J Fourier Anal Appl
DOI 10.1007/s00041-017-9527-9

Mathematical Analysis of a Cauchy Problem for the Time-Fractional Diffusion-Wave Equation with $\alpha \in (0, 2)$

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Received: 4 September 2016 / Revised: 24 November 2016

Abstract This paper deals with a theoretical mathematical analysis of a Cauchy problem for the time-fractional diffusion-wave equation in the upper half-plane, $x \in \mathbb{R}$, $t \in \mathbb{R}^+$, where the Caputo fractional derivative of order $\alpha \in (0, 2)$ is considered. An explicit solution to this Cauchy problem is obtained via separation of variables. A first proof of the validity of the obtained results is provided for a certain kind of initial conditions. Throughout this work a new expression of the solution to this problem and its utility for carrying out rigurous proofs are presented. Finally, several new properties of the solution are obtained.

Keywords Time-fractional diffusion-wave equation · Caputo fractional derivative · Cauchy problem · Mittag–Leffler function · Mainardi function · Fourier transform

Mathematics Subject Classification Primary 26A33 · Secondary 33E12 · 35G10 · 35R11 · 42A38

Communicated by Luis Vega.

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Published online: 12 April 2017

1 Introduction

Although already distinguished mathematicians like Leibniz, Liouville or Riemann had already contributed to establish a consistent definition of fractional derivatives, only in the last three decades fractional analysis has grown in importance. These days fractional derivatives have been applied to most of the branches of analysis, such as calculus of variations (see [1,2,9]), differential equations (see [10,28,39]) or numerical analysis (see [5,29,31]) and as an inmediate consequence it has been tried to generalize mathematical tools and theorems of classical analysis to fractional analysis, being different maximum principles (see [19,34]), Fourier analysis (see [21]) or Hopf's lemma (see [36]) particular examples of this fact.

Furthermore, abundant applications in all scientific areas have emerged from the study of fractional derivatives up to now. An example of this is the study of the behaviour of viscoelastic media, which present an intermediate behaviour between the one of viscous fluids and of elastic solids. Since these two cases are interpreted by the diffusion equation and the wave equation respectively, it seems natural to interpret the intermediate behaviour by an interpolation equation with fractional derivatives, which is called the time-fractional diffusion-wave equation,

$$\frac{\partial^\alpha}{\partial t^\alpha} u\left(x,t\right) = \lambda \frac{\partial^2}{\partial x^2} u\left(x,t\right), \qquad \alpha \in (0,2),$$

where the definition of fractional derivative established by Caputo (see [4]) is considered, since it turned out to be the best mathematical framework of fractional analysis to physical phenomena. This is the main equation this paper is dealing with and partly this paper aims at exhibiting the inner nature of this equation and its solutions.

From the beginning of fractional calculus until now a huge amount of papers dealt with this equation (e.g.[6,8,13,15–18,20,22,24–27,30,32,33,35,37,38]), where the fractional derivative definition, the order of differentiation, the variable to which the derivative is applied (time or space), the domain and the problems in which the equation is considered vary from paper to paper. Among the different applications that have been presented, we want to emphasize the connection to the behaviour of linear viscoelastic media mentioned above, which has primarily been studied by Mainardi (see [23]).

In this paper we study a Cauchy Problem for the one-dimensional time-fractional diffusion-wave equation in the upper half-plane. Both groups of functions, the Mittag–Leffler functions and the Mainardi functions, play a fundamental role in fractional calculus and particularly in the study of the time-fractional diffusion-wave equation. Given their key role, a handful of basic properties of these functions, that will be useful hereinafter, will be presented in Sect. 2. In Sect. 3 an alternative way to obtain the solution already presented in previous papers (see [12,25]) is given, providing for the first time a complete mathematical background and theoretical foundations of these results. Section 4 shows different bounds to the space-L^p norm of the obtained solution. In Sect. 5 the continuity of the solution with respect to the order of differentiation α is proven and in this way the consistency of the differential equation alltogether with its solution with respect to the classical integer order differential equations is exhibited. Several particular cases are finally considered in Sect. 6. Throughout this paper the

Fourier-transform will play a key-role, since almost all results will be obtained by using this integral transform together with its classical properties.

During this work a new expression of the solution to the Cauchy problem, which opens a gate to a new approach to this equation, is obtained. Its evident applicability to strict, meticulous proofs will be made clear in the following sections.

2 Preliminar Definitions and Basic Properties of the Key Functions of Fractional Analysis

There are different definitions of fractional derivatives, like the Riemann–Liouville fractional derivative or the Grünwald–Letnikov fractional derivative, among others (see [7]). In this paper we work with the Caputo fractional derivative since it has numerous applications in a great branch of scientific areas. It has proven itself as an usefull tool while modelling physical phenomena, since Cauchy problems with this fractional derivative require initial data of the solution and its integer order derivatives, whereas other fractional derivatives demand initial data of fractional derivatives of the solution, which lack of physical interpretation.

Definition 1 Let $\alpha \in \mathbb{R}^+$. The Riemann–Liouville fractional integral operator of order α, which will be denoted I_a^α, is defined on $L^1([a, b])$ by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

Definition 2 Let $\alpha \in \mathbb{R}^+$ and $n = \lceil \alpha \rceil$. The Caputo fractional derivation operator of order α, which will be denoted ${}^C D_a^\alpha$, is defined on $W^{n,1}([a, b])$ by

$${}^C D_a^\alpha f(t) = I_a^{n-\alpha} f^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau.$$

In this paper we will work with $a = 0$. To simplify notation, the expressions ${}^C D_0^\alpha f(t) = f^{(\alpha)}(t) = \frac{d^\alpha}{dt^\alpha} f(t)$ will be used.

The Fourier transform plays a mayor role in this work. The most important results—specially the rigorous proofs of previously only heuristically obtained statements—were obtained by applying this usefull tool.

Definition 3 Let $f \in L^2(\mathbb{R})$. The Fourier transform of f, which will be denoted $\mathcal{F}\{f\}$, is defined by

$$\mathcal{F}\{f\}(\varsigma) = \int_{-\infty}^{\infty} e^{i\varsigma y} f(y)\, dy.$$

We also write

$$f \overset{\mathcal{F}}{\longleftrightarrow} \mathcal{F}\{f\}.$$

The Inverse Fourier transform $\overline{\mathcal{F}}$, which verifies $\mathcal{F}\overline{\mathcal{F}}f = \overline{\mathcal{F}}\mathcal{F}f \quad \forall f \in L^2(\mathbb{R})$, is given by

$$\overline{\mathcal{F}}\{f\}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixy} f(y)\,dy.$$

What follows is a brief synopsis of basic properties of two essential groups of functions related to fractional analysis, namely the Mittag–Leffler functions and the Mainardi functions. Some new useful properties—which helped us in the course of this work— will be presented.

Definition 4 Let $\alpha, \beta > 0$. The two–parameter Mittag–Leffler function, which will be denoted $\mathcal{E}_{\alpha,\beta}$, is defined for $z \in \mathbb{C}$ by

$$\mathcal{E}_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

If $\beta = 1$, we write $\mathcal{E}_{\alpha,1} = \mathcal{E}_\alpha$ and call it the one–parameter Mittag–Leffler function.

It has been proven that $\forall\ \alpha, \beta > 0$ the Mittag–Leffler function $\mathcal{E}_{\alpha,\beta}$ is entire (see [11]), nevertheless we will only work with the real domain. This function is known as a generalization of the exponential function due to several properties it verifies, such as $\mathcal{E}_1(z) = e^z$ and the fact that $\mathcal{E}_\alpha(cz^\alpha)$ verifies the differential equation

$$\frac{d^\alpha}{dz^\alpha} f(z) = cf(z). \tag{1}$$

Definition 5 Let $\alpha > -1$ and $\beta \in \mathbb{R}$. The Wright function, which will be denoted $\mathcal{W}(z; \alpha, \beta)$, is defined by

$$\mathcal{W}(z; \alpha, \beta) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\alpha k + \beta)}.$$

Once again it has been proven that the Wright function is an entire function $\forall \alpha > -1$ (see [12]), $\beta \in \mathbb{R}$ and its derivative is given by

$$\frac{d}{dz}\mathcal{W}(z; \alpha, \beta) = \mathcal{W}(z; \alpha, \beta + \alpha).$$

Definition 6 Let $\nu \in (0, 1)$. The Mainardi function, which will be denoted $\mathcal{M}_\nu$, is defined for $z \in \mathbb{C}$ by

$$\mathcal{M}_\nu(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(-\nu k + 1 - \nu)}.$$

For all $\nu \in (0, 1)$ the Mainardi function $\mathcal{M}_\nu$ is entire, which is a consequence of the fact that the Mainardi function is a special case of the Wright function, namely $\mathcal{M}_\nu(z) = \mathcal{W}(-z; -\nu, 1 - \nu)$. As in the case of the Mittag–Leffler function, only the real domain will be considered. Having

$$\mathcal{M}_{\frac{1}{2}}(z) = \frac{1}{\sqrt{\pi}} e^{-\frac{z^2}{4}},$$

(see [27]), this function can be considered to be a generalization of the Gaussian function. The role of the Mainardi function as a fractional heat–kernel—which has been analyzed throughout this paper—also supports this idea.

For the Mainardi function, the following rule for absolute moments in $\mathbb{R}^+$ has been proven:

$$\int_0^\infty t^n \mathcal{M}_\nu(t)\,dt = \frac{n!}{\Gamma(\nu n + 1)}. \tag{2}$$

In fact, a more general property is obtained in [25], replacing the Mainardi function by the Wright function:

$$\int_0^\infty t^n \mathcal{W}(-t; \alpha, \beta)\,dt = \frac{n!}{\Gamma(-\alpha n + \beta - \alpha)}. \tag{3}$$

Using (3) it can be shown that

$$\sup_{x \in \mathbb{R}} \left| x^m \mathcal{M}_\nu^{(n)}(|x|) \right| < \infty$$

$\forall\, m, n \in \mathbb{N}_0$ and $\forall\, \nu \in (0, 1)$. In fact, from the series expansion of the Mainardi function and its derivatives it can easily be derived that

$$\lim_{x \to 0^-} x^m \mathcal{M}_\nu^{(n)}(|x|) = \begin{cases} \frac{1}{\Gamma(1-n\nu)} & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases} \quad \lim_{x \to 0^+} x^m \mathcal{M}_\nu^{(n)}(|x|) = \begin{cases} \frac{(-1)^n}{\Gamma(1-n\nu)} & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases},$$

and together with the fact that

$$\lim_{|x| \to \infty} x^m \mathcal{M}_\nu^{(n)}(|x|) = 0,$$

which is obtained from (3), and the continuity of $x^m \mathcal{M}_\nu^{(n)}(|x|)$ in $\mathbb{R} - \{0\}$ the result is obtained. This shows that the Mainardi function $\mathcal{M}_\nu(|x|)$ has a behaviour similar to a Schwartz–function, i.e. a function all of whose derivatives are continuous and rapidly decreasing. Note that this makes sense if we think of the Mainardi function as a generalization of the Gaussian function, the most common example of a nontrivial function in the Schwartz space. Nevertheless $\mathcal{M}_\nu(|x|) \notin \mathcal{S}(\mathbb{R})$ for $\nu \neq \frac{1}{2}$, since in $x = 0$ its derivative is not continuous. We also have,

Proposition 1 $\mathcal{M}_\nu(|x|) \in L^p(\mathbb{R})$, $\forall\ 1 \leq p \leq \infty$, $\forall\ \nu \in (0, 1)$.

Proof It is an immediate consequence of

$$\sup_{x \in \mathbb{R}} \left| x^m \mathcal{M}_\nu^{(n)}(|x|) \right| < \infty \quad \forall\ m, n \in \mathbb{N}_0.$$

□

The Mainardi function and the Mittag–Leffler function are strongly related by the Fourier transform (see [25]).

Proposition 2 *The following relation between the Mainardi function and the Mittag–Leffler function through the Fourier-transform holds.*

$$\mathcal{M}_\nu(|x|) \overset{\mathcal{F}}{\longleftrightarrow} 2\mathcal{E}_{2\nu}\left(-\xi^2\right).$$

Since $\mathcal{M}_\nu(|x|) \in L^2(\mathbb{R})$ $\forall\ \nu \in (0, 1)$, we now have $\mathcal{E}_{2\nu}\left(-\xi^2\right) \in L^2(\mathbb{R})$ $\forall\ \nu \in (0, 1)$. Another consequence is that the family of functions $\{\mathcal{E}_{2\nu}(-\xi^2)\}_{\nu \in (0,1)}$ is uniformly bounded, since

$$\sup_{\xi \in \mathbb{R}} \mathcal{E}_{2\nu}(-\xi^2) \leq \int_{-\infty}^{\infty} \frac{1}{2} \mathcal{M}_\nu(|x|) dx = 1.$$

In [27], Mainardi already observed that for $\nu \to 1$ the limit of the sequence of functions $\{\mathcal{M}_\nu\}$ in the real positive domain is the Dirac–delta distribution centered in 1, δ_1. Nevertheless no proof of this fact has been provided yet. What follows is a proof of an equivalent proposition in the space of tempered distributions, for which we will need the following lemma.

Lemma 1 *The following pointwise convergence holds.*

$$\lim_{\nu \to 1} \mathcal{E}_{2\nu}(-\xi^2) = \cos(\xi),$$

where the convergence is uniform over compact sets.

Proof Take $0 < \varepsilon < 1$ and ν so that $2\nu > \varepsilon$. And so for sufficiently big $k \in \mathbb{N}$ it is valid that $\frac{1}{\Gamma(2\nu k+1)} < \frac{1}{\Gamma(\varepsilon k+1)}$. We obtain

$$\left|\frac{(-1)^k \xi^{2k}}{\Gamma(2\nu k+1)}\right| \leq \frac{|\xi|^{2k}}{\Gamma(\varepsilon k+1)} := a_k$$

To see that the series $\sum_{k=0}^{\infty} a_k$ is convergent it is sufficient to show that $\left|\frac{a_{k+1}}{a_k}\right|$ converges to 0 when $k \to \infty$.

$$\left|\frac{a_{k+1}}{a_k}\right| = |\xi|^2 \frac{\Gamma(\varepsilon k+1)}{\Gamma(\varepsilon k+\varepsilon+1)} \leq |\xi|^2 \frac{(\varepsilon k+\varepsilon)^{1-\varepsilon}\Gamma(\varepsilon k+\varepsilon)}{(\varepsilon k+\varepsilon)\Gamma(\varepsilon k+\varepsilon)} = \frac{|\xi|^2}{(\varepsilon k+\varepsilon)^{\varepsilon}} \to 0$$

Here we used the property $\Gamma(x+s) \leq x^s \Gamma(x)$, valid for $x > 0$ and $0 \leq s \leq 1$. This finally completes the proof since we now have

$$\lim_{\nu\to 1} \mathcal{E}_{2\nu}(-\xi^2) = \sum_{k=0}^{\infty} \lim_{\nu\to 1} \frac{(-\xi^2)^k}{\Gamma(2\nu k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k \xi^{2k}}{(2k)!} = \cos(\xi).$$

□

For simplicity we will write $\mathcal{M}_\nu := \mathcal{M}_\nu(|\cdot|)$ and obtain the following result.

Proposition 3 *Consider $\mathcal{M}_\nu \in \mathcal{S}'(\mathbb{R})$ to be a tempered distribution. Then*

$$\mathcal{M}_\nu \to \delta_1 + \delta_{-1} \quad \textit{as} \quad \nu \to 1 \quad \textit{in } \mathcal{S}'(\mathbb{R}).$$

Proof Take $\psi \in \mathcal{S}(\mathbb{R})$. Then using proposition 2

$$\langle \mathcal{M}_\nu, \psi\rangle = \langle \mathcal{F}\{\overline{\mathcal{F}}\{\mathcal{M}_\nu\}\}, \psi\rangle = \langle \overline{\mathcal{F}}\{\mathcal{M}_\nu\}, \mathcal{F}\{\psi\}\rangle = \int_{-\infty}^{\infty} 2\mathcal{E}_{2\nu}(-\xi^2)\mathcal{F}\{\psi\}(\xi)d\xi.$$

But now, $|2\mathcal{E}_{2\nu}(-\xi^2)\mathcal{F}\{\psi\}(\xi)| \leq 2|\mathcal{F}\{\psi\}(\xi)| \in L^1(\mathbb{R})$ and so we have

$$\begin{aligned}\lim_{\nu\to 1}\langle \mathcal{M}_\nu, \psi\rangle &= \int_{-\infty}^{\infty} 2\cos(\xi)\mathcal{F}\{\psi\}(\xi)d\xi = \int_{-\infty}^{\infty} e^{i\xi}\mathcal{F}\{\psi\}(\xi)d\xi + \int_{-\infty}^{\infty} e^{-i\xi}\mathcal{F}\{\psi\}(\xi)d\xi \\ &= \langle e^{i\xi}, \mathcal{F}\{\psi\}\rangle + \langle e^{-i\xi}, \mathcal{F}\{\psi\}\rangle = \langle \delta_1, \psi\rangle + \langle \delta_{-1}, \psi\rangle = \langle \delta_1 + \delta_{-1}, \psi\rangle.\end{aligned}$$

Here lemma 1 was used. □

Different other properties of the Mainardi functions help us to make a graphic representation of this family of functions. We have $\mathcal{M}_\nu > 0 \;\forall\, \nu \in (0, 1)$, we know from (2) that $\lim_{x\to\infty} \mathcal{M}_\nu(x) = 0$ and that $\int_0^{\infty} \mathcal{M}_\nu(t)\,dt = 1 \;\forall\, \nu \in (0, 1)$, and it has been shown that for $\nu \in \left(0, \frac{1}{2}\right)$ $\mathcal{M}_\nu$ is a decreasing function (see [10]), whereas for $\nu \in \left(\frac{1}{2}, 1\right)$ $\mathcal{M}_\nu$ has exactly one local maximum (see [27]).

3 The Cauchy Problem for the Time-Fractional Diffusion-Wave Equation

For $\alpha \in (0, 2)$ we consider the Cauchy problem for the one-dimensional time-fractional diffusion-wave equation

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \lambda \frac{\partial^2}{\partial x^2} u(x,t) & \text{if} \quad x \in \mathbb{R}; \qquad t \in \mathbb{R}_0^+ \\ u(x,0) = \varphi(x) & \text{if} \quad x \in \mathbb{R} \\ \text{u bounded} \end{cases} \tag{4}$$

and we want to show via separation of variables that the following function is its solution.

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{2} \sqrt{\lambda t^\alpha}^{-1} \mathcal{M}_{\frac{\alpha}{2}} \left(\sqrt{\lambda t^\alpha}^{-1} |x - \xi| \right) \varphi(\xi)\, d\xi,$$

where $\mathcal{M}_{\frac{\alpha}{2}}$ is the Mainardi function of parameter $\frac{\alpha}{2}$. This expression was obtained by Mainardi in [26] using the Laplace transform in the time-variable and the Fourier transform in the space-variable. The function $K_\alpha(x,t)$ defined as

$$K_\alpha(x,t) := \frac{1}{2} \sqrt{\lambda t^\alpha}^{-1} \mathcal{M}_{\frac{\alpha}{2}} \left(\sqrt{\lambda t^\alpha}^{-1} |x| \right)$$

is also refered to as fractional heat–kernel. Let us consider now a solution of the type $u(x,t) = X(x)\, T(t)$. Such a function will be solution to (4) if and only if:

$$\frac{X''(x)}{X(x)} = \frac{T^{(\alpha)}(t)}{\lambda T(t)} = -c$$

where c is a real constant. We obtain the following two differential equations:

$$T^{(\alpha)}(t) = -c\lambda T(t) \tag{5}$$

$$X''(x) = -cX(x) \tag{6}$$

From (1), the solution to (5) is given by

$$\mathcal{E}_\alpha \left(-c\lambda t^\alpha \right),$$

where $\mathcal{E}_\alpha$ is the one–parameter Mittag–Leffler function of parameter α, and the general solution to (6) is given by:

$$X(x) = Ae^{\sqrt{-c}x} + Be^{-\sqrt{-c}x}.$$

In order to obtain a bounded solution, $\sqrt{-c}$ has to be a pure imaginary number, which means that $c > 0$. Therefore we can set $c = \xi^2$, with $\xi \in \mathbb{R}$. For all ξ we obtain a solution to the differential equation given by

$$u(x, t; \xi) = \left(a(\xi) e^{i\xi x} + b(\xi) e^{-i\xi x}\right) \mathcal{E}_\alpha\left(-\xi^2 \lambda t^\alpha\right).$$

Now we propose as a candidate for the solution the following integral

$$u(x, t) = \int_{-\infty}^{\infty} u(x, t; \xi) \, d\xi$$

$$= \int_{-\infty}^{\infty} a(\xi) e^{i\xi x} \mathcal{E}_\alpha\left(-\xi^2 \lambda t^\alpha\right) d\xi + \int_{-\infty}^{\infty} b(\xi) e^{-i\xi x} \mathcal{E}_\alpha\left(-\xi^2 \lambda t^\alpha\right) d\xi$$

$$= \int_{-\infty}^{\infty} f(\xi) e^{i\xi x} \mathcal{E}_\alpha\left(-\xi^2 \lambda t^\alpha\right) d\xi \tag{7}$$

where $f(\xi) = a(\xi) + b(-\xi)$ is a function yet to be determined. For this purpose, we consider the initial condition:

$$u(x, 0) = \int_{-\infty}^{\infty} f(\xi) e^{i\xi x} d\xi = \mathcal{F}\{f\}(x) = \varphi(x).$$

Due to the properties of the Fourier transform we can state that $f(\xi) = \overline{\mathcal{F}}\{\varphi\}(\xi)$ and use this together with

$$\frac{1}{2}\sqrt{\lambda t^\alpha}^{-1} \mathcal{M}_{\frac{\alpha}{2}}\left(\sqrt{\lambda t^\alpha}^{-1} |x|\right) \overset{\mathcal{F}}{\longleftrightarrow} \mathcal{E}_\alpha\left(-\lambda \xi^2 t^\alpha\right)$$

(which is obtained from proposition 2 using the scaling property) in (7):

$$u(x, t) = \int_{-\infty}^{\infty} \overline{\mathcal{F}}\{\varphi\}(\xi) e^{i\xi x} \mathcal{E}_\alpha\left(-\xi^2 \lambda t^\alpha\right) d\xi$$

$$= \int_{-\infty}^{\infty} \overline{\mathcal{F}}\{\varphi\}(\xi) e^{i\xi x} \mathcal{F}\left\{\frac{1}{2}\sqrt{\lambda t^\alpha}^{-1} \mathcal{M}_{\frac{\alpha}{2}}\left(\sqrt{\lambda t^\alpha}^{-1} |\cdot|\right)\right\}(\xi) \, d\xi$$

$$= \int_{-\infty}^{\infty} \overline{\mathcal{F}}\{\varphi\}(\xi) \mathcal{F}\left\{\frac{1}{2}\sqrt{\lambda t^\alpha}^{-1} \mathcal{M}_{\frac{\alpha}{2}}\left(\sqrt{\lambda t^\alpha}^{-1} |\cdot - x|\right)\right\}(\xi) \, d\xi$$

$$= \int_{-\infty}^{\infty} \mathcal{F}\left\{\overline{\mathcal{F}}\{\varphi\}\right\}(\xi) \frac{1}{2}\sqrt{\lambda t^\alpha}^{-1} \mathcal{M}_{\frac{\alpha}{2}}\left(\sqrt{\lambda t^\alpha}^{-1} |\xi - x|\right) d\xi$$

$$= \int_{-\infty}^{\infty} \varphi(\xi) \frac{1}{2} \sqrt{\lambda t^{\alpha}}^{-1} \mathcal{M}_{\frac{\alpha}{2}} \left(\sqrt{\lambda t^{\alpha}}^{-1} |x - \xi| \right) d\xi. \tag{8}$$

And so, we obtain a candidate for a solution to the Cauchy problem (4) given by

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{2} \sqrt{\lambda t^{\alpha}}^{-1} \mathcal{M}_{\frac{\alpha}{2}} \left(\sqrt{\lambda t^{\alpha}}^{-1} |x - \xi| \right) \varphi(\xi)\, d\xi.$$

Note that this expression is identical to the function obtained in [26,27], and so we provide here an alternative way to obtain the desired solution. Nonetheless this reasoning does not guarantee that the obtained function is actually the solution to our problem. We will now prove this for initial conditions of a special kind.

Theorem 1 *Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ so that $\mathcal{F}\{\varphi\}(x)x^2 \in L^1(\mathbb{R})$. A solution to the Cauchy problem (4) with $\alpha \in (0,2)$ is given by*

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{2} \sqrt{\lambda t^{\alpha}}^{-1} \mathcal{M}_{\frac{\alpha}{2}} \left(\sqrt{\lambda t^{\alpha}}^{-1} |x - \xi| \right) \varphi(\xi)\, d\xi. \tag{9}$$

Proof *Case 1:* $0 < \alpha < 1$.

To prove that the function verifies the differential equation $\frac{\partial^{\alpha}}{\partial t^{\alpha}} u(x,t) = \lambda \frac{\partial^2}{\partial x^2} u(x,t)$ we consider the equalities that lead us to (8). Given that $\mathcal{M}_{\frac{\alpha}{2}} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and under the assumption that $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, these equalities are valid. We therefore have

$$u(x,t) = \int_{-\infty}^{\infty} \overline{\mathcal{F}}\{\varphi\}(\xi)\, e^{i\xi x} \mathcal{E}_{\alpha} \left(-\xi^2 \lambda t^{\alpha} \right) d\xi. \tag{10}$$

The fractional derivative of order α with $0 < \alpha < 1$ of u is now given by

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} u(x,t) = \int_{0}^{t} \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \left[\frac{\partial}{\partial \tau} \int_{-\infty}^{\infty} \overline{\mathcal{F}}\{\varphi\}(\xi)\, e^{i\xi x} \mathcal{E}_{\alpha} \left(-\xi^2 \lambda \tau^{\alpha} \right) d\xi \right] d\tau.$$

In order to interchange the derivative and the integral, we see that

$$\begin{aligned} &\left| \frac{\partial}{\partial \tau} \overline{\mathcal{F}}\{\varphi\}(\xi)\, e^{i\xi x} \mathcal{E}_{\alpha} \left(-\xi^2 \lambda \tau^{\alpha} \right) \right| \\ &= \left| \overline{\mathcal{F}}\{\varphi\}(\xi)\, \xi^2 \lambda \alpha \tau^{\alpha-1} \mathcal{E}_{\alpha,\alpha} \left(-\xi^2 \lambda \tau^{\alpha} \right) \right| = g(\xi). \end{aligned}$$

As a consequence of the asymptotic behaviour of the Mittag–Leffler function given in [14], it can easily be seen that $\mathcal{E}_{\alpha,\alpha} \in L^{\infty}(\mathbb{R})$. Combining this with $\overline{\mathcal{F}}\{\varphi\}(\xi)\xi^2 \in L^1(\mathbb{R})$ and using Hölder inequality we obtain $g \in L^1(\mathbb{R})$, and so

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = -\int_0^t \int_{-\infty}^{\infty} \frac{1}{\Gamma(1-\alpha)} (t-\tau)^{-\alpha} \overline{\mathcal{F}}\{\varphi\}(\xi) \\ e^{i\xi x} \xi^2 \lambda \alpha \tau^{\alpha-1} \mathcal{E}_{\alpha,\alpha}\left(-\xi^2 \lambda \tau^\alpha\right) d\xi d\tau.$$

To compute these two integrals we use Fubini's theorem, for which we prove that we are under the hypothesis of Tonelli's theorem (see [3], chapter 4). We already proved that

$$\int_{-\infty}^{\infty} \left| \frac{1}{\Gamma(1-\alpha)} (t-\tau)^{-\alpha} \overline{\mathcal{F}}\{\varphi\}(\xi) e^{i\xi x} \left(-\xi^2 \lambda\right) \alpha \tau^{\alpha-1} \mathcal{E}_{\alpha,\alpha}\left(-\xi^2 \lambda \tau^\alpha\right) \right| d\xi$$

is finite. To show that the whole double integral is finite, we use once again the fact that $\mathcal{E}_{\alpha,\alpha} \in L^\infty(\mathbb{R})$. We obtain

$$\int_0^t \int_{-\infty}^{\infty} \left| \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \overline{\mathcal{F}}\{\varphi\}(\xi) (-\xi^2) e^{i\xi x} \lambda \alpha \tau^{\alpha-1} \mathcal{E}_{\alpha,\alpha}\left(-\xi^2 \lambda \tau^\alpha\right) \right| d\xi d\tau$$

$$= \int_0^t \int_{-\infty}^{\infty} \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \left| \overline{\mathcal{F}}\{\varphi\}(\xi) \xi^2 \right| \lambda \alpha \tau^{\alpha-1} \left| \mathcal{E}_{\alpha,\alpha}\left(-\xi^2 \lambda \tau^\alpha\right) \right| d\xi d\tau$$

$$\leq \int_0^t \int_{-\infty}^{\infty} \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \left| \overline{\mathcal{F}}\{\varphi\}(\xi) \xi^2 \right| \lambda \alpha \tau^{\alpha-1} \left\| \mathcal{E}_{\alpha,\alpha} \right\|_{L^\infty(\mathbb{R})} d\xi d\tau$$

$$= \frac{\lambda \alpha \left\| \mathcal{E}_{\alpha,\alpha} \right\|_{L^\infty(\mathbb{R})}}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \tau^{\alpha-1} d\tau \int_{-\infty}^{\infty} \left| \overline{\mathcal{F}}\{\varphi\}(\xi) \xi^2 \right| d\xi$$

$$= \frac{\lambda \alpha \left\| \mathcal{E}_{\alpha,\alpha} \right\|_{L^\infty(\mathbb{R})}}{\Gamma(1-\alpha)} \int_0^1 r^{(1-\alpha)-1} (1-r)^{\alpha-1} dr \int_{-\infty}^{\infty} \left| \overline{\mathcal{F}}\{\varphi\}(\xi) \xi^2 \right| d\xi$$

where the last term is finite since the first integral is the Beta function[1] with $x = \alpha > 0$, $y = 1-\alpha > 0$ and the second integral is finite since $\overline{\mathcal{F}}\{\varphi\}(\xi)\xi^2 \in L^1(\mathbb{R})$. Due to Tonelli's theorem we are now allowed to use Fubini's theorem and obtain

[1] The Beta function is defined as $\mathcal{B}(x,y) := \int_0^1 (1-r)^{x-1} r^{y-1} dr$, convergent for $x, y > 0$.

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \int_{-\infty}^{\infty} \overline{\mathcal{F}}\{\varphi\}(\xi)\, e^{i\xi x}\left[\frac{\partial^\alpha}{\partial t^\alpha}\mathcal{E}_\alpha\left(-\xi^2\lambda t^\alpha\right)\right] d\xi$$

$$= -\lambda \int_{-\infty}^{\infty} \overline{\mathcal{F}}\{\varphi\}(\xi)\, e^{i\xi x}\xi^2\mathcal{E}_\alpha\left(-\xi^2\lambda t^\alpha\right) d\xi.$$

In a similar way we prove that it is allowed to pass the second derivative with respect to the variable x inside the integral and so

$$\frac{\partial^2}{\partial x^2}u(x,t) = \int_{-\infty}^{\infty} \overline{\mathcal{F}}\{\varphi\}(\xi)\left[\frac{\partial^2}{\partial x^2}e^{i\xi x}\right]\mathcal{E}_\alpha\left(-\xi^2\lambda t^\alpha\right) d\xi$$

$$= -\int_{-\infty}^{\infty} \overline{\mathcal{F}}\{\varphi\}(\xi)\, e^{i\xi x}\xi^2\mathcal{E}_\alpha\left(-\xi^2\lambda t^\alpha\right) d\xi$$

which completes the proof that the function u verifies the differential equation. To show that u satisfies the initial condition, we see that

$$u(x,0) = \int_{-\infty}^{\infty} \overline{\mathcal{F}}\{\varphi\}(\xi)\, e^{i\xi x}\mathcal{E}_\alpha(0)\, d\xi = \mathcal{F}\left\{\overline{\mathcal{F}}\{\varphi\}\right\}(x) = \varphi(x).$$

Case 2: $1 < \alpha < 2$.

In this case the fractional derivative of order α with $1 < \alpha < 2$ is given by

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \int_0^t \frac{(t-\tau)^{1-\alpha}}{\Gamma(2-\alpha)}\left[\frac{\partial^2}{\partial\tau^2}\int_{-\infty}^{\infty} \overline{\mathcal{F}}\{\varphi\}(\xi)\, e^{i\xi x}\mathcal{E}_\alpha\left(-\xi^2\lambda\tau^\alpha\right) d\xi\right] d\tau.$$

We have already proven in Case 1 that we can pass the first derivative inside the integral. For the second one we see that

$$\left|\frac{\partial}{\partial\tau}\overline{\mathcal{F}}\{\varphi\}(\xi)\,\xi^2\lambda\alpha\tau^{\alpha-1}\mathcal{E}_{\alpha,\alpha}\left(-\xi^2\lambda\tau^\alpha\right)\right|$$

$$= \left|\overline{\mathcal{F}}\{\varphi\}(\xi)\,\xi^2\lambda\alpha\tau^{\alpha-2}\mathcal{E}_{\alpha,\alpha-1}\left(-\xi^2\lambda\tau^\alpha\right)\right| = \tilde{g}(\xi).$$

Also here it can be shown that $\mathcal{E}_{\alpha,\alpha-1} \in L^\infty(\mathbb{R})$ and so finally $\tilde{g} \in L^1(\mathbb{R})$. We have

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \int_0^t\int_{-\infty}^{\infty} \frac{(t-\tau)^{1-\alpha}}{\Gamma(2-\alpha)}\overline{\mathcal{F}}\{\varphi\}(\xi)\, e^{i\xi x}\xi^2\lambda\alpha\tau^{\alpha-2}\mathcal{E}_{\alpha,\alpha-1}(-\xi^2\lambda\tau^\alpha)\, d\xi\, d\tau.$$

Once again we use Tonelli's theorem, and as in Case 1, we already proved that

$$\int_{-\infty}^{\infty} \left| \frac{1}{\Gamma(2-\alpha)} (t-\tau)^{1-\alpha} \overline{\mathcal{F}} \{\varphi\} (\xi) e^{i\xi x} \xi^2 \lambda \alpha \tau^{\alpha-2} \mathcal{E}_{\alpha,\alpha-1} \left(-\xi^2 \lambda \tau^\alpha\right) \right| d\xi < \infty$$

and for the double integral we obtain

$$\int_0^t \int_{-\infty}^{\infty} \left| \frac{(t-\tau)^{1-\alpha}}{\Gamma(2-\alpha)} \overline{\mathcal{F}} \{\varphi\} (\xi) e^{i\xi x} \xi^2 \lambda \alpha \tau^{\alpha-2} \mathcal{E}_{\alpha,\alpha-1} \left(-\xi^2 \lambda \tau^\alpha\right) \right| d\xi d\tau$$

$$\leq \frac{\lambda\alpha \left\| \mathcal{E}_{\alpha,\alpha-1} \right\|_{L^\infty(\mathbb{R})}}{\Gamma(2-\alpha)} \int_0^1 r^{(2-\alpha)-1} (1-r)^{(\alpha-1)-1} d\tau \int_{-\infty}^{\infty} \left| \overline{\mathcal{F}}\{\varphi\}(\xi)\xi^2 \right| d\xi$$

where the first integral is the Beta function with $x = \alpha - 1 > 0$, $y = 2 - \alpha > 0$ and the second integral is finite since $\overline{\mathcal{F}} \{\varphi\} (\xi)\xi^2 \in L^1 (\mathbb{R})$. Once again we are allowed to use Fubini's theorem and obtain

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \int_{-\infty}^{\infty} \overline{\mathcal{F}} \{\varphi\} (\xi) e^{i\xi x} \left[\frac{\partial^\alpha}{\partial t^\alpha} \mathcal{E}_\alpha \left(-\xi^2 \lambda t^\alpha\right) \right] d\xi$$

$$= -\lambda \int_{-\infty}^{\infty} \overline{\mathcal{F}} \{\varphi\} (\xi) e^{i\xi x} \xi^2 \mathcal{E}_\alpha \left(-\xi^2 \lambda t^\alpha\right) d\xi = \lambda \frac{\partial^2}{\partial x^2} u(x,t).$$

To show that u satisfies the initial condition, we see once more that

$$u(x,0) = \int_{-\infty}^{\infty} \overline{\mathcal{F}} \{\varphi\} (\xi) e^{i\xi x} \mathcal{E}_\alpha (0) d\xi = \mathcal{F} \left\{ \overline{\mathcal{F}} \{\varphi\} \right\} (x) = \varphi(x).$$

□

Remark 1 In order to prove this theorem the alternative expression of the solution (10) turned out to be more useful than the classic one (9) due to different properties it posseses. The computation of the derivatives is more simple in this expression due to the fact that it separates the two variables in two different functions whose derivatives are easily calculated. But a more important fact is that in the original expression the series expansion of the Mainardi function cannot be used in order to calculate the fractional time-derivative since here the variable t appears only in negative powers and the fractional derivative is not defined for negative powers. This is the main reason why it was not possible to prove this result using the classical expresion. In (10) the variable t only appears in positive powers.

Remark 2 For $\alpha \in (0, 1)$ uniquness of the solution has been shown (see [19]) using a maximum principle. In [19] the initial condition is required to be continuous, but solutions to the problems are also required to be continuous with respect to both, time and space variables, C^2 within the space domain and W^1 within the time domain. On the contrary, Theorem 1 requires no continuity.

To guarantee uniqueness of the solution for $\alpha \in (1, 2)$, an initial condition for the first derivative in time has to be added to the Cauchy problem. The solution (9) verifies for $\alpha \in (1, 2)$ the initial condition $u_t(x, 0) = 0$.

Remark 3 Note that the requirements of Theorem 1 are satisfied if we force the initial condition to verify $\varphi \in \mathcal{S}(\mathbb{R})$.

4 Bounds to the Space-L^p Norm of the Solution to the Cauchy Problem for the Time-Fractional Diffusion-Wave Equation in Dependence of Time

In this section we will use another representation of the solution to the Cauchy problem

$$(\mathrm{PC}_\alpha) \begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} u(x, t) = \lambda \frac{\partial^2}{\partial x^2} u(x, t) & \text{if} \quad x \in \mathbb{R}; \qquad t \in \mathbb{R}_0^+ \\ u(x, 0) = \varphi(x) & \text{if} \quad x \in \mathbb{R} \\ u \text{ bounded} \end{cases}$$

considering the Fourier transform. An expression of the Fourier transform of the solution u of the original problem (PC_α) is given by $\mathcal{F}\{u\}(\xi, t) = \mathcal{E}_\alpha\left(-\lambda\xi^2 t^\alpha\right)\mathcal{F}\{\varphi\}(\xi)$, which will be very useful in order to obtain bounds for the space-L^p norms. From now on we will consider initial data satisfying the hypothesis of Theorem 1.

Theorem 2 *For $\alpha \in (0, 2)$ the solution of (PC_α) with fixed time-variable t verifies*

$$\int_{-\infty}^{\infty} u(x, t)\, dx = \int_{-\infty}^{\infty} \varphi(x)\, dx.$$

Proof We immediately have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_\alpha(x-\xi, t)\varphi(\xi) d\xi dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_\alpha(x-\xi, t) dx \varphi(\xi) d\xi = \int_{-\infty}^{\infty} \varphi(\xi) d\xi.$$

□

In a similar way it can be shown that

$$\|u(x, t)\|_{L^1(\mathbb{R})} \leq \|\varphi(x)\|_{L^1(\mathbb{R})}$$

where the equality holds for functions φ so that $\varphi \geq 0$ or $\varphi \leq 0$.

Theorem 3 *For $\alpha \in (0,2)$, the space-L^∞ norm of the solution of (PC_α) satisfies the bound*

$$\|u(x,t)\|_{L^\infty(\mathbb{R})} \leq \frac{C}{t^{\frac{\alpha}{4}}}.$$

Proof By properties of the Fourier transform we have

$$\|u(x,t)\|_{L^\infty(\mathbb{R})} \leq \|\mathcal{F}\{u\}(\xi,t)\|_{L^1(\mathbb{R})} = \left\|\mathcal{E}_\alpha\left(-\lambda\xi^2 t^\alpha\right)\mathcal{F}\{\varphi\}(\xi)\right\|_{L^1(\mathbb{R})},$$

where

$$\int_{-\infty}^{\infty}\left|\mathcal{E}_\alpha(-\lambda\xi^2 t^\alpha)\mathcal{F}\{\varphi\}(\xi)\right| d\xi \leq \left(\int_{-\infty}^{\infty}\left|\mathcal{E}_\alpha(-\lambda\xi^2 t^\alpha)\right|^2 d\xi\right)^{\frac{1}{2}} \|\mathcal{F}\{\varphi\}\|_{L^2(\mathbb{R})}^{\frac{1}{2}}$$

$$= \left(\frac{1}{\sqrt{\lambda t^\alpha}}\int_{-\infty}^{\infty}\left|\mathcal{E}_\alpha\left(-s^2\right)\right|^2 ds\right)^{\frac{1}{2}} \|\mathcal{F}\{\varphi\}\|_{L^2(\mathbb{R})}^{\frac{1}{2}} = \frac{C}{t^{\frac{\alpha}{4}}}$$

where the substitution $\sqrt{\lambda t^\alpha}\xi = s$ has been made and the Hölder inequality has been used with $\mathcal{E}_\alpha\left(-s^2\right), \mathcal{F}\{\varphi\}(\xi) \in L^2(\mathbb{R})$. □

Note that for small values of t the bound

$$\|u(x,t)\|_{L^\infty(\mathbb{R})} \leq \int_{-\infty}^{\infty} K_\alpha(x-\xi,t)\,|\varphi(\xi)|\, d\xi \leq \|\varphi(x)\|_{L^\infty(\mathbb{R})}$$

is more precise than the one given in Theorem 3.

Theorem 4 *Let $1 < p < \infty$, $\alpha \in (0,2)$. Then the decay of the L^p–norm of the solution of (PC_α) is given by*

$$\|u(x,t)\|_{L^p(\mathbb{R})} \leq \frac{C}{t^{\frac{\alpha}{4}\left(1-\frac{1}{p}\right)}}.$$

Proof By the interpolation inequality we have

$$\|u(x,t)\|_{L^p(\mathbb{R})} \leq \|u(x,t)\|_{L^1(\mathbb{R})}^{\frac{1}{p}} \|u(x,t)\|_{L^\infty(\mathbb{R})}^{1-\frac{1}{p}}.$$

The result follows from the previous Theorem, as the L^1–norm of u remains bounded by the L^1–norm of the initial condition φ. □

5 Continuity of the Solution of the Cauchy Problem for the Time-Fractional Diffusion-Wave Equation with Respect to the Parameter α

It has already been proven in [10] that

$$\lim_{\alpha\to 1^-} \mathcal{M}_{\frac{\alpha}{2}}(x) = \mathcal{M}_{\frac{1}{2}}(x) = \frac{e^{\frac{-x^2}{4}}}{\sqrt{\pi}}.$$

What follows is a more general proof for the limit $\alpha \to \alpha_0 \in (0, 2)$.

Lemma 2 *Let* $x \in \mathbb{R}$, $\alpha_0 \in (0, 2)$. *Then*

$$\lim_{\alpha\to\alpha_0} \mathcal{M}_{\frac{\alpha}{2}}(x) = \mathcal{M}_{\frac{\alpha_0}{2}}(x),$$

where the convergence is uniform over compact sets.

Proof By definition of the Mainardi function, we have

$$\mathcal{M}_{\frac{\alpha}{2}}(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!\Gamma\left(-\frac{\alpha}{2}k + 1 - \frac{\alpha}{2}\right)}.$$

We take $\alpha \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon) \subset (0, 2)$ and k sufficiently big so that $\frac{\alpha_0-\varepsilon}{2}k + \frac{\alpha_0-\varepsilon}{2}$ is greater than $\frac{3}{2}$. Taking now into account that in $\left(\frac{3}{2}, \infty\right)$ Γ is an increasing function, using $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$ and considering the bound $\Gamma(x+s) \le x^s \Gamma(x)$ (valid for $x > 0$ and $0 \le s \le 1$) with $x = \frac{\alpha}{2}k$ and $0 \le s = \frac{\alpha}{2} \le 1$, we have

$$\left|\frac{(-x)^k}{k!\Gamma\left(-\frac{\alpha}{2}k + 1 - \frac{\alpha}{2}\right)}\right| = \left|\frac{(-x)^k \sin\left(\pi\left(-\frac{\alpha}{2}k + 1 - \frac{\alpha}{2}\right)\right)\Gamma\left(\frac{\alpha}{2}k + \frac{\alpha}{2}\right)}{k!\pi}\right|$$

$$\le \frac{|x|^k}{k!\pi}\Gamma\left(\frac{\alpha}{2}k + \frac{\alpha}{2}\right) \le \frac{|x|^k}{k!\pi}\left(\frac{\alpha}{2}k\right)^{\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}k\right) \le \frac{|x|^k}{k!\pi}\left(\frac{\alpha_0+\varepsilon}{2}k\right)^{\frac{\alpha_0+\varepsilon}{2}}\Gamma\left(\frac{\alpha_0+\varepsilon}{2}k\right) := a_k.$$

To see that the series $\sum_{k=0}^{\infty} a_k$ is convergent it is sufficient to prove that $\left|\frac{a_{k+1}}{a_k}\right| \to 0$ when $k \to \infty$:

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{|x|}{k+1}\left(\frac{k+1}{k}\right)^{\frac{\alpha_0+\varepsilon}{2}} \frac{\Gamma\left(\frac{\alpha_0+\varepsilon}{2}k + \frac{\alpha_0+\varepsilon}{2}\right)}{\Gamma\left(\frac{\alpha_0+\varepsilon}{2}k\right)} \le \frac{|x|}{k+1}\left(\frac{k+1}{k}\right)^{\frac{\alpha_0+\varepsilon}{2}}\left(\frac{\alpha_0+\varepsilon}{2}k\right)^{\frac{\alpha_0+\varepsilon}{2}}$$

$$= \frac{|x|}{(k+1)^{1-\frac{\alpha_0+\varepsilon}{2}}}\left(\frac{\alpha_0+\varepsilon}{2}\right)^{\frac{\alpha_0+\varepsilon}{2}} \to 0,$$

since $\alpha_0 + \varepsilon < 2$. □

Remark 4 In particular, taking $\alpha_0 = 1$ we obtain

$$\lim_{\alpha\to 1} \mathcal{M}_{\frac{\alpha}{2}}(x) = \mathcal{M}_{\frac{1}{2}}(x) = \frac{e^{\frac{-x^2}{4}}}{\sqrt{\pi}}.$$

Theorem 5 *Let $u(x,t;\alpha)$ be the solution to the problem (PC_α) for $\alpha \in (0,2)$. Take $\alpha_0 \in (0,2)$. Then we have*

$$\lim_{\alpha\to\alpha_0} u(x,t;\alpha) = u(x,t;\alpha_0).$$

Proof Consider the solution of (PC$_\alpha$) for $\alpha \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon) = U_\varepsilon$,

$$u(x,t;\alpha) = \int_{-\infty}^{\infty} K_\alpha(x-\xi,t)\,\varphi(\xi)\,d\xi,$$

Set $M = \max_{\alpha\in U_\varepsilon} \|K_\alpha(\xi,t)\|_{L^\infty(\mathbb{R})}$ to obtain

$$|K_\alpha(x-\xi,t)\,\varphi(\xi)| \le \|K_\alpha(\xi,t)\|_{L^\infty(\mathbb{R})}\,|\varphi(\xi)| \le M\,|\varphi(\xi)| \in L^1(\mathbb{R}).$$

Taking the limit of this function when $\alpha \to \alpha_0$ we obtain

$$\lim_{\alpha\to\alpha_0} \int_{-\infty}^{\infty} K_\alpha(x-\xi,t)\,\varphi(\xi)\,d\xi = \int_{-\infty}^{\infty} K_{\alpha_0}(x-\xi,t)\,\varphi(\xi)d\xi,$$

which is what we wanted to prove. □

Remark 5 Once again, taking $\alpha = 1$ we obtain

$$\lim_{\alpha\to 1} u(x,t;\alpha) = u(x,t;1) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\lambda\pi t}} e^{-\frac{(x-\xi)^2}{4\lambda t}}\,\varphi(\xi)\,d\xi,$$

which is the classical solution to the Cauchy problem for the diffusion equation

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = \lambda\frac{\partial^2}{\partial x^2}u(x,t) & \text{if} \quad x \in \mathbb{R}; \qquad t \in \mathbb{R}_0^+ \\ u(x,0) = \varphi(x) & \text{if} \quad x \in \mathbb{R} \\ u \text{ bounded.} \end{cases}$$

For the particular case $\alpha \to 2$, once again the Mainardi function will be considered to be a tempered distribution. For simplicity, we write $K_{\alpha,t} := K_\alpha(x,t)$. Then the following result is valid.

Theorem 6 *Consider $K_{\alpha,t} \in \mathcal{S}'(\mathbb{R})$ to be a tempered distribution. Then*

$$K_{\alpha,t} * \varphi \to \varphi_{\lambda,t} \quad \text{as} \quad \alpha \to 2 \quad \text{in } \mathcal{S}'(\mathbb{R}),$$

where $\varphi_{\lambda,t}$ is defined as

$$\varphi_{\lambda,t}(x) := \frac{\varphi(x+\sqrt{\lambda}t)+\varphi(x-\sqrt{\lambda}t)}{2}.$$

Proof Take $\psi \in \mathcal{S}(\mathbb{R})$. Then

$$\langle K_{\alpha,t} * \varphi, \psi\rangle = \langle \overline{\mathcal{F}}\{\mathcal{F}\{K_{\alpha,t} * \varphi\}\}, \psi\rangle = \langle \mathcal{F}\{K_{\alpha,t}\}\mathcal{F}\{\varphi\}, \overline{\mathcal{F}}\{\psi\}\rangle$$

$$= \int_{-\infty}^{\infty} \mathcal{E}_\alpha(-(\sqrt{\lambda}t\xi)^2)\mathcal{F}\{\varphi\}(\xi)\overline{\mathcal{F}}\{\psi\}(\xi)d\xi.$$

Using once more Lemma 1 and the boundary

$$|\mathcal{E}_\alpha(-(\sqrt{\lambda}t\xi)^2)\mathcal{F}\{\varphi\}(\xi)\overline{\mathcal{F}}\{\psi\}(\xi)| \le |\mathcal{F}\{\varphi\}(\xi)\overline{\mathcal{F}}\{\psi\}(\xi)| \in L^1(\mathbb{R}),$$

we obtain

$$\lim_{\alpha\to 2}\langle K_{\alpha,t} * \varphi, \psi\rangle = \int_{-\infty}^{\infty} \cos(\sqrt{\lambda}t\xi)\mathcal{F}\{\varphi\}(\xi)\overline{\mathcal{F}}\{\psi\}(\xi)d\xi$$

$$= \frac{1}{2}\left[\langle e^{-i\sqrt{\lambda}t\xi}\mathcal{F}\{\varphi\}, \overline{\mathcal{F}}\{\psi\}\rangle + \langle e^{i\sqrt{\lambda}t\xi}\mathcal{F}\{\varphi\}, \overline{\mathcal{F}}\{\psi\}\rangle\right]$$

$$= \frac{1}{2}\left[\langle \mathcal{F}\{(T_{\sqrt{\lambda}t}\varphi)\}, \overline{\mathcal{F}}\{\psi\}\rangle + \langle \mathcal{F}\{(T_{-\sqrt{\lambda}t}\varphi)\}, \overline{\mathcal{F}}\{\psi\}\rangle\right],$$

where $(T_{x_0}\varphi)(x) = \varphi(x+x_0)$. And so we finally obtain

$$\lim_{\alpha\to 2}\langle K_{\alpha,t} * \varphi, \psi\rangle = \frac{\langle \mathcal{F}\{(T_{\sqrt{\lambda}t}\varphi)\} + \mathcal{F}\{(T_{-\sqrt{\lambda}t}\varphi)\}, \overline{\mathcal{F}}\{\psi\}\rangle}{2}$$

$$= \frac{\langle (T_{\sqrt{\lambda}t}\varphi) + (T_{-\sqrt{\lambda}t}\varphi), \psi\rangle}{2},$$

where

$$\frac{(T_{\sqrt{\lambda}t}\varphi) + (T_{-\sqrt{\lambda}t}\varphi)}{2}(x) = \varphi_{\lambda,t}(x) = \frac{\varphi(x+\sqrt{\lambda}t)+\varphi(x-\sqrt{\lambda}t)}{2},$$

the solution to the classical problem for the wave equation,

$$\begin{cases} \frac{\partial^2}{\partial t^2} u\left(x,t\right) = \lambda \frac{\partial^2}{\partial x^2} u\left(x,t\right) & \text{if} \quad x \in \mathbb{R}; \qquad t \in \mathbb{R}_0^+ \\ u\left(x,0\right) = \varphi\left(x\right) & \text{if} \quad x \in \mathbb{R} \\ u'\left(x,0\right) = 0 & \text{if} \quad x \in \mathbb{R}. \end{cases}$$

□

6 The Solution to the Cauchy Problem for the Time-Fractional Diffusion-Wave Equation with Explicit Initial Data

In this section we will compute the solution to the time-fractional diffusion-wave equation with explicit initial conditions, which not necessarily satisfy the hypothesis of Theorem 1. We will now obtain an alternative expression of the solution of (PC_α), which will turn out to be more practical for this purpose. Fixing $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ we get

$$\begin{aligned} u\left(x,t\right) &= \int_{-\infty}^{\infty} \frac{1}{2} \sqrt{\lambda t^\alpha}^{-1} \mathcal{M}_{\frac{\alpha}{2}} \left(\sqrt{\lambda t^\alpha}^{-1} \left|x-\xi\right| \right) \varphi\left(\xi\right) d\xi \\ &= \int_{-\infty}^{x} \frac{1}{2} \sqrt{\lambda t^\alpha}^{-1} \mathcal{M}_{\frac{\alpha}{2}} \left(\sqrt{\lambda t^\alpha}^{-1} \left(x-\xi\right) \right) \varphi\left(\xi\right) d\xi \\ &\quad + \int_{x}^{\infty} \frac{1}{2} \sqrt{\lambda t^\alpha}^{-1} \mathcal{M}_{\frac{\alpha}{2}} \left(\sqrt{\lambda t^\alpha}^{-1} \left(\xi-x\right) \right) \varphi\left(\xi\right) d\xi \\ &= -\frac{1}{2} \int_{\infty}^{0} \mathcal{M}_{\frac{\alpha}{2}}\left(r\right) \varphi\left(x - r\sqrt{\lambda t^\alpha}\right) dr + \frac{1}{2} \int_{0}^{\infty} \mathcal{M}_{\frac{\alpha}{2}}\left(r\right) \varphi\left(x + r\sqrt{\lambda t^\alpha}\right) dr \\ &= \frac{1}{2} \int_{0}^{\infty} \mathcal{M}_{\frac{\alpha}{2}}\left(r\right) \left[\varphi\left(x - r\sqrt{\lambda t^\alpha}\right) + \varphi\left(x + r\sqrt{\lambda t^\alpha}\right) \right] dr \end{aligned} \tag{11}$$

where the substitutions $\sqrt{\lambda t^\alpha}^{-1}\left(x-\xi\right) = r$ and $\sqrt{\lambda t^\alpha}^{-1}\left(\xi-x\right) = r$ have been made respectively.
We now can easily study the following problem

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} u\left(x,t\right) = \lambda \frac{\partial^2}{\partial x^2} u\left(x,t\right) & \text{if} \quad x \in \mathbb{R}; \qquad t \in \mathbb{R}_0^+ \\ u\left(x,0\right) = \cos\left(cx\right) & \text{if} \quad x \in \mathbb{R}. \end{cases}$$

It may be a simple task to prove that $u(x,t) = \cos(cx)\,\mathcal{E}_\alpha\left(-c^2\lambda t^\alpha\right)$ verifies the required conditions, since

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \cos(cx)\frac{\partial^\alpha}{\partial t^\alpha}\mathcal{E}_\alpha(-c^2\lambda t^\alpha) = -c^2\lambda\cos(cx)\mathcal{E}_\alpha(-c^2\lambda t^\alpha) = \lambda\frac{\partial^2}{\partial x^2}u(x,t).$$

Nevertheless the idea is to show how to obtain this solution using the expression of the general solution and replacing $\varphi(x) = \cos(cx)$. We use the series expansion of $\cos(x)$ in (11)

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\int_0^\infty \mathcal{M}_{\frac{\alpha}{2}}(r)\left[\cos\left(c\left(x - r\sqrt{\lambda t^\alpha}\right)\right) + \cos\left(c\left(x + r\sqrt{\lambda t^\alpha}\right)\right)\right]dr \\
&= \frac{1}{2}\int_0^\infty \mathcal{M}_{\frac{\alpha}{2}}(r)\sum_{k=0}^\infty \frac{(-1)^k c^{2k}}{(2k)!}\left[\left(x - r\sqrt{\lambda t^\alpha}\right)^{2k} + \left(x + r\sqrt{\lambda t^\alpha}\right)^{2k}\right]dr \\
&= \frac{1}{2}\int_0^\infty \mathcal{M}_{\frac{\alpha}{2}}(r)\sum_{k=0}^\infty \frac{(-1)^k c^{2k}}{(2k)!}\left[2\sum_{\substack{i=0,\\ i\text{ even}}}^{2k}\binom{2k}{i}x^{2k-i}\left(r\sqrt{\lambda t^\alpha}\right)^i\right]dr \\
&= \sum_{k=0}^\infty \frac{(-1)^k c^{2k}}{(2k)!}\int_0^\infty \mathcal{M}_{\frac{\alpha}{2}}(r)\left[\sum_{i=0}^{k}\binom{2k}{2i}x^{2k-2i}\left(r\sqrt{\lambda t^\alpha}\right)^{2i}\right]dr \\
&= \sum_{k=0}^\infty \frac{(-1)^k c^{2k}}{(2k)!}\left[\sum_{i=0}^{k}\binom{2k}{2i}x^{2(k-i)}\left(\sqrt{\lambda t^\alpha}\right)^{2i}\int_0^\infty \mathcal{M}_{\frac{\alpha}{2}}(r)\,r^{2i}dr\right] \\
&= \sum_{k=0}^\infty \frac{(-1)^k c^{2k}}{(2k)!}\left[\sum_{i=0}^{k}\frac{(2k)!}{(2(k-i))!}\frac{(\lambda t^\alpha)^i}{\Gamma(\alpha i + 1)}x^{2(k-i)}\right] \\
&= \sum_{k=0}^\infty \left[\sum_{i=0}^{k}\frac{(-1)^k c^{2k}}{(2(k-i))!}\frac{(\lambda t^\alpha)^i}{\Gamma(\alpha i + 1)}x^{2(k-i)}\right].
\end{aligned}$$

If we regroup the coefficients of the same power, we obtain

$$\begin{aligned}
u(x,t) &= \sum_{k=0}^\infty\left[\sum_{j=k}^\infty \frac{(-1)^j c^{2j}}{(2k)!}\frac{(\lambda t^\alpha)^{j-k}}{\Gamma(\alpha(j-k)+1)}\right]x^{2k} \\
&= \sum_{k=0}^\infty\left[\sum_{i=0}^\infty \frac{(-1)^{k+i} c^{2k+2i}}{(2k)!}\frac{(\lambda t^\alpha)^{i}}{\Gamma(\alpha i+1)}\right]x^{2k} \\
&= \sum_{k=0}^\infty\left[\sum_{i=0}^\infty \frac{\left(-c^2\lambda t^\alpha\right)^i}{\Gamma(\alpha i+1)}\right]\frac{(-1)^k c^{2k}}{(2k)!}x^{2k}
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \mathcal{E}_\alpha \left(-c^2 \lambda t^\alpha\right) \frac{(-1)^k}{(2k)!} (cx)^{2k} = \cos(cx)\, \mathcal{E}_\alpha \left(-c^2 \lambda t^\alpha\right).$$

In a similar way we can obtain the following results

Proposition 4 *The solution to the Cauchy problem*

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \lambda \frac{\partial^2}{\partial x^2} u(x,t) & \text{if } x \in \mathbb{R}; \quad t \in \mathbb{R}_0^+ \\ u(x,0) = \sin(cx) & \text{if } x \in \mathbb{R} \end{cases} \tag{12}$$

with $c \in \mathbb{R}$ constant, is given by

$$u(x,t) = \sin(cx)\, \mathcal{E}_\alpha \left(-c^2 \lambda t^\alpha\right).$$

Proposition 5 *The solution to the Cauchy problem*

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \lambda \frac{\partial^2}{\partial x^2} u(x,t) & \text{if } x \in \mathbb{R}; \quad t \in \mathbb{R}_0^+ \\ u(x,0) = e^{cx} & \text{if } x \in \mathbb{R} \end{cases} \tag{13}$$

with $c \in \mathbb{R}$ constant, is given by

$$u(x,t) = e^{cx} \mathcal{E}_\alpha \left(c^2 \lambda t^\alpha\right).$$

Proposition 6 *The solution to the Cauchy problem*

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \lambda \frac{\partial^2}{\partial x^2} u(x,t) & \text{if } x \in \mathbb{R}; \quad t \in \mathbb{R}_0^+ \\ u(x,0) = \sum_{k=0}^{n} a_k x^k & \text{if } x \in \mathbb{R} \end{cases} \tag{14}$$

with $n \in \mathbb{N}$ is given by:

$$u(x,t) = \sum_{k=0}^{n} a_k \sum_{\substack{i=0, \\ i \text{ even}}}^{k} \frac{k!}{(k-i)!} \frac{x^{k-i} \left(\sqrt{\lambda t^\alpha}\right)^i}{\Gamma\left(\frac{\alpha}{2} i + 1\right)}.$$

7 Conclusions

We have studied several properties of the Mittag–Leffler functions and the Mainardi functions and used them to obtain the solution to a Cauchy problem for the time-fractional diffusion-wave equation, which happens to be identical to the one previously obtained by other authors. A first complete proof of this result is provided for initial conditions of a certain kind. Finally, bounds of the space-L^p norm of this solution as well as the continuity of the solution with respect to the parameter α and the explicit solution for particular initial conditions have been presented. Within this work, a new expression of the solution to the discussed Cauchy Problem was obtained, which may not be as aesthetical as the well known convolution integral expression, but which has proven itself as a far more practical expression. The problems, which arise when trying to use the original convolution integral in a rigorous mathematical proof, are solved using this new expression. Considering a non homogeneous time-fractional diffusion-wave equation and facing similar problems with a similar mathematical rigorousity applied in this paper may be a task for future work.

Acknowledgements We would like to express our gratitude towards the referees who reviewed this paper for their useful suggestions and helpful comments. This paper has been sponsored by Project ING349 "*Problemas de frontera libre con ecuaciones diferenciales fraccionarias*", from Universidad Nacional de Rosario, Argentina and "*Beca Estímulo a las Vocaciones Científicas en el marco del Plan de Fortalecimiento de la Investigación Científica, el Desarrollo Tecnológico y la Innovación en las Universidades Nacionales*" from Consejo Interuniversitario Nacional, Argentina.

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