Split clique graph complexity

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Abstract. A *complete set* of a graph G is a subset of vertices inducing a complete subgraph. A *clique* is a maximal complete set. Denote by $\mathcal{C}(G)$ the *clique family* of G. The *clique graph* of G, denoted by $K(G)$, is the intersection graph of $\mathcal{C}(G)$. Say that G is *a clique graph* if there exists a graph H such that $G = K(H)$. The clique graph recognition problem, a long-standing open question posed in 1971, asks whether a given graph is a clique graph and it was recently proved to be NP-complete even for a graph G with maximum degree 14 and maximum clique size 12. Hence, if P≠NP, the study of graph classes where the problem can be proved to be polynomial, or of more restricted graph classes where the problem remains NP-complete is justified. We present a proof that given a split graph $G = (V, E)$ with partition (K, S) for V, where K is a complete set and S is a stable set, deciding whether there is a graph H such that G is the clique graph of H is NP-complete. As a byproduct, we prove that a problem about the Helly property on a family of sets is NP-complete. Our result is optimum in the sense that each vertex of the independent set of our split instance has degree at most 3, whereas when each vertex of the independent set has degree at most 2 the problem is polynomial, since it is reduced to check whether the clique family of the graph satisfies the Helly property. Additionally, we show three split graph subclasses for which the problem is polynomially solvable: the subclass where each vertex of S has a private neighbor, the subclass where $|S| \leq 3$, and the subclass where $|K| \leq 4$.

Keywords: clique graph, Helly property, NP-complete, split graphs

1 Introduction

Consider finite, simple and undirected graphs. V and E denote the vertex set and the edge set of the graph G, respectively. A *complete set* of G is a subset of V inducing a complete subgraph. A *clique* is a maximal complete set. The *clique family* of G is denoted by $\mathcal{C}(G)$. The *clique graph* of G is the intersection graph of $\mathcal{C}(G)$.

The *clique operator*, K, assigns to each graph G its clique graph which is denoted by $K(G)$. On the other hand, say that G is *a clique graph* if G belongs to the image of the clique operator, i.e. if there exists a graph H such that $G = K(H)$.

Clique operator and its image were widely studied. First articles focused on recognizing clique graphs [8, 18]. Graphs fixed under the operator K or fixed under the iterated clique operator, K^n, for some positive integer n; and the behavior under these operators of parameters such as number of vertices or diameter were studied in [4, 9, 10]. For several classes of graphs, the image of the class under the clique operator was characterized [5, 11, 16, 19]; and, in some cases, also the inverse image of the class [14, 17]. Results of the previous bibliography can be found in the survey [21]. Clique graphs have been much studied as intersection graphs and are included in several books [6, 12, 15].

The characterization of clique graphs given in [18] proposed the computational complexity of the recognition of clique graphs, a long-standing open question [6, 15, 18, 21] just recently settled as NP-complete [1, 2].

A graph is *split* if its vertex set can be partitioned into a complete set and a stable set. In this paper, we are concerned with the time complexity of the problem of recognizing split clique graphs, for which we establish NP-complete and polynomial results.

SPLIT CLIQUE GRAPH
INSTANCE: A split graph $G = (V, E)$.
QUESTION: Is there a graph H such that $G = K(H)$?

We prove that SPLIT CLIQUE GRAPH is NP-complete. As a byproduct, we prove that a problem about the Helly property is NP-complete. Given a *set family* $\mathcal{F} = (F_i)_{i \in I}$, the sets F_i are called *members* of the family. $F \in \mathcal{F}$ means that F is a member of $\mathcal{F}$. The family is *pairwise intersecting* if the intersection of any two members is not the empty set. The *intersection* or *total intersection* of $\mathcal{F}$ is the set $\bigcap \mathcal{F} = \bigcap_{i \in I} F_i$. The family $\mathcal{F}$ has the *Helly property*, if any pairwise intersecting subfamily has nonempty total intersection. Besides the theoretical interest, the Helly property has applications in many different areas such as optimization and location problems, semantics, coding, computational biology, data bases, image processing and, in special, graph theory where it has been a useful and a natural tool. Please refere to [7] for a survey on the Helly property and its complexity aspects.

Given a family of sets $\mathcal{F}$, say that a family $\mathcal{F}'$ is a *spanning* family for $\mathcal{F}$ if: $\bigcup_{F' \in \mathcal{F}'} F' = \bigcup_{F \in \mathcal{F}} F$; for each $F' \in \mathcal{F}'$, $|F'| > 1$; for each $F' \in \mathcal{F}'$, there exists $F \in \mathcal{F}$ such that $F' \subseteq F$; and for each $F \in \mathcal{F}$, $\bigcup_{F' \subseteq F, F' \in \mathcal{F}'} F' = F$.

SPANNING HELLY FAMILY
INSTANCE: A family of sets $\mathcal{F}$.
QUESTION: Does $\mathcal{F}$ admit a spanning family $\mathcal{F}'$ that satisfies the Helly property?

Our NP-completeness result yields that SPANNING HELLY FAMILY is NP-complete even when restricted to the members of the input family $\mathcal{F}$ having cardinality 2 or 3. Note that the problem is polynomial when all members of $\mathcal{F}$ have cardinality 2, and we leave as open the problem when all members of $\mathcal{F}$ have cardinality exactly 3. Note that the problem 3SAT$_{\overline{3}}$ when restricted to having exactly three literals per clause is polynomial [13].

2 NP-complete split clique graph classes

Theorem 1 is a well known characterization of Clique Graphs. The edge with end vertices u and v is represented by uv. We say that the complete set C *covers* the edge uv when u and v belong to C. A *complete set edge cover* of a graph G is a family of complete sets of G covering all edges of G.

Theorem 1 (Roberts and Spencer [18]). *G is a clique graph if and only if there exists a complete set edge cover of G satisfying the Helly property.*

Notice that for any graph G the clique family $\mathcal{C}(G)$ is a complete set edge cover of G, but, in general, this family does not satisfy the Helly property. Graphs such that $\mathcal{C}(G)$ satisfies the Helly property are called *clique-Helly* graphs. It follows from Theorem 1 that every clique-Helly graph is a clique graph. In [20], clique-Helly graphs are characterized and a polynomial-time algorithm for their recognition is presented. Lemma 2 extends that result and leads to a polynomial-time algorithm to check if a given complete set edge cover of a graph satisfies the Helly property which in turn yields that CLIQUE GRAPH is in NP [1, 2].

A *triangle* is a complete set with exactly 3 vertices. The set of triangles of G is denoted $T(G)$. Let $\mathcal{F}$ be a complete set edge cover of G and T a triangle, and denote by $\mathcal{F}_T$ the subfamily of $\mathcal{F}$ formed by all the members containing at least two vertices of T.

Lemma 2 (Alcón and Gutierrez [3]). *Let $\mathcal{F}$ be a complete set edge cover of G. The following conditions are equivalent:*
i) $\mathcal{F}$ has the Helly property.
ii) For every $T \in T(G)$, the subfamily $\mathcal{F}_T$ has the Helly property.
iii) For every $T \in T(G)$, the subfamily $\mathcal{F}_T$ has nonempty intersection, this means $\bigcap \mathcal{F}_T \neq \emptyset$.

A graph admits a complete set edge cover with the Helly property if and only if the graph admits a complete set edge cover with the Helly property such that no member is contained in another; such cover is called an *RS-family* of the graph. Thus Theorem 1 is equivalent to the following simpler statement: G is a clique graph if and only if G admits an RS-family. The following properties are stated and proved by Roberts and Spencer [18].

Lemma 3 (Lemma 1 and Theorem 3 of [18]). *Let $\mathcal{F}$ be an RS-family of a graph G. Then $\mathcal{F}$ contains a complete set of size 2 if and only if this complete set is a clique of G. If a triangle T is a clique of G, then T is a member of $\mathcal{F}$.*

We show that SPLIT CLIQUE GRAPH is NP-complete by a reduction from the following version of the 3–satisfiability problem with at most 3 occurrences per variable [13]. Let $U = \{u_i, 1 \leq i \leq n\}$ be a set of boolean variables. A literal is either a variable u_i or its complement $\overline{u_i}$. A clause over U is a set of literals of L. Let $C = \{c_j, 1 \leq j \leq m\}$ be a collection of clauses over U. We say that variable u_i occurs in clause c_j (and then in C) if u_i or $\overline{u_i} \in c_j$. We say that variable u_i

occurs in clause c_j as literal u_i (or that literal u_i occurs in c_j) if $u_i \in c_j$, and as literal $\overline{u_i}$ (or that literal $\overline{u_i}$ occurs in c_j) if $\overline{u_i} \in c_j$.

3SAT$_{\overline{3}}$

INSTANCE: $I = (U, C)$, where $U = \{u_i, 1 \leq i \leq n\}$ is a set of boolean variables, and $C = \{c_j, 1 \leq j \leq m\}$ a set of clauses over U such that each clause has two or three variables, each variable occurs at most three times in C.

QUESTION: Is there a truth assignment for U such that each clause in C has at least one true literal?

In order to reduce 3SAT$_{\overline{3}}$ to SPLIT CLIQUE GRAPH, we need to construct in polynomial time a particular instance G_I of SPLIT CLIQUE GRAPH from a generic instance $I = (U, C)$ of 3SAT$_{\overline{3}}$, in such a way that the constructed graph G_I is a clique graph if and only if C is satisfiable. The particular instance G_I is a 3-split graph and we first characterize 3-split clique graphs.

3-split graphs

A split graph admits a *split partition* of its vertex set into a complete set K and a stable set S. The family of cliques of a split graph with split partition (K, S) is composed by the closed neighbourhood $N[s]$, for each $s \in S$, and the complete set K if it is not contained in $N[s]$, for $s \in S$. An *ℓ-cone* is an $\ell + 1$-clique containing a vertex of S that is called its *extreme* vertex and the remaining ℓ vertices are in K composing the *basis* of the cone. The *triangles of an ℓ-cone* are its ℓ triangles that contain the extreme vertex of the cone. The set of the remaining vertices of a triangle of an ℓ-cone are the *basis* of the triangle. Note that a 2-cone is a triangle that is a clique and so by Lemma 3 forced to belong to any RS-family of a split clique graph.

A *3-split* graph admits a split partition where each vertex of the stable set S has degree 2 or 3, in this case (K, S) is called a *3-split partition.*

Theorem 4. *Let G be a 3-split graph with 3-split partition (K, S). The following are equivalent:*

1. *G is a clique graph;*
2. *There exists an RS-family $\mathcal{F}$ of G composed by K, each 2-cone and exactly two triangles of each 3-cone;*
3. *There exists a family of complete sets of G containing each basis of a 2-cone and the bases of exactly two triangles of each 3-cone that satisfies the Helly property;*
4. *There exists a family of edges containing all the edges corresponding to the bases of the 2-cones and the edges of the bases of exactly two triangles of each 3-cone that induces a triangle-free subgraph of $G[K]$.*

Proof. 1. implies 2.: Let G be a 3-split graph with 3-split partition (K, S) and let $\mathcal{F}$ be an RS-family of G. Assume K is not a member of $\mathcal{F}$ and consider $\mathcal{F}'$ the family obtained from $\mathcal{F}$ by the addition of member K. Suppose there exists a pairwise intersecting subfamily of $\mathcal{F}'$ without a common vertex. It is clear this subfamily must contain K, since the original RS-family $\mathcal{F}$ has the Helly property. Let $F_1, F_2, ..., F_\ell, K$ be the pairwise intersecting subfamily without a common vertex. Observe that $\ell \geq 2$. Since $F_1, F_2, ..., F_\ell$ are members of $\mathcal{F}$, they have a common vertex s. It is clear s is not in K, and so $s \in S$. In case $N(s) = \{x, y\}$, then $F_1 = \{s, x\}$ and $F_2 = \{s, y\}$ but this contradicts Lemma 3 since F_1 and F_2 are not cliques of G. Hence, $N(s) = \{x, y, z\}$ and the assumption that $F_1, F_2, ..., F_\ell$ have no common vertex in K forces $\ell = 3$, $F_1 = \{s, y, z\}$, $F_2 = \{x, s, z\}$ and $F_3 = \{x, y, s\}$, Note that F_1, F_2 and F_3 are the three triangles containing vertex s. Now we can eliminate one of these three triangles from $\mathcal{F}'$, the remaining two triangles have a common vertex in K and cover the same set of edges as $\mathcal{F}'$. Observe that in case we have another intersecting subfamily in $\mathcal{F}'$ without a common vertex, it must be the three triangles of another 3-cone. We repeat the same reasoning for each such pairwise intersecting subfamily to obtain an RS-family containing K.

So we may assume that K is a member of the RS-family $\mathcal{F}$. Observe that each 2-cone is a clique and must be a member of $\mathcal{F}$. Let $C_s = \{s, x, y, z\}$ be a 3-cone with extreme s and basis $T = \{x, y, z\}$. In order to cover the edges incident to s, note that $\mathcal{F}$ must contain exactly two triangles of C_s or must contain the 3-cone C_s itself. Suppose $C_s \in \mathcal{F}$. Note that no other member of $\mathcal{F}$ contains s. By Lemma 2, let $u_T \in \bigcap \mathcal{F}_T$. Since $u_T \in K \cap T$, we may assume $u_T = y$. Consider $\mathcal{F}'$ obtained from $\mathcal{F}$ by the removal of cone C_s and the addition of triangles $\{y, x, s\}$ and $\{y, z, s\}$. Now suppose $F_1, F_2, ..., F_\ell, \{y, x, s\}$ is a pairwise intersecting subfamily of $\mathcal{F}'$ without a common vertex. Since $F_i \cap \{y, x, s\} \neq \emptyset$ and $F_i \cap \{y, x, s\} \neq s$, we may assume $x \in F_1$ and $y \notin F_1$, $x \notin F_2$ and $y \in F_2$. Since F_1, F_2, C_s are pairwise intersecting members of $\mathcal{F}$, we must have $z = F_1 \cap F_2 \cap C_s$. Now $z, x \in F_1$ implies $F_1 \in \mathcal{F}_T$, so $y \in F_1$, a contradiction. Suppose $F_1, F_2, ..., F_\ell, \{y, x, s\}, \{y, z, s\}$ is a pairwise intersecting subfamily of $\mathcal{F}'$ without a common vertex. We have $y \notin F_1$ but $F_1 \cap \{y, x, s\} \neq \emptyset$ and $F_1 \cap \{y, z, s\} \neq \emptyset$, which implies $F_1 \in \mathcal{F}_T$, again leading to a contradiction.

4. implies 1.: Let $\mathcal{E}$ be a family of edges containing all the edges corresponding to the bases of the 2-cones and the edges of the bases of exactly two triangles of each 3-cone that induces a triangle-free subgraph of $G[K]$. Let $e = xy$ be an edge of the family $\mathcal{E}$. Call $S_e = \{s \in S | \{x, y\} \subseteq N(s)\}$. Observe that: (1) if s is the extreme vertex of a 2-cone then s belongs to exactly one set S_e; (2) if s is the extreme vertex of a 3-cone then s belongs to exactly two sets S_e. Consider the complete set family $\mathcal{F}$ whose members are K and the triangles $T_{e,s}$, where $e \in \mathcal{E}$ ans $s \in S_e$. By (1) and (2) if a subfamily of triangles $T_{e,s}$ is pairwise intersecting then the corresponding family of edges e is pairwise intersecting. Since by hypothesis the family of edges do not contain a triangle, then they have a common vertex in K, which implies $\mathcal{F}$ is an RS-family.

The remaining implications are simpler to establish and omitted in the extended abstract. □

The family of edges defined in Theorem 4.4 is called an *RS-basis* of a 3-split clique graph.

Construction of $\mathbf{G_I}$ from $\mathbf{I = (U, C)}$

Let $I = (U, C)$ be any instance of 3SAT$_{\overline{3}}$. We assume with no loss of generality that each variable occurs two or three times in C, and no variable occurs twice in the same clause. In addition, if variable u_i occurs twice in C, then we assume it is once as literal u_i and once as literal $\overline{u_i}$; and if variable u_i occurs three times in C, then we assume it is once as literal u_i and twice as literal $\overline{u_i}$.

For each variable u_i, let j_i be the subindex of the unique clause where variable u_i occurs as literal u_i; and $\overline{J}_i = \{j \mid \text{ literal } \overline{u_i} \text{ occurs in } c_j\}$.

For each clause c_j with $|c_j| = 3$, let $I_j = \{i \mid \text{ variable } u_i \text{ occurs in } c_j\}$; and for each clause c_j with $|c_j| = 2$, let $I_j = \{i \mid \text{ variable } u_i \text{ occurs in } c_j\} \cup \{n+1\}$. Notice that in any case $|I_j| = 3$. Given $I_j = \{i_1, i_2, i_3\}$, with $i_1 < i_2 < i_3$, let $i_1^* = i_2$, $i_2^* = i_3$ and $i_3^* = i_1$.

From instance $I = (U, C)$, we construct a graph $G_I = (V, E)$ as follows.

The vertex set V is the union:

$$V = \bigcup_{1 \le i \le n} \{a^i_{j_i}, b^i_{j_i}, c^i_{j_i}, d^i_{j_i}, e^i_{j_i}, f^i_{j_i}, g^i_{j_i}, h^i_{j_i}\} \cup$$

$$\bigcup_{1 \le i \le n} \bigcup_{j \in \overline{J}_i} \{a^i_j, b^i_j, c^i_j, d^i_j, e^i_j, f^i_j, p^i_j, q^i_j\} \cup$$

$$\bigcup_{1 \le j \le m, |c_j| = 2} \{a^{n+1}_j, c^{n+1}_j, d^{n+1}_j\}.$$

In order to have the property that $G_I = (V, E)$ is a split graph, the edge set E is composed so that:

$$K = \bigcup_{1 \le i \le n} \{a^i_{j_i}, d^i_{j_i}, g^i_{j_i}, h^i_{j_i}\} \cup \bigcup_{1 \le i \le n} \bigcup_{j \in \overline{J}_i} \{a^i_j, d^i_j\} \cup \bigcup_{1 \le j \le m, |c_j| = 2} \{a^{n+1}_j, d^{n+1}_j\}.$$

is a complete set and the remaining vertices $S = V \setminus K$ compose the set:

$$S = \bigcup_{1 \le i \le n} \{b^i_{j_i}, c^i_{j_i}, e^i_{j_i}, f^i_{j_i}\} \cup \bigcup_{1 \le i \le n} \bigcup_{j \in \overline{J}_i} \{b^i_j, c^i_j, e^i_j, f^i_j, p^i_j, q^i_j\} \cup \bigcup_{1 \le j \le m, |c_j| = 2} \{c^{n+1}_j\}.$$

and is a stable set.

We finish the definition of the edge set by defining the edges incident to the vertices of the stable set S: For $1 \le i \le n$, $N(b^i_{j_i}) = \{a^{i^*}_{j_i}, d^i_{j_i}\}$, $N(c^i_{j_i}) = \{a^{i^*}_{j_i}, a^i_{j_i}, d^i_{j_i}\}$, $N(e^i_{j_i}) = \{d^i_{j_i}, h^i_{j_i}\}$, $N(f^i_{j_i}) = \{a^{i^*}_{j_i}, g^i_{j_i}\}$. For $1 \le i \le n, j \in \overline{J}_i$, $N(b^i_j) = \{a^{i^*}_j, d^i_j\}$, $N(c^i_j) = \{a^{i^*}_j, a^i_j, d^i_j\}$, $N(e^i_j) = \{d^i_j, h^i_{j_i}\}$, $N(f^i_j) = \{\{a^{i^*}_j, g^i_{j_i}\}$, $N(p^i_j) = \{a^i_{j_i}, g^i_{j_i}, a^i_j\}$, $N(q^i_j) = \{a^i_{j_i}, h^i_{j_i}, a^i_j\}$. For $1 \le j \le m, \mid c_j \mid = 2$, $N(c^{n+1}_j) = \{a^{n+1}_j, a^{n+1^*}_j\}$.

Note that the constructed instance G_I is a 3-split graph. Notice that for each variable u_i, graph G_I contains as induced subgraph, Truth Setting component T_i, the graph depicted in Figure 1 for the case variable u_i has 3 occurrences. Throughout the paper, we shall use the convention in the figures: vertices of K

are black, vertices of S are white; only edges between vertices of the same cone are drawn which means all other edges between black vertices are omitted. For the convenience of the reader, we offer in the Appendix an example of the whole graph G_I obtained from an instance I of 3SAT$_{\overline{3}}$.

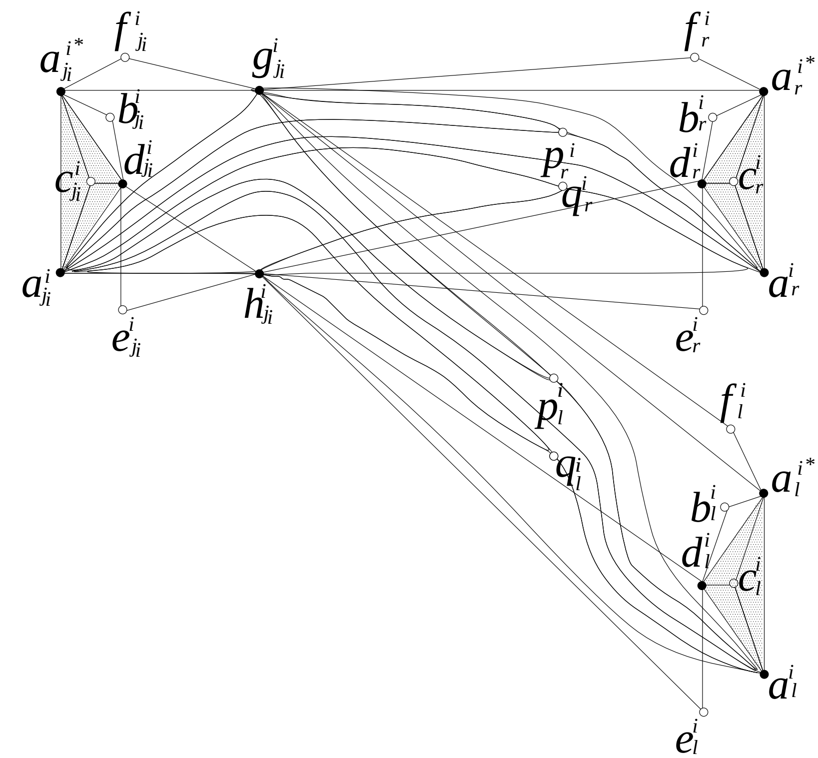

Fig. 1. Graph T_i corresponding to a variable u_i, with $\overline{J}_i = \{r, l\}$.

Please refer to Figure 2 for the proof of Lemma 5.

Lemma 5. (*True edge–False edge*) *Suppose $\mathcal{F}$ be an RS-basis of the constructed graph G_I. For each $j, 1 \le j \le m$, and for each $i \in I_j, i \ne n+1$, exactly one of the edges $a_j^i a_j^{i^*}$, $a_j^i d_j^i$ belongs to $\mathcal{F}$. For each $i, 1 \le i \le n$, and for each $j \in \overline{J}_i$, if $a_j^i d_j^i \in \mathcal{F}$ then $a_{j_i}^i a_{j_i}^{i^*} \in \mathcal{F}$, and if $a_j^i a_j^{i^*} \in \mathcal{F}$ then $a_{j_i}^i d_{j_i}^i \in \mathcal{F}$.*

Proof. Consider any j, $1 \le j \le m$, and $i \in I_j, i \ne n+1$. Assume with no loss of generality, $j = j_i$. By considering the 2-cone $N[b_{j_i}^i]$, notice that edge $a_{j_i}^{i^*} d_{j_i}^i$ must belong to the RS-basis $\mathcal{F}$ which implies that both edges $a_{j_i}^i a_{j_i}^{i^*}$ and $a_{j_i}^i d_{j_i}^i$ cannot belong to $\mathcal{F}$, which implies that exactly one of the edges $a_{j_i}^i a_{j_i}^{i^*}$, $a_{j_i}^i d_{j_i}^i$ belongs to $\mathcal{F}$.

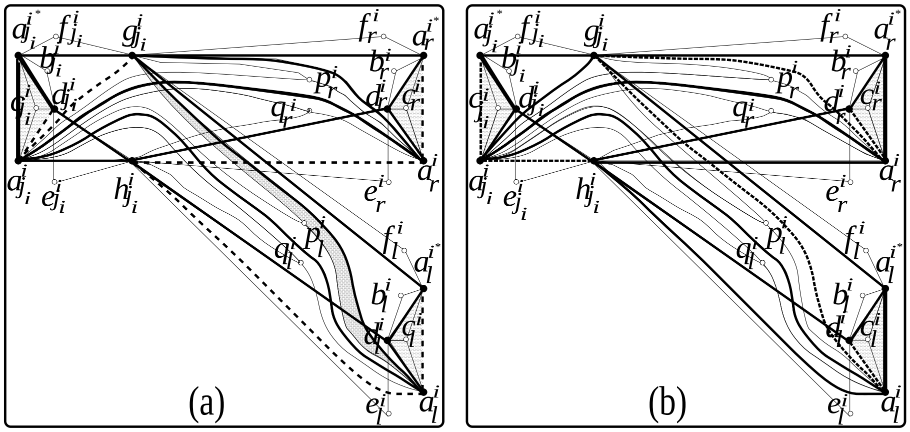

Fig. 2. (a) RS-basis for T_i containing edge $a^i_r d^i_r$ is depicted in bold edges. Dashed edges are the edges of the bases of the 3-cones that are not members of the RS-basis. (b) Respectively for edge $a^i_r a^{i^*}_r$.

Consider any $i, 1 \leq i \leq n$, and $j \in \overline{J}_i = \{r, l\}$. Say $j = r$ and refer to Figure 2(a). Notice that, edge $h^i_{j_i} d^i_r$ must belong to the RS-basis $\mathcal{F}$. Assume that $a^i_r d^i_r \in \mathcal{F}$. Then $a^i_r h^i_{j_i} \notin \mathcal{F}$, and so by considering the 3-cone $N[q^i_r]$, edges $a^i_r a^i_{j_i}, h^i_{j_i} a^i_{j_i} \in \mathcal{F}$. Notice that edge $h^i_{j_i} d^i_{j_i}$ must belong to the RS-basis $\mathcal{F}$. Hence $a^i_{j_i} d^i_{j_i} \notin \mathcal{F}$, and so by the first statement, $a^i_{j_i} a^{i^*}_{j_i} \in \mathcal{F}$. Assume that $a^i_r a^{i^*}_r \in \mathcal{F}$ and refer to Figure 2(b) to obtain an analogous reasoning. □

Lemma 5 is the key for the NP-completeness result. Given any variable u_i and any clause c_j where u_i occurs, any RS-basis of G_I is forced to choose exactly one of the edges $a^i_j a^{i^*}_j$, $a^i_j d^i_j$. If $r \in \overline{J}_i$, then any RS-basis of G_I is forced to choose different types of edges incident to vertices a^i_r and $a^i_{j_i}$, respectively. If $r, \ell \in \overline{J}_i$, then any RS-basis of G_I is forced to choose the same type of edges incident to vertices a^i_r and a^i_ℓ, respectively. The correspondence between the two possible truth assignments of variable u_i and the two possible edges incident to vertex $a^i_{j_i}$ is clear.

Theorem 6. SPLIT CLIQUE GRAPH *is NP-complete.*

Proof. As mentioned in the Introduction, SPLIT CLIQUE GRAPH belongs to NP.

Let G be the constructed 3-split graph obtained from an instance $I = (U, C)$ of 3SAT$_{\overline{3}}$. Suppose G is a clique graph, and we exhibit a truth assignment for U such that C is satisfied. By Theorem 4, let $\mathcal{F}$ be an RS-basis for G. Let $u_i \in U$ be a variable. Set u_i equal to true if and only if edge $a^i_{j_i} d^i_{j_i} \in \mathcal{F}$. To see that this truth assignment for U satisfies C consider a clause c_j and its corresponding triangle $\{a^i_j, a^{i^*}_j, a^{i^{**}}_j\}$. Since $\mathcal{F}$ induces a triangle-free subgraph of $G[K]$, there exists $i \in I_j$ such that the edge $a^i_j a^{i^*}_j$ is not a member of $\mathcal{F}$. Notice that $i \neq n+1$. By Lemma 5, edge $a^i_j a^{i^*}_j \notin \mathcal{F}$ implies that edge $a^i_j d^i_j \in \mathcal{F}$. If $j = j_i$ then variable

u_i is true and clause c_j is satisfied. If $j \neq j_i$, then $j \in \overline{J}_i$, by Lemma 5 edge $a^i_j d^i_j \in \mathcal{F}$ implies edge $a^i_{j_i} a^{i^*}_{j_i} \in \mathcal{F}$, and edge $a^i_{j_i} d^i_{j_i} \notin \mathcal{F}$. It follows that u_i is false, and then c_j is satisfied.

Conversely, given a truth assignment of U that satisfies C, by Theorem 4, it suffices to exhibit an RS-basis $\mathcal{F}$ in order to prove that G is a clique graph.

For each $j, 1 \leq j \leq m$, for each $i \in I_j$, the edges $a^{i^*}_j g^i_j$, $d^i_j h^i_j$, $a^{i^*}_j d^i_j$.

For each $j, 1 \leq j \leq m$, for $i = n+1$, the edges $a^{n+1^*}_j a^{n+1}_j$.

For each $i, 1 \leq i \leq n$, such that variable u_i is true, the edges $d^i_{j_i} a^i_{j_i}$, $a^i_{j_i} g^i_{j_i}$; and for each $j \in \overline{J}_i$, the edges $h^i_{j_i} a^i_j$, $a^i_j a^{i^*}_j$.

For each $i, 1 \leq i \leq n$, such that variable u_i is false, the edges $a^i_{j_i} a^{i^*}_{j_i}$, $a^i_{j_i} h^i_{j_i}$; and for each $j \in \overline{J}_i$, the edges $g^i_{j_i} a^i_j$, $a^i_j d^i_j$.

The proof is completed by showing that the chosen set of edges indeed induces a triangle-free subgraph of $G[K]$ containing all the basis of 2-cones and two edges of the basis of each 3-cone. Details are omitted in the extended abstract. $\square$

For the convenience of the reader, we offer in the Appendix an example of an RS-family defined by a satisfying truth assignment, according to the proof of Theorem 6.

3 Polynomially solvable split clique graph classes

In the following three theorems we present non trivial split graph classes for which clique graphs can be recognized in polynomial time. Let G be a split graph with split partition (K, S), without loss of generality assume $K = \bigcup_{s \in S} N(s)$, to obtain a unique possible split partition.

Fig. 3. (a) w is a private neighbour of s_2. (b) no vertex in S has a private neighbour.

We say that a vertex $x \in K$ is a *private neighbor* of $s \in S$, if s is the only vertex in S adjacent to x, i.e. $N(x) \cap S = \{s\}$. Please refer to Figure 3.

Theorem 7. *If every vertex $s \in S$ has a private neighbor then G is a clique graph.*

Proof. Suppose every vertex $s \in S$ has a private neighbor h_s. Let x and y be vertices of K. We say that x is a twin of y when $N[x] = N[y]$. Observe this is an equivalence relation, and so the equivalence classes define a partition of K.

Let R_s be the class of h_s for $s \in S$; and $R_1, R_2, ..., R_k$ the remaining classes, this means the classes that do not contain any vertex h_s for $s \in S$. We notice that $\left((R_s)_{s\in S}, R_1, R_2, R_3, \ldots R_k\right)$ is a partition of K. Since h_s is a private neighbor of s, if $s' \in S$ and $s' \neq s$ then $R_s \neq R_{s'}$.

For every $s \in S$, we call I_s the set $\{i, 1 \leq i \leq k$ such that $R_i \subseteq N(s)\}$. Let $\mathcal{F}$ be the family of complete sets of G whose members are: K; $F_{s,i} = R_s \cup R_i \cup \{s\}$, for each $s \in S, I_s \neq \emptyset$ and $i \in I_s$; $F_s = R_s \cup \{s\}$, for each $s \in S, I_s = \emptyset$. We claim that $\mathcal{F}$ is an RS-family of G, and so G is a clique graph.

Details are in the Appendix. □

Theorem 8. *Let $|S| \leq 3$. Graph G is a clique graph if and only if G is not the Hajós graph depicted in Figure 3.(b).*

Proof. It is well known that if G is a clique graph then G is not the Hajós graph. Let us prove the reciprocal implication. Assume G is a graph with split partition (K, S), $\mid S \mid \leq 3$ and G is not the Hajós graph. By Theorem 1, if the clique family of G has the Helly property then G is a clique graph. If the clique family does not satisfy the Helly property, then there exists a subfamily of cliques pairwise intersecting without a common vertex.

It is clear that such subfamily must contain $N[s_1]$, $N[s_2]$ and $N[s_3]$ as members, where s_1, s_2 and s_3 are the vertices in S.

For $1 \leq i < j \leq 3$, let $x_{i,j}$ be three vertices of K such that $x_{i,j} \in N[s_i] \cap N[s_j]$. Since G is not the Hajós graph, then K must contain at least one more vertex.

Call it u and suppose u is a private neighbor, for instance of s_1, then $u \in N[s_1] \setminus (N[s_2] \cup N[s_3])$. In this case it is easy to check that the complete set family $\mathcal{F}$ $N[s_1] \setminus N[s_2]$, $N[s_1] \setminus N[s_3]$, $N[s_2]$, $N[s_3]$ and K satisfies the conditions given by Theorem 1, so G is a clique graph. We depict in Figure 4 such family. Details are in the Appendix. □

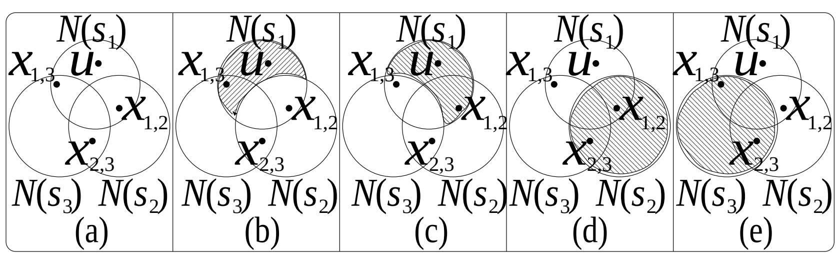

Fig. 4. Case in which u is a private neighbor, assumed of s_1.

Theorem 9. *Let $|K| \leq 4$. Graph G is a clique graph if and only if: (1) There are no three bases of 2-cones forming a triangle; and (2) There are no four bases of cones satisfying: one is the basis $B = \{a, b, d\}$ of a 3-cone, the other three bases $B_1 = \{a, c\}$, $B_2 = \{b, c\}$, and $B_3 = \{d, c\}$ are bases of 2-cones.*

4 Open related problems

We summarize in a table the results and open problems we have managed to state about the complexity of the problem of recognizing clique graphs when restricted to split graphs. Denote by $3split_2$ the class of 3-split graphs, where the vertices of the independent set have degree at least 2 and at most 3, and by $3split_3$ the subclass of 3-split graphs, where the vertices of the independent set have degree exactly 3.

	$3split_3$	$3split_2$	$\forall s \in S, s$ has a private neighbor.	$\lvert S\rvert$ bounded		$\lvert K\rvert$ bounded	
				$\lvert S\rvert \leq 3$	general	$\lvert K\rvert \leq 4$	general
Split graph $G = (V, E)$ partition (K, S).	?	NPC	P	P	?	P	?

The present work presents three distinct sufficient conditions for a split graph to be a clique graph that lead to three non trivial polynomial split clique graph classes. The complexity of recognizing split clique graphs with $|K|$ or $|S|$ bounded remains open.

Several subclasses of clique graphs have been studied for which polynomial-time recognition is known. In particular, for several classes of graphs the corresponding class of clique graphs is known [21]. Note that it is well known that the clique graph of a chordal graph is a dually chordal graph [5, 19] but the complexity of deciding whether a chordal graph is a clique graph was a challenging open problem. We have proved that deciding whether a given split graph is a clique graph is an NP-complete problem. Note that the class of split graphs is the intersection of chordal graphs and complements of chordal graphs.

The NP-completeness of CLIQUE GRAPH [1, 2] suggested the study of the problem restricted to classes of graphs not properly contained in the class of clique graphs. One such class is the class of split graphs, the object of the present paper, and the recognition of split clique graphs is proved NP-complete. Another challenging still open problem is the recognition of planar clique graphs [3].

Let G be a split graph with split partition (K, S). In case G is a 3-split graph, Theorem 4 says G admits an RS-family containing K. We leave as open the complexity of deciding if a split clique graph with split partition (K, S) admits an RS-family containing K.

Our NP-completeness result for split clique graph recognition is optimum in the sense that each vertex of the independent set of our split instance has degree at most 3, whereas when each vertex of the independent set has degree at most 2 the problem is polynomial, since it is reduced to check whether the clique family of the graph satisfies the Helly property. Actually, by Theorem 4 the problem is polynomial when the input is a 3-split graph such that the number of 3-cones is bounded, which implies that 3-split clique graph recognition when $|K|$ is bounded or when $|S|$ is bounded is in P. We leave as open the complexity of recognizing split clique graphs such that every vertex of the independent set has degree exactly 3. Note that the problem $3\text{SAT}_{\overline{3}}$ when restricted to having exactly three literals per clause is polynomial [13].

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Appendix

In Figure 5 we show an 3SPLIT$_3$ instance $G = (V, E)$ obtained from instance 3SAT$_{\overline{3}}$ instance $I = (U, C)$. In Figure 6 we depict a RS-cover for G obtained from a satisfiable truth assignment for U.

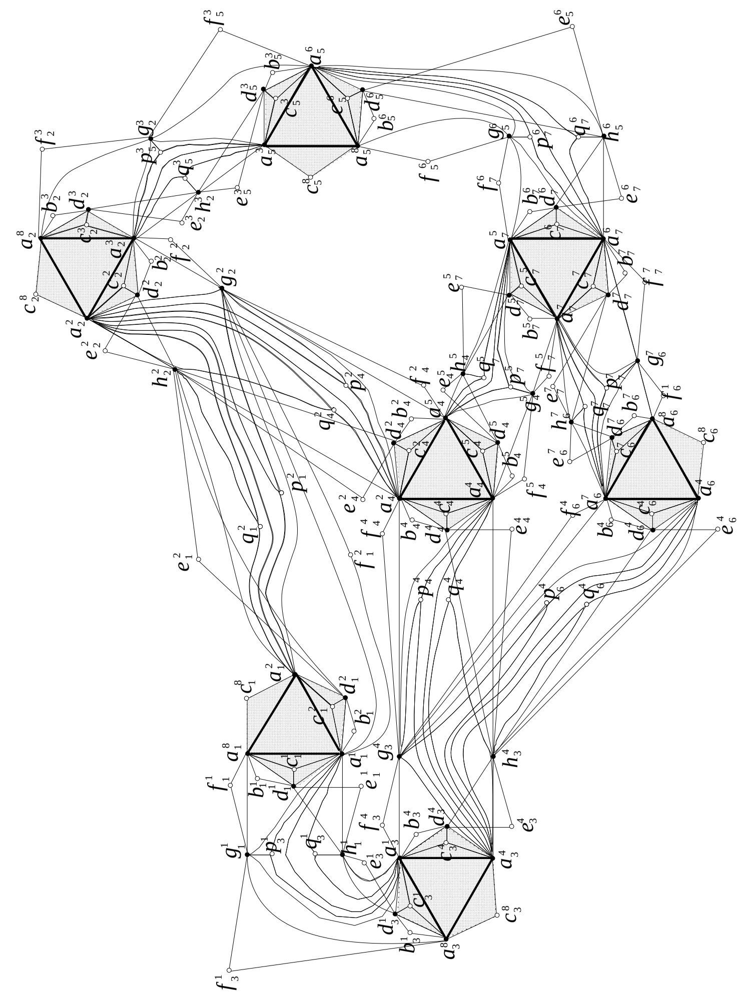

Fig. 5. Instance $G = (V, E)$ obtained from 3SAT$_{\overline{3}}$ instance $I = (U, C)$, where $U = \{u_1, u_2, u_3, u_4, u_5\}$ and $C = \{(u_1, \overline{u}_2), (u_2, u_3), (\overline{u}_1, u_4), (\overline{u}_4, \overline{u}_2, u_5), (\overline{u}_3, \overline{u}_6), (\overline{u}_4, u_7), (\overline{u}_5, \overline{u}_7, u_6)\}$.

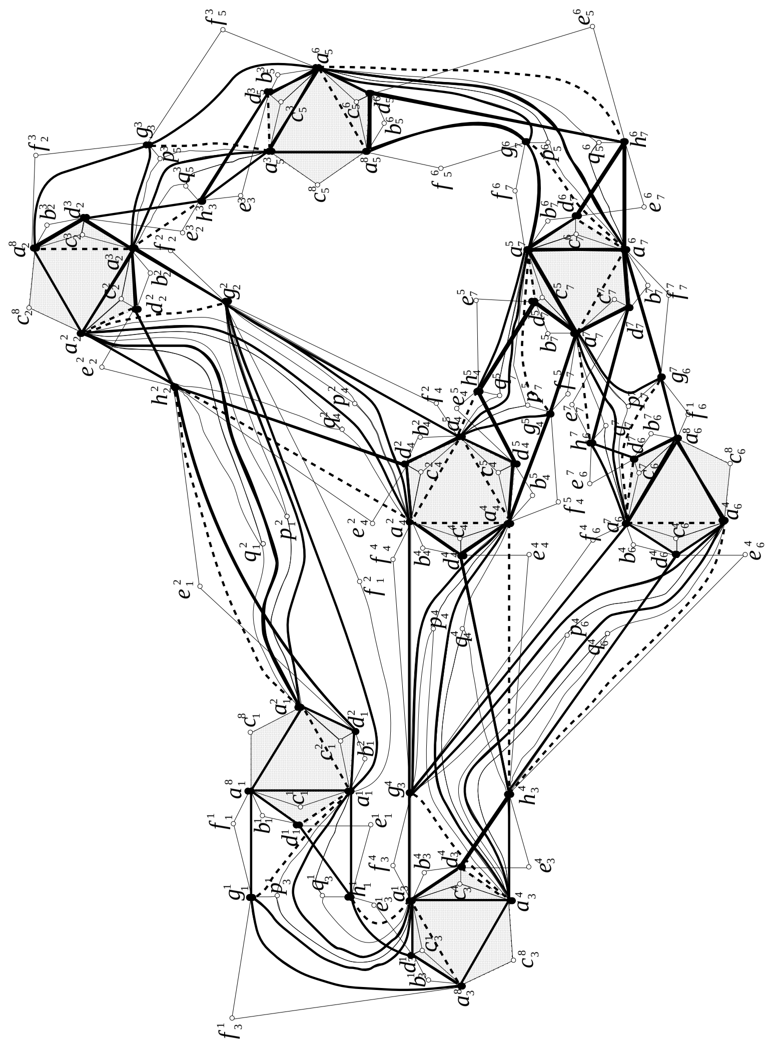

Fig. 6. Satisfying truth assignment for $3\textsc{sat}_{\overline{3}}$ instance $I = (U, C)$, where $\overline{u}_1 = \overline{u}_2 = u_3 = \overline{u}_4 = u_5 = \overline{u}_6 = \overline{u}_7 =$T.

Now we consider the details omitted in the proofs corresponding to Theorems 7, 8 and 9.

Theorem 7

Statement of Theorem 7: If every vertex $s \in S$ has a private neighbor then G is a clique graph.
Proof of Theorem 7: Suppose every vertex $s \in S$ has a private neighbor h_s. Let x and y be vertices of K. We say that x is a twin of y when $N[x] = N[y]$. Observe this is an equivalence relation, and so the equivalence classes define a partition of K. Let R_s be the class of h_s for $s \in S$; and $R_1, R_2, ..., R_k$ the remaining classes, this means the classes that do not contain any vertex h_s for $s \in S$. We notice that $\left((R_s)_{s\in S}, R_1, R_2, R_3, \ldots R_k\right)$ is a partition of K. Since h_s is a private neighbor of s, if $s' \in S$ and $s' \neq s$ then $R_s \neq R_{s'}$.

For every $s \in S$, we call I_s the set $\{i, 1 \leq i \leq k$ such that $R_i \subseteq N(s)\}$. Let $\mathcal{F}$ be the family of complete sets of G whose members are: K; $F_{s,i} = R_s \cup R_i \cup \{s\}$, for each $s \in S, I_s \neq \emptyset$ and $i \in I_s$; $F_s = R_s \cup \{s\}$, for each $s \in S, I_s = \emptyset$. We claim that $\mathcal{F}$ is an RS-family of G, and so G is a clique graph.

Let $e \in E_G$. If both end vertices of e are in K then e is covered by K which is a member of $\mathcal{F}$. If not, since S is a stable set, then $e = sx$ with $s \in S$ and $x \in K$. If there exists i, $1 \leq i \leq k$, such that $x \in R_i$, since every vertex in R_i is a twin of x and x is adjacent to s, then $R_i \subseteq N(s)$. It follows that $F_{s,i}$ covers $e = sx$.

If such i does not exist, there must exist $s' \in S$ such that $x \in R_{s'}$. Then $N[x] = N[s']$ and, since $sx \in E$, it follows that $s = s'$, so e is covered by F_s.

To prove that $\mathcal{F}$ has the Helly property, notice the following facts:

1. F_s is not a member of $\mathcal{F}$ if and only if there exists i such that $F_{s,i}$ is a member of $\mathcal{F}$.
2. $F_{s,i} \cap F_{s',i'} \neq \emptyset$ implies $i = i'$ or $s = s'$.
3. F_s has empty intersection with all members of $\mathcal{F}$ except K.

Now, assume $\mathcal{F}'$ is a pairwise intersecting subfamily with at least three members. Consider the members that are not K. By fact 3, all of them must be of type $F_{s,i}$. Moreover, by fact 2, there must exist i such that all these members have the same subindex i; or there must exist s such that all of them have the same subindex s. In the first case, all members of $\mathcal{F}'$ have the vertices of R_i in common. In the second case, all members of $\mathcal{F}'$ have the vertices of R_s in common. It follows that $\mathcal{F}'$ has non-empty total intersection. This completes the proof. $\square$

Theorem 8

Statement of Theorem 8: Let $|S| \leq 3$. Graph G is a clique graph if and only if G is not the Hajós graph depicted in Figure 3.(b).

Proof of Theorem 8: It is well known that if G is a clique graph then G is not the Hajós graph. Let us prove the reciprocal implication. Assume G is a graph with split partition (K, S), $\mid S \mid \leq 3$ and G is not the Hajós graph. By Theorem 1, if the clique family of G has the Helly property then G is a clique graph. If the clique family does not satisfy the Helly property, then there exists a subfamily of cliques pairwise intersecting without a common vertex.

It is clear that such subfamily must contain $N[s_1]$, $N[s_2]$ and $N[s_3]$ as members, where s_1, s_2 and s_3 are the vertices in S.

For $1 \leq i < j \leq 3$, let $x_{i,j}$ be three vertices of K such that $x_{i,j} \in N[s_i] \cap N[s_j]$. Since G is not the Hajós graph, then K must contain at least one more vertex.

Call it u and suppose u is a private neighbor, for instance of s_1, then $u \in N[s_1] \setminus (N[s_2] \cup N[s_3])$. In this case it is easy to check that the complete set family $\mathcal{F}$

$$N[s_1] \setminus N[s_2], \quad N[s_1] \setminus N[s_3], \quad N[s_2], \quad N[s_3] \quad \text{and} \quad K$$

satisfies the conditions given by Theorem 1, so G is a clique graph. We depict in Figure 4 such family. Observe that if $N[s_1] \setminus N[s_2]$ (Figure 4(b)) belongs to a pairwise intersecting family $\mathcal{F}'$, then $N[s_2] \notin \mathcal{F}'$ (Figure 4(e)), since their intersection is empty. The same occurs between $N[s_1] \setminus N[s_3]$ (Figure 4(c)) and $N[s_3]$ (Figure 4(d)). Hence, three intersecting complete sets of $\mathcal{F}'$ have $x_{1,2}$, or $x_{2,3}$, or u as a common element.

If u is not a private neighbor, we can assume $N(s_i) \setminus (N(s_j) \cup N(s_k)) = \emptyset$ for the three different possible sub-indices. Then u is adjacent to at least two vertices of S; without loss of generality assume $u \in N(s_1) \cap N(s_2)$. In this case it is easy to check that the complete set family $\mathcal{F}$

$$N[s_1] \setminus (N[s_2] - \{u\}), \quad N[s_1] \setminus N[s_3], \quad N[s_2] \setminus (N[s_1] - \{x_{1,2}\}), \quad N[s_2] \setminus N[s_3], \quad N[s_3] \quad \text{and} \quad K$$

satisfies the conditions given by Theorem 1, so G is a clique graph. We depict in Figure 7 such family. Observe that if $N[s_1] \setminus (N[s_2] - \{u\})$ (Figure 7(b)) belongs to a pairwise intersecting family $\mathcal{F}'$, then $N[s_2] \setminus (N[s_1] - \{x_{1,2}\}) \notin \mathcal{F}'$ (Figure 7(d)). The same occurs between $N[s_1] \setminus N[s_3]$ (Figure 7(c)) and $N[s_3]$ (Figure 7(f)). Hence, three intersecting complete sets of $\mathcal{F}'$ have $x_{1,2}$, or $x_{2,3}$, or u as a common element. The proof is complete. $\square$

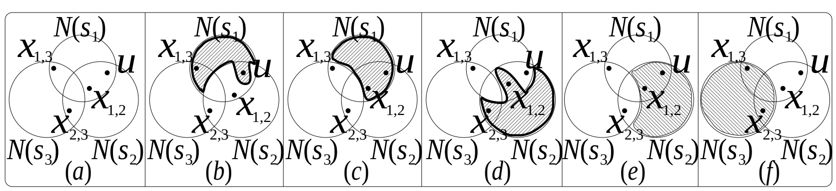

Fig. 7. Case in which u is not a private neighbor.

Theorem 9

Statement of Theorem 9: Let $|K| \leq 4$. Graph G is a clique graph if and only if: (1) There are no three bases of 2-cones forming a triangle; and (2) There are no four bases of cones satisfying: one is the basis $B = \{a, b, d\}$ of a 3-cone, the other three bases $B_1 = \{a, c\}$, $B_2 = \{b, c\}$, and $B_3 = \{d, c\}$ are bases of 2-cones. Proof of Theorem 9: If $|K| \leq 3$, then G is a 3-split graph, and then G is a clique graph if and only if G is an extended triangle that satisfies the Helly property, and then G is a clique graph if and only if there is no subset of the set of bases of cones of G with three bases that correspond to edges of G forming a triangle.

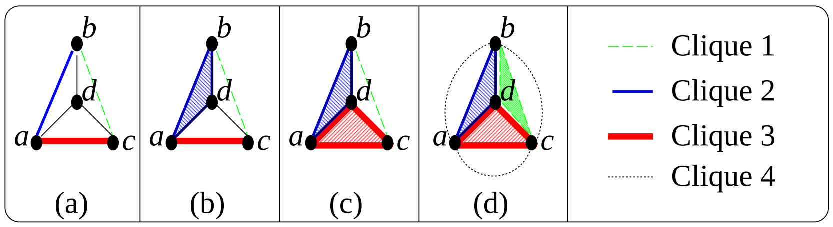

Fig. 8. The 4 possible different non Helly bases of cones for a split graph G with $|K| = 4$.

Suppose $|K| = 4$, and assume $K = \{a, b, c, d\}$. First observe that if for a vertex $s \in S$ we have $N(s) = K$, then G is a clique graph iff $G - s$ is a clique graph, so we can consider G as a 3-split graph. We consider when it occurs exactly one of the five cases below:

1. The set of bases of cones of G satisfy the Helly property - In this case G is a clique Helly graph, and therefore a clique graph.
2. There is a set of 3 bases of cones of G which does not satisfy the Helly property formed by three bases that correspond to edges, say ab, bc, ca (Figure 8(a)) - As these bases correspond to three 2-cones, then G is not a clique graph.
3. There is a set of 3 bases of cones of G which does not satisfy the Helly property formed by two bases that correspond to a pair of edges say ac, bc and by the triangle $\{a, b, d\}$ (Figure 8(b)) - In this case if $\mathcal{F}$ is an RS-cover to G, then the two 2-cones corresponding to the bases $\{a, c\}$,$\{b, c\}$ must belong to $\mathcal{F}$. Hence, neither the 3-cones corresponding to the triangle $\{a, b, d\}$ nor the 2-cone corresponding to basis $\{a, b\}$ can belong to $\mathcal{F}$. Thus, if $\mathcal{B}$ is the RS-basis of G related to $\mathcal{F}$, then $\mathcal{B} = \{ac, bc, ad, bd\}$. Observe that the elements of $\mathcal{B}$ satisfy the Helly property. Notice that every triangle basis T of G contains exactly one pair of edges of $\mathcal{B}$. Hence, the two cones corresponding to the pair of bases of $\mathcal{B}$ contained in T can be used to cover the edges of the 3-cone of T. If an edge e, that is not ab nor cd, is a basis of a 2-cone of G, then e belongs to $\mathcal{B}$. Notice that $\{a, b\}$ is not a basis of

a 2-cone of G, since we would have a triangle with 3 basis edges ac, bc, ab of the case 2. Therefore, G is a clique graph if and only if edge cd does not belong to the set of bases of G.

4. There is a set of 3 bases of cones of G which does not satisfy the Helly property formed by one edge say bc and the two triangles say $\{a, b, d\}$, $\{a, c, d\}$ (Figure 8(c)) - In this case $\{b, c\}$ must belong to any RS-basis of G. If $\{a, b, d\}$ belonged to an RS-basis, then neither $\{a, c\}$ nor $\{c, d\}$ would belong to the RS-basis, because $\{b, c\}$ belongs to the RS-basis, a contradiction with the basis $\{a, c, d\}$. If $\{a, c, d\}$ belonged to an RS-basis, then neither $\{a, b\}$ nor $\{b, d\}$ would belong to the RS-basis, because $\{b, c\}$ belongs to the RS-basis, a contradiction with the basis $\{a, b, d\}$. Hence, neither triangle $\{a, b, d\}$ nor triangle $\{a, c, d\}$ can belong to any RS-basis. Besides, sub-basis $\{a, b\}$ and $\{a, c\}$ can not belong at the same time to an RS-basis. Suppose that $\{a, b\}$ belong to $\mathcal{B}$ an RS-basis. Then, $\{a, c\}$ does not belong to $\mathcal{B}$, and as $\{a, c, d\}$ is a basis of G, we have that $\{a, d\}$ and $\{c, d\}$ are sub-bases of $\mathcal{B}$. Thus, if $\{a, b\}$ belongs to an RS-basis $\mathcal{B}_1$, then $\mathcal{B}_1 = \{ab, bc, ad, cd\}$. Analogously, if $\{a, c\}$ belongs to an RS-basis $\mathcal{B}_2$, then $\mathcal{B}_2 = \{ac, ad, bc, bd\}$. Observe that ab and ac can not belong to the same time to the set bases of cones of G, otherwise we have the 3 basis edges ac, bc, ab of the case 2 not of the case 4. Hence, in this case G is a clique graph.
5. There is a set of 4 bases of cones of G which does not satisfy the Helly property formed by the four triangles $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$ - In this case if triangle $\{a, b, c\}$ belongs to an RS-basis $\mathcal{B}$, then no pair of edges among ad, bd, cd belongs to $\mathcal{B}$. Then we have a contradiction with triangles $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$. Hence, no triangle among $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$ belongs to $\mathcal{B}$. Since, $\{a, b, c\}$ is a basis of a cone of G, exactly two among the edges ab, bc, ca are sub-bases of $\mathcal{B}$. Assume that ab, bc are sub-bases of $\mathcal{B}$. Because ac is not a sub-basis of $\mathcal{B}$, then ad and cd are sub-bases of $\mathcal{B}$. Hence, if $\mathcal{B}$ is an RS-basis for G, then
   - If ab, bc are sub-bases of $\mathcal{B}$, then $\mathcal{B} = \{ab, bc, ad, cd\}$, and analogously
   - If ab, ca are sub-bases of $\mathcal{B}$, then $\mathcal{B} = \{ab, ca, bd, cd\}$, and
   - If ac, bc are sub-bases of B, then $\mathcal{B} = \{ac, bc, ad, bd\}$.

   Notice that, if there is an edge belonging to $\mathcal{B}$, then we have case 4. Therefore, in this case G is clique graph.

□