Comparing trees characteristic to chordal and dually chordal graphs

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Abstract

Chordal and dually chordal graphs possess characteristic tree representations, namely, clique trees and compatible trees, respectively. The following problem is studied: given a chordal graph G, it has to be determined if the clique trees of G are exactly the compatible trees of $K(G)$. This does not always happen. A necessary and sufficient condition so that it is true, in terms of the minimal vertex separators of the graph, is found.

Keywords: Chordal graph, dually chordal graph, clique tree, compatible tree.

1 Introduction

Chordal graphs have a characteristic tree representation, the clique tree. A clique tree T of a graph G has vertex set equal to the family of cliques of G and, for any vertex, the set of cliques to which that vertex belongs induces a subtree of T. Another related class is that of dually chordal graphs, i.e., the clique graphs of chordal graphs. If G is a chordal graph with clique tree T, then it is easy to verify that every clique of $K(G)$ induces a subtree of T.

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A spanning tree T of a graph such that every clique of the graph induces a subtree of T is named compatible tree. Compatible trees are characteristic to dually chordal graphs and, by the previous paragraph, any clique tree of a chordal graph is compatible with its clique graph. The converse is usually false and no research on conditions that make it true is known to us. This is the question the present paper approaches, with the major results appearing in Section 3. There we find a necessary and sufficient condition for the converse to be true in terms of the minimal separators of the chordal graph. On the way to the proof, we deal with the subsets inducing subtrees in clique trees and compatible trees, leading to the concept of basis. We also find, given a dually chordal graph G, all the chordal graphs whose clique trees are just the compatible trees of G.

2 Definitions

For a graph G, $V(G)$ is the set of its vertices and $E(G)$ that of its edges. A *complete* is a set of pairwise adjacent vertices. A *clique* is a maximal complete and the family of cliques of G is denoted by $\mathcal{C}(G)$. The *neighborhood* of $v \in V(G)$, or $N[v]$, is composed of v and the vertices adjacent to it. The subgraph *induced* by $A \subseteq V(G)$, $G[A]$, has A as vertex set and two vertices are adjacent in $G[A]$ if and only if they are adjacent in G.

Given two nonadjacent vertices u and v, a *uv-separator* is a set $S \subseteq V(G)$ such that u and v are in different connected components of $G - S := G[V(G) - S]$. It is *minimal* if no proper subset of S has the same property. $\mathtt{S(G)}$ will denote the family of all the minimal vertex separators of the graph. Two cliques C_1 and C_2 are a *separating pair* if $C_1 \cap C_2$ is a minimal vertex separator.

Let $\mathcal{F}$ be a family of nonempty sets. If $F \in \mathcal{F}$, F is called a *member* of $\mathcal{F}$. If $v \in \bigcup_{F \in \mathcal{F}} F$, we say that v is an *element* of $\mathcal{F}$. $\mathcal{F}$ is *separating* if, for all $v \in \bigcup_{F \in \mathcal{F}} F$, $\bigcap_{v \in F} F = \{v\}$. The *intersection graph* of $\mathcal{F}$, $L(\mathcal{F})$, has the members of $\mathcal{F}$ as vertices, two of them being adjacent if and only if they are not disjoint. The *clique graph* $K(G)$ is the intersection graph of $\mathcal{C}(G)$. The *two section* of $\mathcal{F}$, $S(\mathcal{F})$, is a graph whose vertices are the elements of $\mathcal{F}$, two of them being adjacent if and only if there exists a member of $\mathcal{F}$ to which both belong. The dual family $D\mathcal{F}$ consists of the sets of the form $D_v = \{F \in \mathcal{F} : \ v \in F\}$, with v being an element of $\mathcal{F}$. When we refer to $\mathcal{C}(G)$, C_v will be used instead of D_v. Given $A \subseteq V(G)$, the set $\{C \in \mathcal{C}(G) : \ A \subseteq C\}$ is denoted by C_A.

All the graphs considered will be assumed to be connected.

3 Comparing clique trees and compatible trees

Given a cycle C, a *chord* is defined as an edge joining two nonconsecutive vertices of C. *Chordal graphs* are mostly defined as those for which any cycle of length greater than or equal to four has a chord. But there are some additional characterizations for them. A *clique tree* of a graph G is a tree T whose vertex set is $\mathcal{C}(G)$ and such that, for every $v \in V(G)$, $T[C_v]$ is a subtree. The most important characterization of chordal graphs for the purpose of this paper says that a graph is chordal if and only if it has a clique tree [4].

With respect to minimal vertex separators, they are closely related to clique trees. An edge of the clique graph of a chordal graph G is in at least one clique tree if and only if its endpoints form a separating pair [3]; and, given a clique tree T of G, every minimal vertex separator of G is the intersection of a separating pair which is an edge of T.

A graph is *dually chordal* if it is the clique graph of a chordal graph. A spanning tree T of a given graph G such that each clique of G induces a subtree of T is said to be *compatible* with G. Equivalently, a tree is compatible with G if and only if the neighborhood of each $v \in V(G)$ induces a subtree. Compatible trees are characteristic to dually chordal graphs, i.e., a graph is dually chordal if and only if it has a compatible tree [1]. The concepts of clique tree for chordal graphs and compatible tree can be connected via clique graphs:

Proposition 3.1 *Any clique tree of a chordal graph G is compatible with $K(G)$.*

Proof. Let T be a clique tree of G. For every $C \in \mathcal{C}(G)$, $N_{K(G)}[C] = \bigcup_{v \in C} C_v$, which induces a subtree because T is a clique tree. □

The main goal set for this paper is to develop tools to answer as easily as possible if, given a chordal graph G, each clique tree of G is a compatible tree of $K(G)$ and vice versa. In view of Proposition 3.1, the problem reduces to studying if each tree compatible with $K(G)$ is a clique tree of G.

3.1 Subtree inducing sets and the concept of basis

Clique trees and compatible trees are characterized by the fact that some particular sets induce subtrees. It will be our interest to find all the sets inducing subtrees in them.

Given a graph G, if G is chordal then $\mathcal{SC}(G)$ will denote the family of all sets F such that, for any clique tree T of G, $T[F]$ is a subtree of T. For instance, each member of $D\mathcal{C}(G)$ is in $\mathcal{SC}(G)$.

If G is dually chordal, then $\mathcal{SDC}(G)$ will denote the family of all sets F such that, for any tree T compatible with G, $T[F]$ is a subtree of T. Some of the subfamilies of $\mathcal{SDC}(G)$ are $\mathcal{C}(G)$, $\mathtt{S(G)}$ and $\{N[v]\}_{v\in V(G)}$ [1,2].

A comparison between these definitions plus Proposition 3.1 yield that, for G chordal, $\mathcal{SDC}(K(G)) \subseteq \mathcal{SC}(G)$.

It is not always easy to list the members of $\mathcal{SC}(G)$ or $\mathcal{SDC}(G)$ because there is no polynomial bound for their cardinality. But it would be desirable to know a procedure generating them all if only "a few" members are known.

Given a family $\mathcal{F}$ of sets, the union $\bigcup_{F\in\mathcal{F}} F$ is said to be *connected* if $L(\mathcal{F})$ is a connected graph. If $\mathcal{F}$ is closed under connected unions, we call a subfamily $\mathcal{F}'$ *generating* if each member of $\mathcal{F}$ with more than one element can be expressed as the connected union of members of $\mathcal{F}'$. A subfamily $\mathcal{B}$ is a *basis* for $\mathcal{F}$ if it is generating and no $\mathcal{B}' \subsetneq \mathcal{B}$ generates $\mathcal{F}$. It is not hard to see that bases are unique, consisting of all the members with at least two elements that cannot be expressed as connected union of others.

Before we continue, we relate these concepts to the original problem:

Theorem 3.2 *Let G be a chordal graph. The following are equivalent:*

(a) *Every tree compatible with $K(G)$ is a clique tree of G.*

(b) *$\mathcal{SC}(G) = \mathcal{SDC}(K(G))$.*

(c) *$\mathcal{SC}(G)$ and $\mathcal{SDC}(K(G))$ have the same basis.*

We can first find a basis for $\mathcal{SC}(G)$ in terms of minimal vertex separators:

Proposition 3.3 *Let G be a chordal graph and $A \in \mathcal{SC}(G)$. If C_1, C_2 is a separating pair contained in A, then $C_{C_1\cap C_2} \subseteq A$.*

Theorem 3.4 *For a chordal graph G, $\{C_S,\ S \in \mathtt{S(G)}\}$ is the basis of $\mathcal{SC}(G)$.*

3.2 Tree correspondence and the basis of $\mathcal{SDC}(G)$

It is a logical next step to attempt, given a dually chordal graph G, to compute the basis for $\mathcal{SDC}(G)$. We do it by finding chordal graphs that contain information about G and whose bases we already know how to compute. Say that a chordal graph H is *in correspondence* with G if $K(H)$ is isomorphic to G and any tree compatible with $K(H)$ is a clique tree of H.

Theorem 3.5 *Let G be a dually chordal graph and H a chordal graph. Then H is in correspondence with G if and only if H is isomorphic to the intersection graph of a separating subfamily $\mathcal{F}$ of $\mathcal{SDC}(G)$ such that $S(\mathcal{F}) = G$.*

Theorem 3.5 can be used to reduce any problem about the compatible trees of a dually chordal graph to a problem about the clique trees of a chordal graph. We use it here, given G dually chordal graph, for computing the basis for $\mathcal{SDC}(G)$ with the help of Proposition 3.3 and Theorem 3.4.

Theorem 3.6 *Let G be a dually chordal graph, T compatible with G, $\mathcal{F}$ a separating subfamily of $\mathcal{SDC}(G)$ such that $S(\mathcal{F}) = G$ and $D\mathcal{F} = \{D_v\}_{v \in V(G)}$. Then:*

(a) *If $A \in \mathcal{SDC}(G)$, $uv \in E(T)$ and $\{u, v\} \subseteq A$, then $\bigcap_{F \in D_u \cap D_v} F \subseteq A$.*

(b) *$\{\bigcap_{F \in D_u \cap D_v} F,\ uv \in E(T)\}$ is the basis of $\mathcal{SDC}(G)$.*

One basic example of a family with the characteristics mentioned in Theorem 3.6 consists of the members of $\mathcal{C}(G)$ and the unit sets of vertices. Applying part *(b)* of Theorem 3.6 to this family leads to the following conclusion:

Theorem 3.7 *Let G be a dually chordal graph and T compatible with G. Then $\{\bigcap_{C \in C_u \cap C_v} C,\ uv \in E(T)\}$ is the basis of $\mathcal{SDC}(G)$.*

If we think of a dually chordal graph as a clique graph of a chordal graph, the expression for the basis becomes the following:

Proposition 3.8 *Let G be a chordal graph and, for $S \in \mathtt{S(G)}$, $B_S := \bigcap_{C \cap S \neq \emptyset} N_{K(G)}[C]$. Then $\{B_S :\ S \in \mathtt{S(G)}\}$ is the basis for $\mathcal{SDC}(K(G))$.*

If each tree compatible with $K(G)$ is a clique tree of G, then Theorem 3.2 implies that $\{B_S :\ S \in \mathtt{S(G)}\} = \{C_S :\ S \in \mathtt{S(G)}\}$. We go one step further to show that $B_S = C_S$ for all $S \in \mathtt{S(G)}$, leading to a new characterization.

Lemma 3.9 *Let G be a dually chordal graph, T a tree compatible with G, $uv \in E(T)$ and $\mathcal{F}$ any separating subfamily of $\mathcal{SDC}(G)$ such that $S(\mathcal{F}) = G$. Then $B_{\mathcal{F}} := \bigcap_{F \in D_u \cap D_v} F$ does not depend on the choice of $\mathcal{F}$.*

Theorem 3.10 *Let G be a chordal graph. Then any tree compatible with $K(G)$ is a clique tree of G if and only if, for every $S \in \mathtt{S(G)}$, $B_S = C_S$.*

Sketch of proof. Suppose that, for each $S \in \mathtt{S(G)}$, $B_S = C_S$. Then the bases for $\mathcal{SC}(G)$ and $\mathcal{SDC}(K(G))$ are equal. Now apply Theorem 3.2.

Conversely, suppose that every tree compatible with $K(G)$ is a clique tree of G. Let $S \in \mathtt{S(G)}$, T a clique tree of G and $C_1C_2 \in E(T)$ such that $C_1 \cap C_2 = S$. Set $\mathcal{F}_1 = \mathcal{C}(K(G)) \cup \{\{C\} :\ C \in K(G)\}$ and $\mathcal{F}_2 = \{C_v :\ v \in V(G)\}$. Apply Lemma 3.9 to $C_1, C_2, \mathcal{F}_1, \mathcal{F}_2$ to get that $B_S = C_S$. $\square$

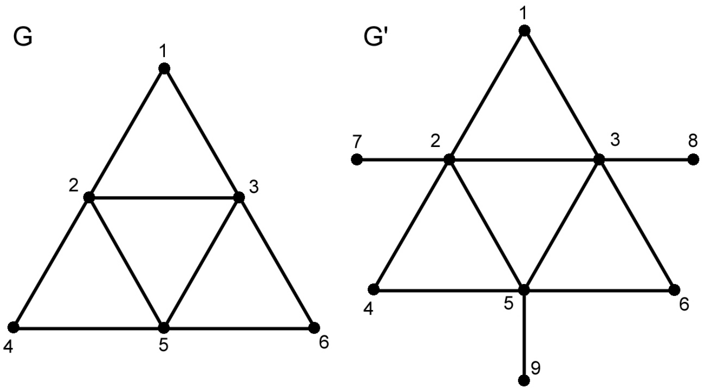

Fig. 1. Every tree compatible with $K(G')$ is a clique tree of G'. The same cannot be said about G.

We end this paper with an example. Consider the chordal graph G in Figure 1. $K(G)$ is the complete graph on 4 vertices, so any spanning tree of $K(G)$ is trivially a compatible tree. However, G has only one clique tree T, where $\{2, 3, 5\}$ is a universal vertex and any other clique of G is a leaf. By Theorem 3.10, we could find $S \in \mathtt{S(G)}$ such that $B_S \neq C_S$. This is verified if we set $S = \{2, 3\}$. Thus, $C_S = \{\{1, 2, 3\}, \{2, 3, 5\}\}$. However, $B_S = \mathcal{C}(G)$.

Although a similar graph, G' satisfies that $B_S = C_S$ for all $S \in \mathtt{S(G')}$. This is not difficult to check. Therefore, the clique trees of G' and the compatible trees of $K(G')$ are the same.

References

[1] Brandstädt A., V. Chepoi, F. Dragan and V. Voloshin, *Dually chordal graphs*, SIAM Journal on Discrete Mathematics, **11** (1998), 437-455.

[2] De Caria P. and M. Gutierrez, *Minimal vertex separators and new characterizations for dually chordal graphs*, Electronic Notes in Discrete Mathematics, Volume **35** (2009), 127-132.

[3] Habib M. and J. Stacho, *Reduced clique graphs of chordal graphs*, submitted to European Journal of combinatorics.

[4] Shibata Y., *On the tree representation of chordal graphs*, J. Graph Theory, **12** (1988), 421-428.