t \in \Delta}, \Omega, \Omega_{\text{wk}})$ be a weakly continuous Hilbert bundle over Δ . Given $\nu \in \Omega_{\text{wk}}$, the function h that arises in Lemma 2.2 will be denoted by $\langle \nu, \nu \rangle$. There is no ambiguity in so doing because if $h_1, h_2 \in C(\Delta)$ and if $h_1(s) = h_2(s)$ for all $s \notin (M_1 \cup M_2)$ for some meagre subsets M_1 and M_2 , then h_1 and h_2 agree on Δ . (If not, then by continuity, h_1 and h_2 would

differ on an open set U ; but $\emptyset \neq U \subset M_1 \cup M_2$ is in contradiction to the fact that no meagre set in a Baire space can contain a nonempty open set.)

Now use the polarisation identity to define $\langle \nu_1, \nu_2 \rangle \in C(\Delta)$ for any pair $\nu_1, \nu_2 \in \Omega_{\text{wk}}$. This gives Ω_{wk} the structure of pre-inner product module over $C(\Delta)$ whereby for each $\nu_1, \nu_2 \in \Omega_{\text{wk}}$ there is a meagre subset $M_{\nu_1, \nu_2} \subset \Delta$ such that the continuous function $\langle \nu_1, \nu_2 \rangle$ satisfies

$$\langle \nu_1, \nu_2 \rangle (s) = \langle \nu_1(s), \nu_2(s) \rangle, \quad \forall s \in \Delta \setminus M_{\nu_1, \nu_2}.$$

In particular, if $\nu \in \Omega_{\text{wk}}$ and $\omega \in \Omega$, then

$$\langle \nu, \omega \rangle (s) = \langle \nu(s), \omega(s) \rangle, \quad \forall s \in \Delta.$$

In fact, Ω_{wk} is an inner product module over $C(\Delta)$, for if $\nu \in \Omega_{\text{wk}}$ satisfies $\langle \nu, \nu \rangle = 0$, then Lemma 2.2 yields $\|\nu(s)\|^2 = 0$ for all $s \in \Delta$. Therefore,

$$\|\nu\| = \|\langle \nu, \nu \rangle\|^{1/2}, \quad \nu \in \Omega_{\text{wk}},$$

defines a norm on Ω_{wk} , where

$$\|\nu\|^2 = \sup_{s \in \Delta} \langle \nu(s), \nu(s) \rangle = \|\langle \nu, \nu \rangle\|. \quad (1)$$

Recall that given a C^* -algebra B , a *Hilbert C^* -module over B* is a left B -module E together with a B -valued definite sesquilinear map $\langle \cdot, \cdot \rangle$ such that E is complete with the norm $\|\nu\| = \|\langle \nu, \nu \rangle\|^{1/2}$ (we refer to [17] for a detailed account on Hilbert modules).

Note that if $\nu \in \Omega_{\text{wk}}$, then $|\nu|(s) := \langle \nu, \nu \rangle^{1/2}(s) \geq \|\nu(s)\|$ for $s \in \Delta$ and there exists a meagre set $M \subset \Delta$ with $|\nu|(s) = \|\nu(s)\|$ if $s \in (\Delta \setminus M)$ (Lemma 2.2). These facts will be used repeatedly from now on.

Proposition 2.3. Ω_{wk} is a C^* -module over $C(\Delta)$ and Ω is a C^* -submodule of Ω_{wk} .

Proof. The only Hilbert C^* -module axiom that is not obviously satisfied by Ω_{wk} is the axiom of completeness. Let $\{\nu_i\}_{i \in \mathbb{N}}$ be a Cauchy sequence in Ω_{wk} . By the equality (1), $\{\nu_i(s)\}_{i \in \mathbb{N}}$ is a Cauchy sequence in H_s for every $s \in \Delta$. Let $\nu(s) \in H_s$ denote the limit of this sequence so that $\nu : \Delta \rightarrow \bigsqcup_{s \in \Delta} H_s$ is a vector field.

Choose $\omega \in \Omega$ and consider the function $g_{i, \omega} \in C(\Delta)$ given by $g_{i, \omega}(s) = \langle \omega(s), \nu_i(s) \rangle$. Let $\varepsilon > 0$. Then there is $N_\varepsilon \in \mathbb{N}$ such that $\|\nu_i - \nu_j\| < \varepsilon$, for all $i, j \geq N_\varepsilon$. Therefore, the Cauchy-Schwarz inequality yields

$$\sup_{s \in \Delta} |g_{i, \omega}(s) - g_{j, \omega}(s)| < \varepsilon \|\omega\|, \quad \forall i, j \geq N_\varepsilon.$$

Thus, the sequence $\{g_{i,\omega}\}_i$ is Cauchy in $C(\Delta)$; let $g_\omega \in C(\Delta)$ denote its limit. Observe that $g_\omega(s) = \lim_i \langle \nu_i(s), \omega(s) \rangle = \langle \nu(s), \omega(s) \rangle$, for all $s \in \Delta$. As the choice of $\omega \in \Omega$ is arbitrary, this shows that ν is weakly continuous. The Cauchy sequence $\{\nu_i\}_{i \in \mathbb{N}}$ is necessarily uniformly bounded by, say, $\rho > 0$, and then $\|\nu(s)\| \leq \rho$ for every $s \in \Delta$. That is, the function $s \rightarrow \|\nu(s)\|$ is bounded and so $\nu \in \Omega_{\text{wk}}$. Finally, if $i, j \geq N_\varepsilon$, then for any $s \in \Delta$ we have $\|\nu(s) - \nu_i(s)\| \leq \|\nu(s) - \nu_j(s)\| + \|\nu_j(s) - \nu_i(s)\| \leq \|\nu(s) - \nu_j(s)\| + \varepsilon$, and so letting $j \rightarrow \infty$ yields $\|\nu(s) - \nu_i(s)\| \leq \varepsilon$ for every $s \in \Delta$. That is, $\|\nu - \nu_i\| \rightarrow 0$, which proves that Ω_{wk} is complete.

For the case of Ω , let $\{\omega_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in Ω . For each $s \in \Delta$, $\{\omega_n(s)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in H_s ; let $\omega(s)$ denote the limit. Since the limit is uniform, it is in particular locally uniform, and so $\omega \in \Omega$. Hence, Ω is complete. \square

Definition 2.4. A Hilbert C^* -module E over a C^* -algebra B is called a Kaplansky–Hilbert module if in addition B is an abelian AW^* -algebra and the following three properties hold [16, p. 842] (Kaplansky’s original term for such a module was “faithful AW^* -module”):

- i.* if $e_i \cdot \nu = 0$ for some family $\{e_i\}_i \subset B$ of pairwise-orthogonal projections and $\nu \in E$, then also $e \cdot \nu = 0$, where $e = \sup_i e_i$;
- ii.* if $\{e_i\}_i \subset B$ is a family of pairwise-orthogonal projections such that $1 = \sup_i e_i$, and if $\{\nu_i\}_i \subset E$ is a bounded family, then there is a $\nu \in E$ such that $e_i \cdot \nu = e_i \cdot \nu_i$ for all i ;
- iii.* if $\nu \in E$, then $g \cdot \nu = 0$ for all $g \in B$ only if $\nu = 0$.

Remark 2.5. The element $\nu \in E$ obtained in the situation described in (ii) will sometimes be denoted as $\sum_i e_i \nu_i$. It should be emphasized that this is not a pointwise sum.

Theorem 2.6. Ω_{wk} is a Kaplansky–Hilbert module over $C(\Delta)$.

Proof. For property (i), assume that $\nu \in \Omega_{\text{wk}}$ and $\{e_i\}_i \subset C(\Delta)$ is a family of pairwise-orthogonal projections with supremum $e \in C(\Delta)$ for which $e_i \cdot \nu = 0$ for all i . Because projections in $C(\Delta)$ are the characteristic functions of clopen sets, there are pairwise-disjoint clopen sets $U_i \subset \Delta$ such that $e_i = \chi_{U_i}$. Thus, for each i , using Lemma 2.2,

$$\begin{aligned} 0 &= \|e_i \cdot \nu\|^2 = \max_{s \in \Delta} \langle e_i \cdot \nu, e_i \cdot \nu \rangle(s) = \sup_{s \in \Delta} \langle e_i(s)\nu(s), e_i(s)\nu(s) \rangle \\ &= \max_{s \in \Delta} e_i(s) [\langle \nu, \nu \rangle(s)] = \max_{s \in U_i} \langle \nu, \nu \rangle(s), \end{aligned}$$

and so $\langle \nu, \nu \rangle(s) = 0$ for every $s \in U_i$. Let $U = \bigcup_i U_i$. The set \bar{U} is clopen and $\chi_{\bar{U}} = \sup_i e_i = e$ [5, §8]. As $\langle \nu, \nu \rangle$ is a continuous function that vanishes on U , it also vanishes on \bar{U} . Hence,

$$\|e \cdot \nu\|^2 = \max_{s \in \Delta} e(s) [\langle \nu, \nu \rangle(s)] = \max_{s \in \bar{U}} \langle \nu, \nu \rangle(s) = 0,$$

which yields property (i).

For the proof of property (ii), assume that $\{e_i\}_i \subset C(\Delta)$ is a family of pairwise-orthogonal projections such that $1 = \sup_i e_i$ and that $\{\nu_i\}_i \subset \Omega_{\text{wk}}$ is a family such that $K = \sup \|\nu_i\| < \infty$; we aim to prove that there is a $\nu \in \Omega_{\text{wk}}$ such that $e_i \cdot \nu = e_i \cdot \nu_i$ for all i . As before, assume that $e_i = \chi_{U_i}$ and $U = \bigcup_i U_i$. Then $1 = \sup_i e_i$ implies that $\bar{U} = \Delta$.

For each $\omega \in \Omega$, consider the unique function $f_\omega \in C(\Delta)$ such that $e_i f_\omega = e_i \langle \omega, \nu_i \rangle$ for all i (its existence guaranteed by the fact that Δ is the Stone–Čech compactification of U). Note that for $s \in U_i$ we have that $f_\omega(s) = \langle \omega(s), \nu_i(s) \rangle$. Hence, $|f_\omega(s)| \leq K \|\omega(s)\|$ for $s \in U$; the same inequality holds for all $s \in \Delta$ because $\bar{U} = \Delta$ and both sides of the inequality are continuous functions of s . Moreover, if $\omega_1, \omega_2 \in \Omega$ and $\alpha \in \mathbb{C}$ then, for $s \in U$ we get that $f_{\alpha \omega_1 + \omega_2}(s) = \alpha f_{\omega_1}(s) + f_{\omega_2}(s)$ and, therefore, that $f_{\alpha \omega_1 + \omega_2} = \alpha f_{\omega_1} + f_{\omega_2}$. Thus, for each $s \in \Delta$ the function $\omega(s) \mapsto f_\omega(s)$ is a well-defined, bounded linear functional on H_s . Let $\nu(s) \in H_s$ be the representing vector for this functional, yielding a vector field $\nu : \Delta \rightarrow \bigsqcup_{s \in \Delta} H_s$. Since $\langle \nu(s), \omega(s) \rangle = \overline{f_\omega(s)}$, for every $\omega \in \Omega$, ν is weakly continuous. It remains to show that ν is a bounded vector field. If $s \in U$,

$$\|\nu(s)\| = \sup_{\omega \in \Omega, \|\omega(s)\|=1} |\langle \omega(s), \nu(s) \rangle| = \sup_{\omega \in \Omega, \|\omega(s)\|=1} |f_\omega(s)| \leq \sup_i \|\nu_i\| = K,$$

which shows that $\|\nu(s)\|$ is uniformly bounded on U . Thus, since U is dense, the lower semicontinuous function $s \mapsto \|\nu(s)\|^2$ is bounded on Δ . Therefore, $\nu \in \Omega_{\text{wk}}$.

Now we show that $e_i \cdot \nu = e_i \cdot \nu_i$, for all i . Fix i and $s \in U_i$ and consider $\omega \in \Omega$. Then,

$$\begin{aligned} \langle \omega(s), e_i(s) \nu(s) \rangle &= \langle \omega(s), \nu(s) \rangle = f_\omega(s) \\ &= e_i(s) f_\omega(s) = e_i(s) \langle \omega(s), \nu_i(s) \rangle \\ &= \langle \omega(s), e_i(s) \nu_i(s) \rangle. \end{aligned}$$

Since $(e_i \cdot \nu)(s) = 0 = (e_i \cdot \nu_i)(s)$ for $s \in \Delta \setminus U_i$ we conclude that $e_i \cdot \nu = e_i \cdot \nu_i$.

For the proof of property (iii), assume that $\nu \in \Omega_{\text{wk}}$ satisfies $g \cdot \nu = 0$ for all $g \in C(\Delta)$. Then, in particular, $\langle \nu, \nu \rangle \cdot \nu = 0$, so $\langle \nu, \nu \rangle = 0$. Hence, from $\|\nu\| = \|\langle \nu, \nu \rangle\|^{1/2} = 0$ we conclude that $\nu = 0$. \square

3 Endomorphisms of Ω and Ω_{wk}

Throughout this section A will denote the Fell C^* -algebra of the continuous Hilbert bundle $(\Delta, \{H_s\}_{s \in \Delta}, \Omega)$, as described in Definition 1.4, with Δ Stonean. Let $B(\Omega)$ and $B(\Omega_{\text{wk}})$ denote, respectively, the C^* -algebras of adjointable $C(\Delta)$ -endomorphisms of Ω and Ω_{wk} . Since, by Theorem 2.6, Ω_{wk} is a Kaplansky–Hilbert AW^* -module over $C(\Delta)$, $B(\Omega_{\text{wk}})$ coincides with the set of all $C(\Delta)$ -endomorphisms of Ω_{wk} [16, Theorem 6] and is a type I AW^* -algebra with centre $C(\Delta)$ [16, Theorem 7].

In the particular case where Ω is given by the trivial Hilbert bundle $(\Delta, \{H\}_{s \in \Delta}, C(\Delta, H))$ with H is a fixed Hilbert space, Hamana [15] proved that $B(\Omega_{\text{wk}}) \cong C(\Delta) \overline{\otimes} B(H)$, the monotone complete tensor product of $C(\Delta)$ and $B(H)$.

For each $\nu_1, \nu_2 \in \Omega_{\text{wk}}$, consider the endomorphism Θ_{ν_1, ν_2} on Ω_{wk} defined by

$$\Theta_{\nu_1, \nu_2}(\nu) = \langle \nu, \nu_2 \rangle \cdot \nu_1, \quad \nu \in \Omega_{\text{wk}}.$$

For a Hilbert bundle Ω_0 , let

$$F(\Omega_0) = \left\{ \sum_{j=1}^n \Theta_{\omega_j, \omega'_j} : n \in \mathbb{N}, \omega_j, \omega'_j \in \Omega \right\}.$$

We will consider both $F(\Omega)$ and $F(\Omega_{\text{wk}})$.

If $\omega_1, \omega_2 \in \Omega$, then $\Theta_{\omega_1, \omega_2}(\omega) \in \Omega$ for all $\omega \in \Omega$, and so $F(\Omega) \subset B(\Omega)$. In fact, $F(\Omega)$ and $F(\Omega_{\text{wk}})$ are algebraic ideals in $B(\Omega)$ and $B(\Omega_{\text{wk}})$ respectively. The norm-closures of these algebraic ideals, namely $K(\Omega)$ and $K(\Omega_{\text{wk}})$, are essential ideals in each of $B(\Omega)$ and $B(\Omega_{\text{wk}})$ —called the ideals of compact endomorphisms—and the multiplier algebras of $K(\Omega)$ and $K(\Omega_{\text{wk}})$ are, respectively, $B(\Omega)$ and $B(\Omega_{\text{wk}})$ [17].

When referring to rank-1 operators x acting on a Hilbert space H , we will use the notation $x = \xi \otimes \eta$ for such an operator—the action on $\gamma \in H$ given by $\gamma \mapsto \langle \gamma, \eta \rangle \xi$ —and we reserve the notation $\Theta_{\xi, \eta}$ for “rank-1” operators acting on a Hilbert module.

The term “homomorphism” will be used to mean a $*$ -homomorphism between C^* -algebras.

For any C^* -algebra B , we denote the injective envelope [13], [18, Chapter 15] of B by $I(B)$ (and we consider $I(B)$ as a C^* -algebra rather than as an operator system).

The main result of the present section is the following.

Theorem 3.1. *There exist C^* -algebra embeddings such that*

$$K(\Omega) \subset A \subset B(\Omega) \subset B(\Omega_{\text{wk}}) = I(K(\Omega)). \quad (2)$$

In particular, $I(K(\Omega)) = I(A) = I(B(\Omega)) = B(\Omega_{\text{wk}})$.

The proof of Theorem 3.1 and a description of the inclusions in (2) begin with the following set of results.

Lemma 3.2. *For every $a \in A$ and $\omega \in \Omega$, the vector field $a \cdot \omega$ defined by $a \cdot \omega(s) = a(s)\omega(s)$ is an element of Ω .*

Proof. Let $a \in A$. Then $a^*a \in A_+$ and since all fields in A are weakly continuous, for every $\omega \in \Omega$ the map $s \mapsto \|a(s)\omega(s)\| = \langle a^*a \cdot \omega(s), \omega(s) \rangle^{1/2}$ is continuous.

Suppose $s_0 \in \Delta$ and $\varepsilon > 0$. Because $H_{s_0} = \{\mu(s_0) : \mu \in \Omega\}$, there is a $\mu \in \Omega$ such that $a(s_0)\omega(s_0) = \mu(s_0)$. Since

$$\|a \cdot \omega(s) - \mu(s)\|^2 = \|a(s)\omega(s)\|^2 + \|\mu(s)\|^2 - 2\text{Re} \langle a(s)\omega(s), \mu(s) \rangle$$

is continuous on Δ and vanishes at s_0 , there is an open set $U \subset \Delta$ containing s_0 such that $\|a \cdot \omega(s) - \mu(s)\| < \varepsilon$ for all $s \in U$. As Ω is closed under local uniform approximation, this proves that $a \cdot \omega \in \Omega$. \square

Proposition 3.3. *The map $\varrho : A \rightarrow B(\Omega)$ given by $\varrho(a)\omega = a \cdot \omega$, for $a \in A$ and $\omega \in \Omega$ is an isometric homomorphism. Furthermore, $K(\Omega) \subset \varrho(A) \subset B(\Omega)$ as C^* -algebras.*

Proof. It is clear that ϱ is a homomorphism, and so we only need to verify that it is one-to-one. To this end, assume that $\varrho(a) = 0$. Thus, $a(s)\omega(s) = 0$ for every $\omega \in \Omega$ and every $s \in \Delta$. Because $H_s = \{\omega(s) : \omega \in \Omega\}$, this implies that $a(s) = 0$ for all $s \in \Delta$, and so $a = 0$.

To show $K(\Omega) \subset \varrho(A) \subset B(\Omega)$ as C^* -algebras, consider $\Theta_{\omega_1, \omega_2}$ with $\omega_1, \omega_2 \in \Omega$. The map $s \mapsto \|\Theta_{\omega_1(s), \omega_2(s)}\|$ is continuous because $\|\Theta_{\omega_1(s), \omega_2(s)}\| = \|\omega_1(s)\| \|\omega_2(s)\|$. For any $\eta_1, \eta_2 \in \Omega$, the map

$$\langle \Theta_{\omega_1, \omega_2} \cdot \eta_1, \eta_2 \rangle(s) = \langle \eta_1, \omega_2 \rangle(s) \langle \omega_1, \eta_2 \rangle(s) = \langle \eta_1(s), \omega_2(s) \rangle \langle \omega_1(s), \eta_2(s) \rangle$$

is continuous. So $\Theta_{\omega_1, \omega_2}$ is also finite dimensional and weakly continuous, which shows that $\Theta_{\omega_1, \omega_2} \in A$ and $K(\Omega) \subset \varrho(A)$. \square

Lemma 3.4. *With respect to the inclusion $\Omega \subset \Omega_{\text{wk}}$, we have $\Omega^\perp = \{0\}$.*

Proof. Let $\nu \in \Omega_{\text{wk}}$ be such that $\langle \nu, \omega \rangle = 0$, for every $\omega \in \Omega$. That is, for every $\omega \in \Omega$ and for every $s \in \Delta$, $\langle \nu(s), \omega(s) \rangle = 0$. If $\nu \neq 0$, there exists $s_0 \in \Delta$ such that $\nu(s_0) \neq 0$. By axiom (II) in Definition 1.1, there exists $\omega \in \Omega$ such that $\omega(s_0) = \nu(s_0)$, in contradiction to $\langle \nu(s_0), \omega(s_0) \rangle = 0$. \square

Lemma 3.5. *If $t_0 \in \Delta$ and $\xi \in H_{t_0}$, then there exists $\omega \in \Omega$ such that $\omega(t_0) = \xi$ and $\|\omega\| = \|\xi\|$.*

Proof. The case $\xi = 0$ is trivial. So assume that $\|\xi\| > 0$. Let $\omega' \in \Omega$ with $\omega'(t_0) = \xi$. Fix a clopen neighbourhood V of t_0 such that $V \subset \{t \in T : \|\omega'(t)\| \geq \|\omega'(t_0)\|/2\}$. Let $h'(\cdot) = \|\xi\| \cdot \|\omega'(\cdot)\|^{-1} \in C(V)$; then h' extends to a continuous function $h \in C(\Delta)$ with $h|_{\Delta \setminus V} = 0$. It is now straightforward to show that $\omega = h \cdot \omega' \in \Omega$ has the desired properties. \square

Proposition 3.6. *There exists an isometric homomorphism $\vartheta : B(\Omega) \rightarrow B(\Omega_{\text{wk}})$ such that for $a \in A$, $\nu \in \Omega_{\text{wk}}$,*

$$(\vartheta(\varrho(a))\nu)(s) = a(s)\nu(s), \quad s \in \Delta. \quad (3)$$

Proof. Assume that $b \in B(\Omega)$ and $\omega \in \Omega$, $s \in \Delta$. By Lemma 3.5,

$$\begin{aligned} \|(b\omega)(s)\| &= \sup_{\xi \in H_s, \|\xi\|=1} |\langle (b\omega)(s), \xi \rangle| = \sup_{\eta \in \Omega, \|\eta\|=1} |\langle (b\omega)(s), \eta(s) \rangle| \\ &= \sup_{\eta \in \Omega, \|\eta\|=1} |\langle b\omega, \eta \rangle(s)| = \sup_{\eta \in \Omega, \|\eta\|=1} |\langle \omega(s), (b^*\eta)(s) \rangle| \\ &\leq \|\omega(s)\| \sup_{\eta \in \Omega, \|\eta\|=1} \|b^*\eta\| \leq \|\omega(s)\| \|b^*\| = \|\omega(s)\| \|b\|. \end{aligned}$$

Therefore the function $\omega(s) \mapsto (b\omega)(s)$ is well defined and induces a bounded linear operator $b(s) \in B(H_s)$ such that $(b\omega)(s) = b(s)\omega(s)$, for $s \in \Delta$ and $\omega \in \Omega$, with $\sup_{s \in \Delta} \|b(s)\| \leq \|b\|$. Moreover,

$$\begin{aligned} \|b\| &= \sup_{\|\omega\|=1} \|b \cdot \omega\| = \sup_{\|\omega\|=1} \sup_s \|b \cdot \omega(s)\| = \sup_{\|\omega\|=1} \sup_s \|b(s)\omega(s)\| \\ &\leq \sup_{\|\omega\|=1} \sup_s \|b(s)\| \|\omega(s)\| \leq \sup_s \|b(s)\| \leq \|b\|, \end{aligned}$$

and so $\sup_{s \in \Delta} \|b(s)\| = \|b\|$. Suppose now that $\nu \in \Omega_{\text{wk}}$ and $s \in \Delta$, and define a vector field $\vartheta b \nu$ by $(\vartheta b \nu)(s) = b(s)\nu(s)$. If $\eta \in \Omega$, then

$$\langle (\vartheta b \nu)(s), \eta(s) \rangle = \langle \nu(s), b(s)^*\eta(s) \rangle = \langle \nu(s), (b^*\eta)(s) \rangle$$

is continuous, which shows that $\vartheta b \nu$ is weakly continuous with respect to Ω . Since $\vartheta b \nu$ is also uniformly bounded, we conclude that $\vartheta b \nu \in \Omega_{\text{wk}}$.

It is straightforward to show that the map $\nu \mapsto \vartheta b \nu$ is a bounded $C(\Delta)$ -endomorphism of Ω_{wk} and hence it gives rise to an element $\vartheta b \in B(\Omega_{\text{wk}})$. It is clear that ϑ is a homomorphism. If $\vartheta b = 0$, then $b(s)\omega(s) = 0$ for all $\omega \in \Omega$, $s \in \Delta$ and so $b(s) = 0$ for all s ; then $\|b\| = \sup_s \|b(s)\| = 0$, and $b = 0$. So ϑ is one-to-one, and thus isometric. Finally, it is clear that (3) holds by construction. \square

One consequence of the proof of Proposition 3.6 is that for every $b \in B(\Omega)$ there exists an operator field $\{b(s)\}_{s \in \Delta}$ acting on the Hilbert bundle $\{H_s\}_{s \in \Delta}$ such that $(b\omega)(s) = b(s)\omega(s)$, for every $s \in \Delta$. This property, however, is not shared by all elements of $B(\Omega_{\text{wk}})$.

Lemma 3.7. *If $z \in B(\Omega_{\text{wk}})$ and $\Theta_{\omega,\omega} z \Theta_{\mu,\mu} = 0$ for all $\omega, \mu \in \Omega$, then $z = 0$.*

Proof. For any $\xi, \omega, \mu \in \Omega$ we have that

$$0 = \Theta_{\omega,\omega} z \Theta_{\mu,\mu} \xi = \langle \xi, \mu \rangle \langle z\mu, \omega \rangle \omega.$$

Hence, we get that

$$0 = \langle \xi, \mu \rangle |\langle z\mu, \omega \rangle|^2 = \langle \xi, \mu \rangle |\langle \mu, z^*\omega \rangle|^2.$$

We are free to choose $\xi, \mu \in \Omega$. Fix s , and choose μ with $\mu(s) = z^*\omega(s)$; let $\xi = \mu$. Then, as $\mu \in \Omega$, we get $0 = \langle \mu, \mu \rangle(s) = \langle \mu(s), \mu(s) \rangle$, so $z^*\omega(s) = \mu(s) = 0$. As $s \in \Delta$ is arbitrary, $z^*\omega = 0$ for every $\omega \in \Omega$. For any $\nu \in \Omega_{\text{wk}}$ and every $\omega \in \Omega$, $\langle z\nu, \omega \rangle = \langle \nu, z^*\omega \rangle = 0$. By Lemma 3.4 we conclude that $z\nu = 0$ for $\nu \in \Omega_{\text{wk}}$ and hence $z = 0$. \square

Proof of Theorem 3.1. We consider the embeddings $A \xrightarrow{\varrho} B(\Omega)$ and $B(\Omega) \xrightarrow{\vartheta} B(\Omega_{\text{wk}})$ defined in Propositions 3.3 and 3.6. In this way, we get the inclusions in (2).

Because $B(\Omega_{\text{wk}})$ is a type I AW*-algebra, it is injective [14, Proposition 5.2]. To show that $B(\Omega_{\text{wk}})$ is the injective envelope $I(K(\Omega))$ of $K(\Omega)$, we need to show that the embedding $\vartheta \circ \varrho$ of $K(\Omega)$ into $B(\Omega_{\text{wk}})$ is rigid [18, Theorem 15.8]: that is, we aim to prove that if $\phi : B(\Omega_{\text{wk}}) \rightarrow B(\Omega_{\text{wk}})$ is a unital completely positive linear map for which $\phi|_{K(\Omega)} = \text{id}_{K(\Omega)}$, then $\phi = \text{id}_{B(\Omega_{\text{wk}})}$.

Let $\phi : B(\Omega_{\text{wk}}) \rightarrow B(\Omega_{\text{wk}})$ be such a ucp map with $\phi|_{K(\Omega)} = \text{id}_{K(\Omega)}$. Suppose that $z \in B(\Omega_{\text{wk}})$ and $\omega, \mu \in \Omega$. Then $\Theta_{\omega,\omega} z \Theta_{\mu,\mu} = \Theta_{\langle z\mu, \omega \rangle \omega, \mu} \in K(\Omega)$. Because $K(\Omega)$ is in the multiplicative domain of ϕ , we have that $\phi(axb) = a\phi(x)b$ for all $x \in B(\Omega_{\text{wk}})$ and $a, b \in K(\Omega)$. This implies that

$$\Theta_{\omega,\omega} \phi(z) \Theta_{\mu,\mu} = \phi(\Theta_{\omega,\omega} z \Theta_{\mu,\mu}) = \phi(\Theta_{\langle z\mu, \omega \rangle \omega, \mu}) = \Theta_{\langle z\mu, \omega \rangle \omega, \mu} = \Theta_{\omega,\omega} z \Theta_{\mu,\mu},$$

and so $\Theta_{\omega,\omega}(z-\phi(z))\Theta_{\mu,\mu} = 0$. Since ω, μ were arbitrary, Lemma 3.7 implies that $z - \phi(z) = 0$ and so $\phi = \text{id}_{B(\Omega_{\text{wk}})}$.

We have shown above that the inclusion $K(\Omega) \subset B(\Omega_{\text{wk}})$ is rigid. Moreover, $K(\Omega)$ is an essential ideal of $B(\Omega)$ and $K(\Omega) \subset A \subset B(\Omega)$. Hence, $I(K(\Omega)) = I(A) = I(B(\Omega)) = B(\Omega_{\text{wk}})$. \square

We conclude this section with a remark about the ideal $K(\Omega_{\text{wk}})$ of $B(\Omega_{\text{wk}})$. In type I AW*-algebras, the ideal generated by the abelian projections has a prominent role. As it happens, $K(\Omega_{\text{wk}})$ is precisely this ideal.

Proposition 3.8. *The C*-algebra $K(\Omega_{\text{wk}})$ coincides with the ideal $J \subset B(\Omega_{\text{wk}})$ generated by the abelian projections of $B(\Omega_{\text{wk}})$. So $K(\Omega_{\text{wk}})$ is a liminal C*-algebra with Hausdorff spectrum.*

Proof. By [16, Lemma 13], a projection $e \in B(\Omega_{\text{wk}})$ is abelian if and only if there exists $\nu \in \Omega_{\text{wk}}$ such that $|\nu|$ is a projection in $C(\Delta)$ and $e = \Theta_{\nu,\nu}$. Hence, $J \subset K(\Omega_{\text{wk}})$.

To show that $K(\Omega_{\text{wk}}) \subset J$, assume $\nu \in \Omega_{\text{wk}}$ is nonzero. Let $\varepsilon > 0$. We will show that there is an $x_\varepsilon \in J$ such that $\|\Theta_{\nu,\nu} - x_\varepsilon\| < \varepsilon$. Let $V \subset \Delta$ be the (clopen) closure of $\{s \in \Delta : |\nu|(s) < \varepsilon^{1/2}\}$, $U = \Delta \setminus V$ (also clopen) and let $g = (1/|\nu|)\chi_U \in C(\Delta)_+$. Then $g|\nu| = \chi_U$ and $\|\chi_{\Delta \setminus U}|\nu|\| < \varepsilon^{1/2}$. Let $\nu' = g \cdot \nu$ so that $|\nu'| = \chi_U$. Hence, $\Theta_{\nu',\nu'} \in J$ and $\Theta_{\nu',\nu'} = g^2 \cdot \Theta_{\nu,\nu}$. Let $x_\varepsilon = |\nu|^2 \cdot \Theta_{\nu',\nu'} \in J$. Then

$$x_\varepsilon = |\nu|^2 \cdot \Theta_{\nu',\nu'} = |\nu|^2 g^2 \Theta_{\nu,\nu} = \chi_U \Theta_{\nu,\nu},$$

and $x_\varepsilon - \Theta_{\nu,\nu} = \chi_{\Delta \setminus U} \cdot \Theta_{\nu,\nu}$. Then

$$\begin{aligned} \|x_\varepsilon - \Theta_{\nu,\nu}\| &= \sup_{\eta \in (\Omega_{\text{wk}})_1} \|\chi_{\Delta \setminus U} \cdot \Theta_{\nu,\nu} \eta\| = \sup_{\eta \in (\Omega_{\text{wk}})_1} \|\chi_{\Delta \setminus U} \cdot \langle \eta, \nu \rangle \nu\| \\ &= \sup_{\eta \in (\Omega_{\text{wk}})_1} \max_{s \in \Delta \setminus U} |\langle \eta, \nu \rangle(s)| \|\nu(s)\| \\ &\leq \sup_{\eta \in (\Omega_{\text{wk}})_1} \max_{s \in \Delta \setminus U} |\eta|(s) |\nu|(s) \|\nu(s)\| \leq \max_{s \in \Delta \setminus U} |\nu|(s)^2 < \varepsilon. \end{aligned}$$

As ε was arbitrary and J is closed, we conclude that $\Theta_{\nu,\nu} \in J$. The polarisation identity then shows that $\Theta_{\nu_1,\nu_2} \in J$ for all $\nu_1, \nu_2 \in \Omega_{\text{wk}}$. Hence, $F(\Omega_{\text{wk}}) \subset J$, and so $K(\Omega_{\text{wk}}) \subset J$.

It remains to justify the last assertion in the statement. By the main result of [12], the ideal generated by the abelian projections in a type I AW*-algebra is liminal and has Hausdorff spectrum. Hence, this is true of $K(\Omega_{\text{wk}})$. \square

4 Multiplier and Local Multiplier Algebras

In the previous section we established the inclusions $K(\Omega) \subset A \subset B(\Omega) \subset B(\Omega_{\text{wk}})$, as C^* -subalgebras, and we showed that $I(A) = B(\Omega_{\text{wk}})$. The present section refines these inclusions to incorporate multiplier algebras and local multiplier algebras.

Given a C^* -algebra C , we denote by $M(C)$ and $M_{\text{loc}}(C)$ its multiplier and local multiplier algebra [2] respectively.

The second order local multiplier algebra of C is $M_{\text{loc}}(M_{\text{loc}}(C))$, the local multiplier algebra of $M_{\text{loc}}(C)$. By [11, Corollary 4.3], the local multiplier algebras (of all orders) of C are C^* -subalgebras of the injective envelope $I(C)$ of C . In particular, $C \subset M_{\text{loc}}(C) \subset M_{\text{loc}}(M_{\text{loc}}(C)) \subset I(C)$ as C^* -subalgebras.

By a well known theorem of Kasparov [2, Theorem 1.2.33], [17, Theorem 2.4], $M(K(\Omega)) = B(\Omega)$. We remark that all the subalgebras we consider are essential in $B(\Omega_{\text{wk}})$ (i.e. the annihilator is zero), and so whenever we write $M(C)$ for one of these subalgebras $C \subset B(\Omega_{\text{wk}})$, we mean the concrete realization [20]

$$M(C) = \{x \in B(\Omega_{\text{wk}}) : xC + Cx \subset C\}.$$

The following theorem is the main result of this section.

Theorem 4.1. *With the notations from the previous sections, we have the equality $M_{\text{loc}}(A) = M_{\text{loc}}(K(\Omega))$ and the following inclusions (as C^* -subalgebras):*

$$\begin{aligned} M(A) &\subset M(K(\Omega)) = B(\Omega) \\ &\subset M_{\text{loc}}(K(\Omega)) \subset M_{\text{loc}}(M_{\text{loc}}(K(\Omega))) = B(\Omega_{\text{wk}}). \end{aligned} \quad (4)$$

In particular, $M_{\text{loc}}(M_{\text{loc}}(A)) = I(A)$.

Ara and Mathieu have presented examples of Stonean spaces Δ and trivial Hilbert bundles Ω where the inclusion $M_{\text{loc}}(K(\Omega)) \subset M_{\text{loc}}(M_{\text{loc}}(K(\Omega)))$ in (4) is proper [3, Theorem 6.13]. As a consequence of Theorem 4.1 and the fact that $B(\Omega_{\text{wk}}) = I(K(\Omega))$, we see that this gap cannot occur for higher local multiplier algebras, i.e. for all $k \geq 2$, $M_{\text{loc}}^{k+1}(K(\Omega)) = M_{\text{loc}}^k(K(\Omega))$ — where $M_{\text{loc}}^{k+1}(K(\Omega)) = M_{\text{loc}}(M_{\text{loc}}^k(K(\Omega)))$ for $k \geq 1$.

The proof of Theorem 4.1 is achieved through a number of lemmas.

Lemma 4.2. *The set*

$$F_+ = \left\{ \sum_{j=1}^n \Theta_{\omega_j, \omega_j} : n \in \mathbb{N}, \omega_j \in \Omega \right\}$$

is dense in the positive cone of $K(\Omega)$.

Proof. Assume that $h \in K(\Omega)_+$ and let $\varepsilon > 0$ be arbitrary. For each $s_0 \in \Delta$ consider the positive compact operator $h(s_0) \in K(H_{s_0})$. Then there are vectors $\xi_1, \dots, \xi_{n_{s_0}} \in H_{s_0}$ such that

$$\|h(s_0) - \sum_{j=1}^{n_{s_0}} \xi_j \otimes \xi_j\| < \varepsilon.$$

Using (II) in Definition 1.1, choose $\omega_1, \dots, \omega_{n_{s_0}} \in \Omega$ such that $\omega_j(s_0) = \xi_j$, $1 \leq j \leq n_{s_0}$, and let $\kappa_{s_0} = \sum_{j=1}^{n_{s_0}} \Theta_{\omega_j, \omega_j}$. By continuity of the operator fields in A , there is an open set $U_{s_0} \subset \Delta$ containing s_0 such that $\|h(s) - \kappa_{s_0}(s)\| < \varepsilon$ for all $s \in U_{s_0}$.

This procedure leads to an open cover $\{U_s\}_{s \in \Delta}$ of Δ , from which (by compactness) there exists a finite subcover $\{U_1, \dots, U_m\}$ and corresponding fields $\kappa_i = \sum_{j=1}^{n_i} \Theta_{\omega_j^{[i]}, \omega_j^{[i]}}$. Let $\{\psi_1, \dots, \psi_m\} \subset C(\Delta)$ be a partition of unity subordinate to $\{U_1, \dots, U_m\}$ and note that $\psi_i \cdot \Theta_{\omega_j^{[i]}, \omega_j^{[i]}} = \Theta_{\psi_i^{1/2} \cdot \omega_j^{[i]}, \psi_i^{1/2} \cdot \omega_j^{[i]}}$ for all j and i . Hence, the field $\kappa = \sum_{i=1}^m \psi_i \cdot \kappa_i$ is in F_+ , and for each $s \in \Delta$,

$$\|h(s) - \kappa(s)\| = \left\| \sum_{i=1}^m \psi_i \cdot (h - \kappa_i)(s) \right\| \leq \sum_{i=1}^m \psi_i(s) \|h - \kappa_i(s)\| < \varepsilon.$$

Hence, h is in the norm-closure of F_+ . \square

Lemma 4.3. *Let $\{U_i\}_{i \in \Lambda}$ be a family of pairwise disjoint clopen subsets of Δ whose union U is dense in Δ , and let $c_i = \chi_{U_i} \in C(\Delta)$, for each $i \in \Lambda$. Suppose that $\{\omega_i\}_{i \in \Lambda}$ is any bounded family in Ω and let $\tilde{\omega} = \sum_{i \in \Lambda} c_i \omega_i \in \Omega_{\text{wk}}$, in the sense of Remark 2.5. If $f \in C(\Delta)$ is such that $f(s) = 0$ for $s \in \Delta \setminus U$, then $f \cdot \tilde{\omega} \in \Omega$.*

Proof. Fix $s_0 \in \Delta$ and let $\varepsilon > 0$. If $s_0 \in \Delta \setminus U$, then by the continuity of f and the fact that $f(s_0) = 0$ there exists an open subset $U_{s_0} \subset \Delta$ containing s_0 such that $|f(s)| < \varepsilon \|\tilde{\omega}\|^{-1}$ for all $s \in U_{s_0}$. Hence, the vector field $f \cdot \tilde{\omega}$ is within ε of the zero vector field $0 \in \Omega$ on the open set U_{s_0} .

On the other hand, if $s_0 \in U$, then there exists $j \in \Lambda$ such that $s_0 \in U_j$. By construction, $c_j \cdot \tilde{\omega} = c_j \cdot \omega_j$ and so $\tilde{\omega}(s) = \omega_j(s)$ for all $s \in U_j$. Because

$\|(f \cdot \tilde{\omega})(s) - (f \cdot \omega_j)(s)\| = 0$ for all $s \in U_j$, the vector field $f \cdot \tilde{\omega}$ is within ε of the vector field $f \cdot \omega_j \in \Omega$ on the open set U_j . Thus, by the local uniform approximation property (axiom (IV) in Definition 1.1), $f \cdot \tilde{\omega} \in \Omega$. \square

The fact that $\Omega^\perp = \{0\}$ in Ω_{wk} (Lemma 3.4) suggests that Ω is somehow dense in Ω_{wk} . The next proposition makes this relation more explicit.

Proposition 4.4. *If $\nu \in \Omega_{\text{wk}}$ and $\varepsilon > 0$, then there exist a family $\{c_i\}_{i \in \Lambda}$ of pairwise orthogonal projections in $C(\Delta)$ with supremum 1 and a bounded family $\{\omega_i\}_{i \in \Lambda} \subset \Omega$ such that $\|\nu - \sum_{i \in \Lambda} c_i \cdot \omega_i\| < \varepsilon$.*

Proof. By Lemma 2.2, the function $s \mapsto \|\nu(s)\|$ is lower semicontinuous; hence, there exists a meagre set M_ν such that the function $s \mapsto \|\nu(s)\|$ is continuous in the relative topology of $\Delta \setminus M_\nu$. Observe that $\overline{(\Delta \setminus M_\nu)} = \Delta$.

Fix $s_0 \in \Delta \setminus M_\nu$ and let $\omega \in \Omega$ be such that $\omega(s_0) = \nu(s_0)$. Since

$$\|\nu(s) - \omega(s)\|^2 = \|\nu(s)\|^2 + \|\omega(s)\|^2 - 2\text{Re} \langle \nu, \omega \rangle(s),$$

the continuity in the relative topology of $\Delta \setminus M_\nu$ guarantees the existence of an open subset U_{s_0} of Δ containing s_0 such that $\|\nu(s) - \omega(s)\| < \varepsilon/2$ for all $s \in (\Delta \setminus M_\nu) \cap U_{s_0}$. Hence, again by continuity we get that $\|\nu - \omega\|(s) < \varepsilon$ for all $s \in \overline{U_{s_0}}$. The set $\overline{U_{s_0}}$ is a clopen subset of Δ and $\Delta' = \Delta \setminus \overline{U_{s_0}}$ is also a Stonean space. Further, $M_\nu \cap \Delta' = M_\nu \cap (\Delta \setminus \overline{U_{s_0}})$ is a meagre set such that the function $s \mapsto \|\nu(s)\|$, for $s \in \Delta' \setminus (M_\nu \cap \Delta')$, is continuous in the relative topology.

An application of Zorn's Lemma yields a maximal family $\{(\chi_{U_i}, \omega_i)\}_{i \in \Lambda}$ such that $U_i \cap U_j = \emptyset$ for $i \neq j$ and such that $\|\chi_{U_i}(\nu - \omega_i)\| < \varepsilon$. Maximality ensures that $\overline{(\cup_{i \in I} U_i)} = \Delta$, for otherwise we can enlarge this family by the previous procedure in the Stonean space $\Delta \setminus \overline{(\cup_{i \in \Lambda} U_i)}$. If we let $c_i = \chi_{U_i}$ for $i \in \Lambda$ then it is clear by Lemma 2.2 that $\|\nu - \sum_{i \in \Lambda} c_i \cdot \omega_i\| < \varepsilon$ as for every $j \in \Lambda$ we have that $\|c_j(\nu - \sum_{i \in \Lambda} c_i \cdot \omega_i)\| = \|c_j(\nu - \omega_j)\| < \varepsilon$ and $\bigvee_{i \in \Lambda} c_i = 1$. \square

The next result is the key step in the proof of Theorem 4.1.

Proposition 4.5. *For every abelian projection $e \in B(\Omega_{\text{wk}})$ and $\varepsilon > 0$ there is an essential ideal $I \subset K(\Omega)$ and $x \in M(I)$ such that $\|e - x\| < \varepsilon$.*

Proof. Assume that $e \in B(\Omega_{\text{wk}})$ is an abelian projection and let $\varepsilon > 0$. Thus, by [16, Lemma 13], $e = \Theta_{\nu, \nu}$ for some $\nu \in \Omega_{\text{wk}}$ for which $\langle \nu, \nu \rangle$ is a projection of $C(\Delta)$. By Proposition 4.4, there is a family $\{c_i\}_{i \in \Lambda}$ of pairwise orthogonal projections in $C(\Delta)$ with supremum 1 and a bounded family

$\{\omega_j\}_{j \in \Lambda} \subset \Omega$ such that $\|\nu - \tilde{\omega}\| < \varepsilon/(2\|\nu\|)$, where $\tilde{\omega} = \sum_{j \in \Lambda} c_j \cdot \omega_j \in \Omega_{\text{wk}}$. Each c_j is the characteristic function of a clopen set U_j and the union U of these sets U_j is dense in Δ .

Let $I = \{a \in K(\Omega) : a(s) = 0, \forall s \in \Delta \setminus U\}$, which is an essential ideal of $K(\Omega)$. Define $F^I \subset F_+ \subset K(\Omega)_+$ to be the set

$$F^I = \left\{ \sum_{i=1}^n \Theta_{\mu_i, \mu_i} : n \in \mathbb{N}, \mu_i \in \Omega, \mu_i|_{\Delta \setminus U} = 0, i = 1, \dots, n \right\}.$$

Suppose that $\eta \in \Omega$ satisfies $\|\eta(s)\| = 0$ for all $s \in \Delta \setminus U$, and consider $\Theta_{\eta, \eta} \in F^I$. Observe that $\Theta_{\tilde{\omega}, \tilde{\omega}} \Theta_{\eta, \eta} = \Theta_{\langle \eta, \tilde{\omega} \rangle \cdot \tilde{\omega}, \eta}$, which is an element of I because $\langle \eta, \tilde{\omega} \rangle(s) = \langle \eta(s), \tilde{\omega}(s) \rangle = 0$ for all $s \in \Delta \setminus U$ and $\langle \eta, \tilde{\omega} \rangle \cdot \tilde{\omega} \in \Omega$ by Lemma 4.3. Hence, $\Theta_{\tilde{\omega}, \tilde{\omega}}$ maps the set F^I back into I . Because F^I is dense in I_+ , as we shall show below, $\Theta_{\tilde{\omega}, \tilde{\omega}} I \subset I$ and a similar computation shows that $I \Theta_{\tilde{\omega}, \tilde{\omega}} \subset I$. Furthermore, writing $x = \Theta_{\tilde{\omega}, \tilde{\omega}}$,

$$\|e - x\| = \|\Theta_{\nu, \nu} - \Theta_{\tilde{\omega}, \tilde{\omega}}\| \leq (\|\nu\| + \|\tilde{\omega}\|) \|\nu - \tilde{\omega}\| < \varepsilon.$$

It remains to show that F^I is dense in I_+ . To this end, assume $\varepsilon' > 0$ and $\kappa \in I_+$. Thus, $\kappa(s) = 0$ for all $s \in \Delta \setminus U$. Furthermore, by Lemma 4.2, there exists $h \in F_+$ such that $\|\kappa - h\| < \varepsilon'$. Let $\tilde{h} = \chi_{\Delta \setminus U} \cdot h$ and note that, as $\kappa \in I$, it is also true that $\|\kappa - \tilde{h}\| < \varepsilon'$. Now if h has the form $\sum_{j=1}^n \Theta_{\mu_j, \mu_j}$ for some $\mu_j \in \Omega$, then $\tilde{h} = \sum_{j=1}^n \Theta_{\chi_{\Delta \setminus U} \mu_j, \chi_{\Delta \setminus U} \mu_j} \in F^I$. \square

Proof of Theorem 4.1. Because $K(\Omega)$ is an ideal of A , we have $M(A) \subset M(K(\Omega))$. Moreover, as $K(\Omega)$ is an essential ideal of A we conclude that $M_{\text{loc}}(A) = M_{\text{loc}}(K(\Omega))$ [2, Proposition 2.3.6]. On the other hand, the inclusions

$$B(\Omega) = M(K(\Omega)) \subset M_{\text{loc}}(K(\Omega)) \subset M_{\text{loc}}(M_{\text{loc}}(K(\Omega))) \subset B(\Omega_{\text{wk}})$$

hold by [11, Theorem 4.6].

Therefore, we are left to show that $M_{\text{loc}}(M_{\text{loc}}(K(\Omega))) = B(\Omega_{\text{wk}})$. By [11, Corollary 4.3], an element $z \in I(K(\Omega)) = B(\Omega_{\text{wk}})$ belongs to $M_{\text{loc}}(K(\Omega))$ if and only if for every $\varepsilon > 0$ there is an essential ideal $I \subset K(\Omega)$ and a multiplier $x \in M(I)$ such that $\|z - x\| < \varepsilon$. By Proposition 3.8, $K(\Omega_{\text{wk}})$ is the (essential) ideal of $B(\Omega_{\text{wk}})$ generated by the abelian projections of $B(\Omega_{\text{wk}})$; thus, by Proposition 4.5, $K(\Omega_{\text{wk}}) \subset M_{\text{loc}}(K(\Omega))$. Hence, $K(\Omega_{\text{wk}})$ is an essential ideal of $M_{\text{loc}}(K(\Omega))$ and so $M(K(\Omega_{\text{wk}})) \subset M_{\text{loc}}(M_{\text{loc}}(K(\Omega)))$. However, $B(\Omega_{\text{wk}}) = M(K(\Omega_{\text{wk}}))$ by Kasparov's Theorem [17, Theorem 2.4] (or by a theorem of Pedersen [20]); hence,

$$B(\Omega_{\text{wk}}) = M(K(\Omega_{\text{wk}})) \subset M_{\text{loc}}(M_{\text{loc}}(K(\Omega))) \subset B(\Omega_{\text{wk}}),$$

which yields $M_{\text{loc}}(M_{\text{loc}}(K(\Omega))) = B(\Omega_{\text{wk}})$. \square

Somerset has shown that every separable postliminal (that is, type I) C^* -algebra A has the property that $M_{\text{loc}}(M_{\text{loc}}(A)) = I(A)$ [22, Theorem 2.8]. Theorem 4.1 demonstrates that the same behavior occurs with (certain) nonseparable type I C^* -algebras. Somerset's methods are different from ours in at least two ways: he employs the Baire $*$ -envelope of a C^* -algebra where we use the injective envelope and he uses properties of Polish spaces—spaces that arise from the separability of the algebras under study. It is reasonable to conjecture that $M_{\text{loc}}(M_{\text{loc}}(A)) = I(A)$ for all C^* -algebras A that possess a postliminal essential ideal. To prove such a statement, it would be enough to prove it for any continuous trace C^* -algebra A .

5 Direct Sum Decompositions

A Kaplansky–Hilbert module E over $C(\Delta)$ is said to be *homogeneous* [16] if there is a subset $\{\nu_j\}_{j \in \Lambda} \subset E$ – called an *orthonormal basis* – such that $\langle \nu_i, \nu_j \rangle = 0$ for all $j \neq i$, $|\nu_j| = 1$ for all j , and $\{\nu_j\}_{j \in \Lambda}^\perp = \{0\}$, where for any $\nu \in E$, $|\nu|$ is the continuous real-valued function $|\nu| = \langle \nu, \nu \rangle^{1/2} \in C(\Delta)$.

Kaplansky introduced the notion of homogeneous AW^* -module with the aim of reducing the study of abstract AW^* -modules to the slightly more concrete setting in which the modules have an orthonormal basis. This is justified by the following result:

Theorem 5.1 ([16]). *Let E be a Kaplansky–Hilbert module over $C(\Delta)$. Then there exist orthogonal projections $\{c_i\}_{i \in I} \subset C(\Delta)$ with supremum 1 such that $c_i E$ is a homogenous AW^* -module over $c_i C(\Delta)$.*

Note that in the situation of Theorem 5.1, for each i there exists a clopen set $\Delta_i \subset \Delta$ with $c_i = \chi_{\Delta_i}$. The sets $\{\Delta_i\}$ are pairwise disjoint, and $\cup_i \Delta_i$ is dense in Δ .

In this section we consider the effect of a direct sum decomposition in the structures that have been studied in the previous sections, namely the Fell algebra A of the weakly continuous Hilbert bundle $(\Delta, \{H_s\}_{s \in \Delta}, \Omega, \Omega_{\text{wk}})$, and its local multiplier algebra $M_{\text{loc}}(A)$. We show that a decomposition of Ω_{wk} into a direct sum $\oplus_i c_i \Omega_{\text{wk}}$ given by a partition of the identity $\{c_i\}$ in $C(\Delta)$ leads one to consider two corresponding direct sum C^* -algebras: $\oplus_i A_i$ and $\oplus_i M_{\text{loc}}(A_i)$, where A_i is a subalgebra of A for all i . We prove that A need not be isomorphic to $\oplus_i A_i$, yet $M_{\text{loc}}(A) \cong \oplus_i M_{\text{loc}}(A_i)$. The latter result is especially interesting if one recalls that $M_{\text{loc}}(A)$ is generally not an AW^* -algebra [3, Theorem 6.13].

Theorem 5.2. *Let $(\Delta, \{H_s\}_{s \in \Delta}, \Omega)$ be a continuous Hilbert bundle over the Stonean space Δ . Assume that $\{\Delta_i\}_{i \in I}$ is a family of pairwise-disjoint clopen subsets of Δ whose union is dense in Δ , and for each $i \in I$ let $c_i = \chi_{\Delta_i} \in C(\Delta)$ and $\Omega_i = \{\omega|_{\Delta_i} : \omega \in \Omega\}$. Then:*

- i.* $(\Delta_i, \{H_s\}_{s \in \Delta_i}, \Omega_i)$ is a continuous Hilbert bundle;
- ii.* $(\Omega_i)_{\text{wk}} \cong c_i \cdot \Omega_{\text{wk}}$ as C^* -modules;
- iii.* $\Omega_{\text{wk}} \cong \bigoplus_i (\Omega_i)_{\text{wk}}$ as C^* -modules;
- iv.* $B((\Omega_i)_{\text{wk}}) \cong c_i \cdot B(\Omega_{\text{wk}})$ as C^* -algebras;
- v.* $B(\Omega_{\text{wk}}) \cong \bigoplus_i B((\Omega_i)_{\text{wk}})$ as C^* -algebras.

In *ii* and *iii*, the isomorphism is considered together with the identification $C(\Delta_i) \simeq c_i C(\Delta)$.

Proof. Being clopen in Δ , each Δ_i is itself a Stonean space, and it is easy to see that $C(\Delta_i) \cong c_i C(\Delta)$

i. For axiom (I) in Definition 1.1, we aim to show that Ω_i is a $C(\Delta_i)$ module. Let $\omega \in \Omega$ and consider $\omega_i = \omega|_{\Delta_i}$. Choose any $f_i \in C(\Delta_i)$. As Δ_i is clopen, f_i can be extended to $F_i \in C(\Delta)$ such that $f_i = F_i|_{\Delta_i}$, and $F_i|_{\Delta \setminus \Delta_i} = 0$. The action $f_i \cdot \omega_i = (F_i \cdot \omega)|_{\Delta_i}$ gives Ω_i the structure of a $C(\Delta_i)$ module. Axioms (II) and (III) of Definition 1.1 are trivially satisfied.

For axiom (IV), let $\xi : \Delta_i \rightarrow \bigsqcup_{s \in \Delta_i} H_s$ be a vector field such that for every $s_0 \in \Delta_i$ and $\varepsilon > 0$ there is an open set $U_i \subset \Delta_i$ containing s_0 and a $\omega_i \in \Omega_i$ with $\|\omega_i(s) - \xi(s)\| < \varepsilon$ for all $s \in U_i$. Let $\Xi : \Delta_i \rightarrow \bigsqcup_{s \in \Delta} H_s$ be the vector field that coincides with ξ on Δ_i and is identically zero off Δ_i . By the definition of Ω_i , there is $\omega \in \Omega$ such that $\omega_i = \omega|_{\Delta_i}$. The set U_i is also open in Δ , and $\|\omega(s) - \Xi(s)\| < \varepsilon$ for all $s \in U_i$. If $s_0 \notin \Delta_i$ choose any open set V_i containing s_0 such that $V_i \cap U_i = \emptyset$ and let $\omega \in \Omega$ be arbitrary; then $0 = \|\chi_{\Delta_i}(s)\omega(s) - \Xi(s)\| < \varepsilon$ for all $s \in V_i$. Since $\chi_{\Delta_i} \cdot \omega \in \Omega$ and since Ω is closed under local uniform approximation, $\Xi \in \Omega$, whence $\xi \in \Omega_i$.

ii. Let $T_i : c_i \Omega_{\text{wk}} \rightarrow (\Omega_i)_{\text{wk}}$ be given by $T_i(c_i \nu) = \nu|_{\Delta_i}$. It is clear that T_i is well defined, linear, bounded, and has trivial kernel; to show that it is onto, note that if $\nu_i \in (\Omega_i)_{\text{wk}}$, then—since Δ_i is clopen—the vector field $\nu : \Delta \rightarrow \bigsqcup_{s \in \Delta} H_s$ defined by $\nu(s) = 0$, for $s \notin \Delta_i$, and $\nu(s) = \nu_i(s)$, for $s \in \Delta_i$, has the property that $\langle \omega, \nu \rangle \in C(\Delta)$, for all $\omega \in \Omega$; so $\nu \in \Omega_{\text{wk}}$ and $\nu_i = T_i(c_i \nu)$. It is also easy to check that T_i preserves inner products.

iii. Let $T : \Omega_{\text{wk}} \rightarrow \bigoplus_i (\Omega_i)_{\text{wk}}$, given by $T\nu = (T_i(c_i \nu))_{i \in I}$. The previous paragraph and Lemma 2.1 show that T is an isometry; we show now that T

is onto. Suppose that $\nu' = (\nu_i)_{i \in I} \in \bigoplus_i (\Omega_i)_{\text{wk}}$. For each $i \in I$ let $\tilde{\nu}_i$ denote the vector field on Δ that coincides with ν_i on Δ_i and vanishes elsewhere. Then $\tilde{\nu}_i \in \Omega_{\text{wk}}$ and $T_i(c_i \tilde{\nu}_i) = \nu_i$. Hence, if $\nu = \sum_i c_i \tilde{\nu}_i$ as in Remark 2.5, we have $\nu \in \Omega_{\text{wk}}$ and $T\nu = \nu'$. Thus, Ω_{wk} and $\bigoplus_i (\Omega_i)_{\text{wk}}$ are isomorphic Banach spaces. Similar arguments show that $\bigoplus_i (\Omega_i)_{\text{wk}}$ is a $C(\Delta)$ -module and that T is module isomorphism. Hence, $\Omega_{\text{wk}} \cong \bigoplus_i (\Omega_i)_{\text{wk}}$ as C^* -modules.

iv. Let $\rho_i : c_i B(\Omega_{\text{wk}}) \rightarrow B((\Omega_i)_{\text{wk}})$ be given by $\rho_i(c_i b) T_i(c_i \nu) = (b\nu)|_{\Delta_i}$. This map is well-defined because if $c_i b_1 = c_i b_2$ then for any $\nu \in \Omega_{\text{wk}}$ we have $(b_1 \nu)|_{\Delta_i} = (c_i b_1 \nu)|_{\Delta_i} = (c_i b_2 \nu)|_{\Delta_i} = (b_2 \nu)|_{\Delta_i}$. A similar computation shows that ρ_i is one-to-one, and linearity is clear. To see that ρ_i is onto, let $b_i \in B((\Omega_i)_{\text{wk}})$. Consider the injection $\tilde{\cdot} : (\Omega_i)_{\text{wk}} \rightarrow \Omega_{\text{wk}}$ where $\tilde{\nu}_i \in \Omega_{\text{wk}}$ is the vector field that agrees with ν_i on Δ_i and is 0 elsewhere. Let $b \in B(\Omega_{\text{wk}})$ be the operator given by $b\nu = \widetilde{b_i(\nu|_{\Delta_i})}$. Then $\rho_i(c_i b)(T_i c_i \nu) = (b\nu)|_{\Delta_i} = \widetilde{b_i(\nu|_{\Delta_i})}|_{\Delta_i} = b_i(\nu|_{\Delta_i}) = b_i(T_i c_i \nu)$, so $\rho_i(c_i b) = b_i$.

v. Let $\rho : B(\Omega_{\text{wk}}) \rightarrow \bigoplus_i B((\Omega_i)_{\text{wk}})$ be the map $\rho(b) = (\rho_i(c_i b))_{i \in I}$. It is clear that ρ is a homomorphism. If $\rho(b) = 0$ for some $b \in B(\Omega_{\text{wk}})$, then – as each ρ_i is one-to-one – $c_i b = 0$ for all i ; this implies that $b^* b = b^*(\sup_i (c_i \cdot I))b = \sup_i (b^* c_i b) = 0$ by [14, Corollary 4.10], so $b = 0$ and ρ is one-to-one. To show that ρ is onto, let $(b_i)_i \in \bigoplus_i B((\Omega_i)_{\text{wk}})$; as each ρ_i is onto, there exist operators $b^i \in B(\Omega_{\text{wk}})$ with $\rho_i(c_i b^i) = b_i$. Define $b \in B(\Omega_{\text{wk}})$ by $b\nu = \sum_i c_i b^i \nu$ (in the sense of Remark 2.5; that is, $c_i b\nu = c_i b^i \nu$). Then $\rho_i(c_i b)\nu|_{\Delta_i} = (c_i b\nu)|_{\Delta_i} = (c_i b^i \nu)|_{\Delta_i} = \rho_i(c_i b^i)\nu|_{\Delta_i} = b_i \nu|_{\Delta_i}$. So $\rho(b) = (b_i)_i$. \square

Proposition 5.3. *Assume the notation, hypotheses, and conclusions of Theorem 5.2. Then there exists an example where the canonical embedding $\Omega \hookrightarrow \bigoplus_i \Omega_i$ (via the isometry T from the proof of **iii** in Theorem 5.2) is not onto. In particular, Ω is properly contained in Ω_{wk} .*

Proof. Take Δ and the family of clopen subsets $\{\Delta_i\}_{i \in I}$ in Theorem 5.2 to be such that $\bigcup_{i \in I} \Delta_i \neq \Delta$. Thus, I is an infinite set. Let H be a Hilbert space with orthonormal basis $\{e_i\}_{i \in I}$ and consider the trivial Hilbert bundle $\Omega = C(\Delta, H)$ of all continuous functions $\omega : \Delta \rightarrow H$. As in Theorem 5.2, let $\Omega_i = C(\Delta_i, H)$.

For each $i \in I$, set $\omega_i \in \Omega$ with $\omega_i(s) = e_i$ for all s and consider $(\omega_i)_{i \in I} \in \bigoplus_i \Omega_i$. Under the isomorphism of Theorem 5.2, this element $(\omega_i)_{i \in I}$ is identified with $\omega = \sum_{i \in I} \chi_{\Delta_i} \cdot \tilde{\omega}_i \in \Omega_{\text{wk}}$ (in the sense of Remark 2.5), where $\tilde{\omega}_i$ is any element of Ω that agrees with ω_i on Δ_i and vanishes off Δ_i . Under this identification, $\omega \notin \Omega$; that is, the function $s \mapsto \|\omega(s)\|$ fails to be continuous on Δ . We argue this by contradiction.

Assume that $s \mapsto \|\omega(s)\|$ is continuous on Δ . Because $\|\omega(s)\| = 1$ for all $s \in \cup_{i \in I} \Delta_i$, continuity implies that $\|\omega(s)\| = 1$ for $s \in \Delta$. Choose $s_0 \in \Delta \setminus (\cup_{i \in I} \Delta_i)$ and let $(s_\alpha)_{\alpha \in \Lambda} \subset \cup_{i \in I} \Delta_i$ be a net such that $s_\alpha \rightarrow s_0$. Let $\eta \in \Omega$ be the constant field $\eta(s) = \omega(s_0)$, for all $s \in \Delta$. Since $\omega \in \Omega_{\text{wk}}$, we have

$$\lim_{\alpha} \langle \omega(s_\alpha), \eta(s_\alpha) \rangle = \langle \omega(s_0), \eta(s_0) \rangle = \langle \omega(s_0), \omega(s_0) \rangle = 1. \quad (5)$$

For each $\alpha \in \Lambda$ let $i(\alpha) \in I$ be such that $s_\alpha \in \Delta_{i(\alpha)}$. Thus, for every $\alpha \in \Lambda$, $I_\alpha = \{i(\beta) : \beta \in \Lambda, \beta \geq \alpha\}$ is an infinite set (for otherwise $s_0 \in \Delta_i$ for some $i \in I$). Therefore,

$$\lim_{\alpha} \langle \omega(s_\alpha), \eta(s_\alpha) \rangle = \lim_{\alpha} \langle e_{i(\alpha)}, \omega(s_0) \rangle = 0. \quad (6)$$

As (5) and (6) cannot be true simultaneously, we obtain a contradiction. Hence, $\omega \notin \Omega$. \square

Our second reduction theorem below notes some consequences of Theorem 5.2 when applied to the injective envelope and local multiplier algebras of the Fell algebra A associated to a continuous Hilbert bundle.

Theorem 5.4. *Let $(\Delta, \{H_t\}_{t \in \Delta}, \Omega)$ be a continuous Hilbert bundle over the Stonean space Δ and let $A = (\Delta, \{K(H_t)\}, \Gamma)$ denote the associated continuous trace C^* -algebra of Fell. Assume that $\{\Delta_i\}_{i \in I}$ is a family of pairwise-disjoint clopen subsets of Δ whose union is dense in Δ , and for each $i \in I$ let $c_i = \chi_{\Delta_i} \in C(\Delta)$ and $\Omega_i = \{\omega|_{\Delta_i} : \omega \in \Omega\}$. Then:*

- i.* if A_i denotes the Fell algebra of $(\Delta_i, \{H_s\}_{s \in \Delta_i}, \Omega_i)$, then $A_i \cong c_i \cdot A$;
- ii.* $I(A_i) = B((\Omega_i)_{\text{wk}})$;
- iii.* $I(A) \cong \bigoplus_{i \in I} I(A_i)$;
- iv.* $M_{\text{loc}}(A) \cong \bigoplus_{i \in I} M_{\text{loc}}(A_i)$.

Proof. Let $A_i = (\Delta_i, \{K(H_s)\}_{s \in \Delta_i}, \Gamma_i)$ denote the Fell C^* -algebra associated to the Hilbert bundle $(\Delta_i, \{H_s\}_{s \in \Delta_i}, \Omega_i)$. That is, Γ_i consists of all weakly continuous almost finite-dimensional operator fields $a_i : \Delta_i \rightarrow \bigsqcup_{s \in \Delta_i} K(H_s)$ such that $s \mapsto \|a_i(s)\|$ is continuous. We have that $B((\Omega_i)_{\text{wk}})$ is a type I AW*-algebra with centre $C(\Delta_i)$.

i. For each $a_i \in \Gamma_i$ there is an $a \in \Gamma$ such that $a_i = a|_{\Delta_i}$. To verify this, let $a : \Delta \rightarrow \bigsqcup_{s \in \Delta} K(H_s)$ be the operator field defined by $a(s) = a_i(s)$, for $s \in \Delta_i$, and $a(s) = 0$, for $s \notin \Delta_i$. Since Δ_i is a clopen set, the maps

$s \rightarrow \|a(s)\|$ and $s \mapsto \langle a(s)\omega_1(s), \omega_2(s) \rangle$ are continuous for every $\omega_1, \omega_2 \in \Omega$. The operator field a is also locally finite-dimensional, again because Δ_i is clopen and a_i has the property on Δ_i . Hence, $a \in \Gamma$. Next, let $\pi_i : A_i \rightarrow c_i A$ be defined by $\pi_i(a_i) = c_i a$, where $a \in A$ is any operator field that restricts to a_i on Δ_i . This map is clearly well-defined, and a homomorphism.

ii. By Theorem 3.1, $B((\Omega_i)_{\text{wk}}) = I(A_i) = I(c_i A)$.

iii. By [14, Lemma 6.2], $I(c_i A) = c_i I(A)$. Hence, $I(A_i) = B((\Omega_i)_{\text{wk}})$ and Theorem 5.2 immediately yields $I(A) \cong \bigoplus_{i \in I} I(A_i)$.

iv. We take each $M_{\text{loc}}(A_i)$ to be a C^* -subalgebra of $B((\Omega_i)_{\text{wk}})$. First we remark that the isomorphism ρ from Theorem 5.2 sends A into $\bigoplus_i A_i$. To see why, recall that $a\nu(s) = a(s)\nu(s)$, for all $a \in A$, $\nu \in \Omega_{\text{wk}}$, and $s \in \Delta$ (Proposition 3.6). Since, for a given $i \in I$, the action of $\rho_i(a)$ on $\nu_i \in (\Omega_i)_{\text{wk}}$ is defined by $\nu_i \mapsto (a\nu)|_{\Delta_i}$, where $\nu \in \Omega_{\text{wk}}$ is any vector with $\nu|_{\Delta_i} = \nu_i$, it is easy to verify that $\rho_i(a)$ is a weakly continuous almost finite-dimensional operator field on Δ_i .

To show that $\rho(M_{\text{loc}}(A)) \subset \bigoplus_i M_{\text{loc}}(A_i)$, let $x \in M_{\text{loc}}(A) \subset I(A)$ and suppose that $\varepsilon > 0$. Thus, there is an essential ideal $J \subset A$ and a multiplier $x \in M(J)$ such that $\|x - y\| < \varepsilon$. Further, there exists an open dense subset $U \subset \Delta$ such that

$$J = \{a \in A : a(s) = 0, s \in \Delta \setminus U\}. \quad (7)$$

For $i \in I$, let $U_i = \Delta_i \cap U$, which is an open dense set in Δ_i . Therefore,

$$J_i = \{a_i \in A_i : a(s) = 0, s \in \Delta_i \setminus U_i\} \quad (8)$$

is an essential ideal in A_i . We aim to show that $\rho_i(y) \in M(J_i)$. To this end, select $a_i \in J_i$. As $A_i \cong c_i \cdot A$, there is an $a \in A$ such that $a_i(s) = a(s)$ for all $s \in \Delta_i$. Moreover, $a \in A$ can be chosen so that $a(s) = 0$ for all $s \in \Delta \setminus U$.

Because $a_i \in J_i$, we conclude that $a(s) = 0$ for all $s \in \Delta \setminus U$; that is, $a \in J$. Therefore, $ya \in J$, which implies that $ya(s) = 0$ for all $s \in \Delta \setminus U$. In particular, $ya(s) = 0$ for all $s \in \Delta_i \setminus U_i$. The element $\rho_i(y)a_i \in B((\Omega_i)_{\text{wk}})$ is in fact an operator field since $\rho_i(y)a_i = \rho_i(y)\rho_i(c_i a) = \rho_i(c_i(ya)) \in A_i$. Then, for all $s \in \Delta_i \setminus U_i$ and $\nu \in \Omega_{\text{wk}}$,

$$\begin{aligned} [\rho_i(y)a_i](s)(T_i c_i \nu)(s) &= \rho_i(y)a_i(T_i c_i \nu)(s) = \rho_i(c_i ya)(T_i c_i \nu)(s) \\ &= (ya)\nu|_{\Delta_i}(s) = (ya)(s)\nu|_{\Delta_i}(s) = 0. \end{aligned}$$

With ν being arbitrary, we conclude that $\rho_i(y)a_i(s) = 0$, that is $\rho_i(y)a_i \in J_i$, and so $\rho_i(y)$ is a left multiplier of J_i . By a similar argument, $\rho_i(y)$ is a right multiplier of J_i , and so $\rho_i(y) \in M(J_i)$. Thus, $\rho(y) \in \bigoplus_i M_{\text{loc}}(A_i)$ and

$\|\rho(x) - \rho(y)\| = \|x - y\| < \varepsilon$. As $\varepsilon > 0$ was chosen arbitrarily, this proves that $\rho(x) \in \bigoplus_i M_{\text{loc}}(A_i)$.

Conversely, let us show that $\bigoplus_i M_{\text{loc}}(A_i) \subset \rho(M_{\text{loc}}(A))$. Let $(x_i)_i \in \bigoplus_i M_{\text{loc}}(A_i)$; thus for each $i \in I$, there exist an essential ideal $J_i \subset A_i$ and $y_i \in M(J_i)$ such that $\|x_i - y_i\| < \varepsilon$ for all $i \in I$. For each $i \in I$, there exists an open dense subset $U_i \subset \Delta_i$ such that J_i is given as in (8). Define $U = \bigcup_{i \in I} U_i$, which is an open dense subset of Δ and let J be the essential ideal of A defined as in (7) (for our present choice of U). Let $y \in B(\Omega_{\text{wk}})$ be such that $\rho(y) = (y_i)_i$.

For each $\omega \in \Omega$, we have that $y\omega \in \Omega_{\text{wk}}$.

CLAIM 1. If $\omega \in \Omega$ is such that $\omega(s) = 0$ for all $s \in \Delta \setminus U$, then $y\omega \in \Omega$ and $y\omega(s) = 0$ for $s \in \Delta \setminus U$.

Assuming Claim 1, consider the set $F_+ = \text{span} \{\Theta_{\omega, \omega} : \omega \in \Omega, \omega(s) = 0 \text{ for } s \in \Delta \setminus U\}$, which by Lemma 4.2 is dense in K_+ , where K is the essential ideal of $K(\Omega)$ defined by $K = K(\Omega) \cap J$. By the Claim, $y\Theta_{\omega, \omega} = \Theta_{y\omega, \omega} \in K$ for all $\omega \in \Omega$. Therefore, y is a left multiplier of K . Similarly, y is a right multiplier of K , which yields $y \in M(K)$. Hence, $(x_i)_{i \in I}$ is within ε of a multiplier—namely, $\rho(y)$ —of an essential ideal of $\rho(K(\Omega))$. Thus, by the Frank–Paulsen description of local multiplier algebras [11], $(x_i)_{i \in I} \in \rho(M_{\text{loc}}(K(\Omega)))$. By Theorem 4.1, $M_{\text{loc}}(A) = M_{\text{loc}}(K(\Omega))$, so $(x_i)_{i \in I} \in \rho(M_{\text{loc}}(A))$.

We are now left with proving Claim 1. Assume that $\omega \in \Omega$ with $\omega(s) = 0$ for all $s \in \Delta \setminus U$. Let $i \in I$ and let $\omega_i = \omega|_{\Delta_i} \in \Omega_i$. Note that for every $\eta_i \in \Omega_i$, $\Theta_{\omega_i, \eta_i} \in J_i$, and hence $\Theta_{y_i \omega_i, \eta_i} = y_i \Theta_{\omega_i, \eta_i} \in J_i$. Also, $y_i \omega_i \in \Omega_i$. Indeed, suppose that $s_0 \in \Delta_i$ and let $\eta_i \in \Omega_i$ such that $\|\eta_i(s_0)\| = 1$. Choose a clopen subset $V_i \subset \Delta_i$ of s_0 for which $\|\eta_i(s)\| \geq 1/2$ for all $s \in V_i$ and define $f(s) = \chi_{V_i}(s) \|\eta_i(s)\|^{-2}$. Thus, $f \in C(\Delta_i)$ and so $f \cdot \eta_i \in \Omega_i$. Then, since $\Theta_{y_i \omega_i, \eta_i} \in J_i \subset A_i$, we have $\Theta_{y_i \omega_i, \eta_i}(f \cdot \eta_i) \in \Omega_i$. So $\chi_{V_i} \cdot y_i \omega_i = \Theta_{y_i \omega_i, \eta_i}(f \cdot \eta_i) \in \Omega_i$. Thus, $y_i \omega_i$ is a local uniform limit of vectors fields in Ω_i and hence, $y_i \omega_i \in \Omega_i$. Moreover, since $\Theta_{y_i \omega_i, \eta_i} \in J_i$ for any $\eta_i \in \Omega_i$, we have $y_i \omega_i(s) = 0$ for $s \in \Delta_i \setminus U_i$.

Since $(y\omega)(s) = (y_i \omega_i)(s)$ for $s \in \Delta_i$, the lower semicontinuous function $s \mapsto \|(y\omega)(s)\|$ is continuous on $\bigcup_i \Delta_i$ and vanishes on $(\bigcup_i \Delta_i) \setminus U$.

CLAIM 2. There exists $C > 0$ such that $\|y\omega(s)\| \leq C \|\omega(s)\|$, $s \in \Delta_i$, $i \in I$.

We will use Claim 2 to show that the function $s \mapsto \|(y\omega)(s)\|$ is continuous on Δ . Let $s \in \Delta \setminus (\bigcup_i \Delta_i)$ and let $(s_\alpha)_\alpha \subset \bigcup_i \Delta_i$ be a net such that $s_\alpha \rightarrow s$ in Δ . This implies that $\lim_\alpha \|\omega(s_\alpha)\| = 0$. By lower semicontinuity

of the function $s \mapsto \|(y\omega)(s)\|$,

$$0 \leq \|y\omega(s)\| \leq \lim_{\alpha} \|y\omega(s_{\alpha})\| \leq C \lim_{\alpha} \|\omega(s_{\alpha})\| = 0,$$

and it follows that $s \mapsto \|(y\omega)(s)\|$ is continuous on Δ and vanishes in $\Delta \setminus U$. This establishes Claim 1.

We finish the proof by proving Claim 2. Fix $s \in \Delta_i$, and let $C = \sup_i \|y_i\|$. We already know that $y_i\omega_i \in \Omega_i$, and so

$$\begin{aligned} \|y\omega(s)\| &= \|y_i\omega_i(s)\| = \|y_i\omega_i\|(s) \leq \|y_i\| \|\omega_i\|(s) \\ &\leq C \|\omega_i\|(s) = C \|\omega_i(s)\| = C \|\omega(s)\|. \square \end{aligned}$$

Local multiplier algebras behave well under direct sums: $M_{\text{loc}}(\oplus_i A_i) \cong \oplus_i M_{\text{loc}}(A_i)$ [2, Proposition 2.3.6]. However, the isomorphism of local multiplier algebras in Theorem 5.4 cannot be established via that generic result:

Proposition 5.5. *Assume the notation, hypotheses, and conclusions of Theorem 5.4. Although ρ sends A into $\bigoplus_i A_i$, it need not be true that $A \cong \bigoplus_i A_i$.*

Proof. If Δ and Ω are as in Proposition 5.3, then $\rho(\Theta_{\omega,\omega}) = (\Theta_{\omega_i,\omega_i})_{i \in I} \in \bigoplus_{i \in I} A_i$, but $\rho(\Theta_{\omega,\omega}) \notin \rho(A)$. \square

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