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PATHS ON GRAPHS AND ASSOCIATED QUANTUM GROUPOIDS

R. TRINCHERO

Abstract. Given any simple biorientable graph it is shown that there exists a weak *-Hopf algebra constructed on the vector space of graded endomorphisms of essential paths on the graph. This construction is based on a direct sum decomposition of the space of paths into orthogonal subspaces one of which is the space of essential paths. Two simple examples are worked out with certain detail, the ADE graph A_3 and the affine graph $A_{[2]}$. For the first example the weak *-Hopf algebra coincides with the so called double triangle algebra. No use is made of Ocneanu's cell calculus.

1. Introduction

One of the most interesting developments in mathematical physics of the last decades has been the classification of SU(2)-type rational conformal field theories by ADE graphs[1][1]. In relation to the present work a possible way to look at this classification is the following[2]. The tensor category of representations of a weak *-Hopf algebra[5] constructed out of the corresponding ADE graph G is summarized by another graph $Oc(G)$, called the Ocneanu graph of quantum symmetries[6]. Knowledge of this last graph encodes information on the conformal field theory when considered in various environments, the corresponding generalized partition functions can be obtained from this graph[1, 7, 8]. In addition the weak *-Hopf algebras mentioned above can be given a physical interpretation as the algebras of quantum mechanical symmetries of certain quantum statistical models, known as face models[9].

For the case of ADE graphs the weak *-Hopf algebra mentioned above is known as the the double triangle algebra(DTA)[6, 10, 11]. The construction of this algebra out of the corresponding ADE graph starts from something called quantum 6-j symbols[12] that can be computed employing Ocneanu's cell calculus. These objects describe the representation theory of the DTA[13, 14, 15]. No direct derivation of

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[1]Analog classifications exist for SU(3)-type[2] and SU(4)-type[3] rational conformal field theories, however the construction of the corresponding weak Hopf algebras out of the analog of the ADE graphs is not known.

[2]Another way closer to the historical path is given in [4].

this weak Hopf algebra out of paths on the corresponding graph is available in the literature. One of the aims of this work is to fill this gap[3].

The key ingredient in this derivation is a direct sum decomposition of the space of paths of a given length into orthogonal subspaces, one of which is the space of essential paths. The space of essential paths can be defined in terms of a representation of the Temperley-Lieb-Jones algebra in the space of paths over the graph. The other terms in the above mentioned decomposition are obtained by means of the application of Ocneanu creation operators to spaces of essential paths of a given length. The product in the resulting weak Hopf algebra is defined using a projection of the concatenation factor by factor of endomorphism of paths. This projection sends graded endomorphism of paths into graded endomorphism of essential paths.

The derivation mentioned above can be done for any simple bioriented graph. This provides a generalization of the construction to simple bioriented graphs that are not ADE. In that cases the resulting weak *-Hopf algebra is infinite dimensional. For illustrative purposes a pair of simple examples are considered in this work. One of which is ADE and the other not.

Some interesting further research arise in relation to this work. The representation theory of these weak *-Hopf algebras has not been considered in this work. The detailed study of all the affine graphs($\beta = 2$) weak *-Hopf algebras remains to be done. Also the case of non-affine non-ADE graphs($\beta > 2$) is missing. Furthermore the relation of these weak *-Hopf algebras with conformal field theory deserves to be considered.

This paper is organized as follows. Sections 2, 3 and 4 set up the scenario and give the basic definitions. Section 5 presents the decomposition and section 6 the projection mentioned above. Sections 7, 8 and 9 deal with the weak Hopf algebra structure.

2. Paths

Let G denote a simple biorientable graph. Just to remind the reader some basic definitions to be employed in what follows are included[4].

Definition 1. *Adjacency matrix.* Let the graph G have N_v vertices, its adjacency matrix M is the $N_v \times N_v$ matrix whose $v_i v_j$ entry is 1 if the vertex v_i is connected to the vertex v_j by an edge belonging to G, 0 if it is not connected.

Definition 2. *Elementary path, length.* An elementary path is a succession of consecutive vertices in G. The number of these vertices -1 is called the length of the path.

Definition 3. *Space of paths $\mathcal{P}$.* The inner product vector space of paths $\mathcal{P}$ is defined by saying that elementary paths provide a orthonormal basis of this space.

[3] The question of whether such a derivation exists or not was posed by Oleg Ogievetsky in relation to joint work with the author.

[4] Further definitions and basic results on graph theory can be found in any textbook on graph theory; a short account of these matters related to this work are presented in appendix A of ref. [9].

Definition 4. *Concatenation product in $\mathcal{P}$.* Given two elementary paths $\eta = (v_0, v_1, \cdots, v_n)$ and $\eta' = (v'_0, v'_1, \cdots, v'_n)$ their concatenation product $\eta \star \eta'$ is given by,

$$\eta \star \eta' = \delta_{v_n v_{0'}} (v_0, v_1, \cdots, v_n, v'_1, \cdots, v'_n)$$

3. Creation and annihilation operators on $\mathcal{P}$

Let $\eta = (v_0, v_1, \cdots, v_n)$ denote a elementary path of length n.

Definition 5. *Creation and annihilation operators $c_i^\dagger : \mathcal{P}_n \to \mathcal{P}_{n+2}$ and $c_i : \mathcal{P}_n \to \mathcal{P}_{n-2}$*

$$\begin{aligned} c_i \, \eta &= c_i \, (v_0, v_1, \cdots, v_i, v_{i+1}, v_{i+2}, v_{i+3}, \cdots, v_n) \\ &= \delta_{v_i v_{i+2}} \sqrt{\frac{\mu_{v_{i+1}}}{\mu_{v_i}}} \, (v_o, v_1, \cdots, v_i, v_{i+3}, \cdots, v_n) \;\; if \; 0 \le i \le n-2, \;\; 0 \; otherwise \end{aligned}$$

$$\begin{aligned} c_i^\dagger \eta &= c_i^\dagger (v_0, v_1, \cdots, v_i, v_{i+1}, \cdots, v_n) \\ &= \sum_{v \; n.n. v_i} \sqrt{\frac{\mu_v}{\mu_{v_i}}} \, (v_0, v_1, \cdots, v_i, v, v_i, v_{i+1}, \cdots, v_n) \;\; if \; 0 \le i \le n, \;\; 0 \; otherwise \end{aligned} \tag{3.1}$$

where $\mathcal{P}_n$ is the inner product vector space of paths of length n, μ_v denotes the components of the Perron-Frobenius eigenvector[5] and $n.n.$ denotes nearest neighbours in G.

Proposition 6. For $i \le n$,

$$c_i c_i^\dagger = \beta 1_n$$

where β stands for the highest eigenvalue of the adjacency matrix of G and 1_n denotes the identity in the space $\mathcal{P}_n$.

Proposition 7. The following operators $e_i : \mathcal{P}_n \to \mathcal{P}_n \;\; , i = 0, \cdots, n-2$,

$$e_i = \frac{1}{\beta} c_i^\dagger c_i$$

give a representation of the Temperley-Lieb-Jones algebra with $n-1$ generators. This algebra is defined by the following relations:

$$e_i^2 = e_i, \qquad e_i^\dagger = e_i, \qquad e_i e_j = e_j e_i \;, |i-j| > 1, \;\; e_i e_{i\pm 1} e_i = \frac{1}{\beta^2} e_i.$$

[5] I.e., the eigenvector of the adjacency matrix M with greatest eigenvalue β and with its smallest components taken to be 1.

4. Essential paths

Definition 8. *Essential Paths Subspace.* For each n we denote by $\mathcal{E}_n$ the subspace of $\mathcal{P}_n$ defined by the relations

$$\xi \in \mathcal{E}_n \Leftrightarrow c_i \xi = 0, \ i = 0, \cdots, n-2,$$

the essential paths subspace $\mathcal{E}$ is defined by

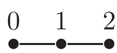
$$\mathcal{E} = \bigoplus_n \mathcal{E}_n.$$

This definition implies that

Proposition 9. For all the ADE graphs $\mathcal{E}$ is finite dimensional[6].

In what follows we will denote by $\{\xi_a\}$ an orthonormal basis of $\mathcal{E}$ (with respect to the restriction to $\mathcal{E}$ of the scalar product in $\mathcal{P}$), i.e. $(\xi_a, \xi_b) = \delta_{ab}$.

Example 10. *Essentials paths for the graph A_3.* The graph A_3, its adjacency matrix, Perron-Frobenius eigenvalue and eigenvector are given below:



$$, \quad M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \beta = \sqrt{2}, \ \ \mu = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}.$$

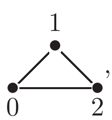
Figure 4.1. The graph A_3

There are ten essential paths in A_3, which are:

- Length zero: $(0), (1), (2)$.
- Length one: $(01), (12), (10), (21)$.
- Length two: $(012), \gamma = \frac{1}{\sqrt{2}}[(121) - (101)], (210)$.

Therefore the maximum length of essential paths over A_3 is $L = 2$.

Example 11. *Essentials paths for the graph $A_{[2]}$.* The graph $A_{[2]}$, its adjacency matrix, Perron-Frobenius eigenvalue and eigenvector are given below:



$$, \quad M \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \ \beta = 2, \ \ \mu = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Figure 4.2. The graph $A_{[2]}$

There are essential paths of any length in $A_{[2]}$. Any cyclic sequence of consecutive vertices defines an essential path, as for example $(120120 contiguous 120120 \cdots)$.

[6]See for example ref.[16] for a proof of this result.

5. Decomposition of the space of paths

Definition 12. *Maximum length of essential paths.* L is the maximum length of essential paths in a graph G iff any path of length greater than L is necessarily non-essential.

The following result will be used to write the above mentioned decomposition.

Proposition 13.

$$c_i^\dagger c_j^\dagger = c_{j+2}^\dagger c_i^\dagger \quad \forall\, j \geq i\, (\Rightarrow c_j c_i = c_i c_{j+2} \quad \forall\, j \geq i).$$

Proof.

$$c_i^\dagger c_j^\dagger(v_0, v_1, \cdots, v_i, \cdots, v_j, \cdots, v_n) =$$

$$= \sum_v \sqrt{\frac{\mu_v}{\mu_{v_j}}}\, c_i^\dagger(v_0, v_1, \cdots, v_i, \cdots, v_j, v, v_j, \cdots, v_n)$$

$$= \sum_{v,v'} \sqrt{\frac{\mu_v \mu_{v'}}{\mu_{v_j}\mu_{v_i}}}(v_0, v_1, \cdots, v_i, v', v_i, \cdots, v_j, v, v_j, \cdots, v_n),$$

on the other hand,

$$c_{j+2}^\dagger c_i^\dagger(v_0, v_1, \cdots, v_i, \cdots, v_j, \cdots, v_n) =$$

$$= \sum_{v'} \sqrt{\frac{\mu_{v'}}{\mu_{v_i}}}\, c_{j+2}^\dagger(v_0, v_1, \cdots, v_i, v', v_i, \cdots, v_j, \cdots, v_n)$$

$$= \sum_{v,v'} \sqrt{\frac{\mu_v \mu_{v'}}{\mu_{v_j}\mu_{v_i}}}(v_0, v_1, \cdots, v_i, v', v_i, \cdots, v_j, v, v_j, \cdots, v_n).$$

□

Proposition 14. The operators $c_i c_j^\dagger : \mathcal{P}_n \to \mathcal{P}_n$ satisfy

$$c_i c_j^\dagger = c_{j-2}^\dagger c_i \quad if \;\; i < j-1 \tag{5.1}$$

$$c_i c_j^\dagger = c_j^\dagger c_{i-2} \quad if \;\; i > j+1 \tag{5.2}$$

$$c_i c_{i\pm1}^\dagger = 1_n \;\; , i \pm 1 \leq n \tag{5.3}$$

$$c_i c_i^\dagger = \beta 1_n \;\; , i \leq n, \tag{5.4}$$

which imply

$$\begin{aligned} c_i c_j^\dagger &= (\beta\, \delta_{i,j} + \delta_{i,j+1} + \delta_{i,j-1}) 1_n + \theta(j-(i+2)) c_{j-2}^\dagger c_i + \theta(-j+(i-2)) c_j^\dagger c_{i-2} \\ &= (\beta\, \delta_{i-j,0} + \delta_{i-j,1} + \delta_{i-j,-1}) 1_n + \theta(2-(i-j)) c_{j-2}^\dagger c_i + \theta((i-j)-2) c_j^\dagger c_{i-2}, \end{aligned} \tag{5.5}$$

where 1_n is the identity operator in $\mathcal{P}_n$ and the function θ is defined by

$$\theta(i) = \begin{cases} 1 \; if \; i \geq 0 \\ 0 \; otherwise. \end{cases}$$

Proof. It follows from definition 5. □

The second equality in (5.5) has been written to emphasize the fact that the coefficients of the different terms depend only on the difference $i-j$.

Theorem 15. *The following decomposition holds*[7]

$$\begin{aligned} \mathcal{P}_n &= \mathcal{E}_n \bigoplus_{i\leq n-2} c_i^\dagger(\mathcal{E}_{n-2}) \bigoplus_{i_1<i_2\leq n-2} c_{i_2}^\dagger c_{i_1}^\dagger(\mathcal{E}_{n-4}) \bigoplus \cdots \\ & \bigoplus_{i_1<i_2\cdots<i_{[n/2]}\leq n-2} c_{i_{[n/2]}}^\dagger c_{i_{[n/2]-1}}^\dagger \cdots c_{i_1}^\dagger(\mathcal{E}_{1|0}), \end{aligned} \tag{5.6}$$

where [ ] *denotes the integer part and in the last summand one should take* 1 *for* n *odd and* 0 *for* n *even.*

Proof. The following important lemma will be employed in this proof:

Lemma 16. For all $\eta \in \mathcal{P}_n$ such that $c_i(\eta) \neq 0$ for some i and $c_j(\eta) = 0\ \forall j$ such that $i<j<n-2$ there exist coefficients α_k , $k=i,\cdots,n$ such that

$$\eta = \sum_{k=i}^{n-2} \alpha_k\, c_k^\dagger(c_i(\eta)) + \xi^{(i)} \tag{5.7}$$

with $\xi^{(i)}$ satisfying $c_j(\xi^{(i)}) = 0\ \forall j$ such that $i-1<j<n-2$.

Proof. Consider the application of c_i to eq.(5.7)

$$\begin{aligned} c_i(\eta) &= \sum_{k=i}^{n-2} \alpha_k\, c_i c_k^\dagger(c_i(\eta)) + c_i(\xi^{(i)}) \\ &= \sum_{k=i}^{n-2} \alpha_k \,\{(\beta\,\delta_{i,k} + \delta_{i,k-1}) + \theta(k-(i+2))c_{k-2}^\dagger c_i](c_i(\eta)) + c_i(\xi^{(i)}) \\ &= (\beta\alpha_i + \alpha_{i+1})c_i(\eta) + c_i(\xi^{(i)}), \end{aligned}$$

where proposition 14 was employed in the second equality and proposition 13 in the third. Therefore if we choose α_i and α_{i+1} such that

$$\beta\alpha_i + \alpha_{i+1} = 1,$$

then $c_i(\xi^{(i)}) = 0$. In general the application of c_{i+l} , $l=0,\cdots,n-2-i$ to eq. (5.7) is considered:

[7]Each term in this decomposition can be characterized by the number of non-essential back and forth subpaths. It has certain similarities with what is called Fock's space in quantum field theory, however they are quite different in some interesting respects. The role of the vacuum is played here by essential paths, so the analogy would be a theory with many non-equivalent vaccums, the number of which could be infinite as for example in the case of $A_{[2]}$. Excitations are created out of the vacuum by means of the creation operators $c_i^\dagger$. The algebra of these creation and annihilation operators being given by (5.5) which depends on the shape of the graph and which differs significantly from the canonical one, which is given in terms of commutators, appearing in the case of Fock's space.

$$
\begin{aligned}
c_{i+l}(\eta) &= \sum_{k=i}^{n-2} \alpha_k\, c_{i+l} c_k^{\dagger}(c_i(\eta)) + c_{i+l}(\xi^{(i)}) \\
&= \sum_{k=i}^{n-2} \alpha_k \,\{(\beta\, \delta_{i+l,k} + \delta_{i+l,k-1} + \delta_{i+l,k+1}) + \theta(2-(i+l-k))c_{k-2}^{\dagger} c_{i+l} \\
&\quad +\theta(l-2)\theta((i+l-k)-2))c_k^{\dagger} c_{i+l-2}](c_i(\eta)) + c_{i+l}(\xi^{(i)}) \\
&= (\beta\alpha_{i+l} + \alpha_{i+l+1} + \alpha_{i+l-1})c_i(\eta) + c_{i+l}(\xi^{(i)}).
\end{aligned}
$$

Therefore if the coefficients α_k can be chosen such that

$$
\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \beta & 1 & 0 & \cdots & 0 \\ 1 & \beta & 1 & \cdots & 0 \\ & 1 & \beta & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & \beta \end{pmatrix} \begin{pmatrix} \alpha_i \\ \alpha_{i+1} \\ \alpha_{i+2} \\ \vdots \\ \alpha_{n-2} \end{pmatrix},
$$

then the result follows because $c_{i+l}(\eta) = 0 \ \ , l = 1, \cdots, n-2-i$ by hypothesis. The determinant of this $(n-1-i) \times (n-1-i)$ matrix can be calculated recursively[8] leading to

$$
D_{n-1-i}(\beta) = \det \begin{vmatrix} \beta & 1 & 0 & \cdots & 0 \\ 1 & \beta & 1 & \cdots & 0 \\ & 1 & \beta & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & \beta \end{vmatrix} = \beta^{n-1-i}\, \frac{\lambda_+^{n-i} - \lambda_-^{n-i}}{\lambda_+ - \lambda_-}, \tag{5.8}
$$

where

$$
\lambda_\pm = \frac{1 \pm \sqrt{1-4\beta^{-2}}}{2}. \tag{5.9}
$$

For $\beta = 2$ the two eigenvalues coincide and taking the limit $\beta \to 2$ in (5.8) gives

$$
\lim_{\beta\to 2} D_{n-1-i}(\beta) = (n-i),
$$

which does not vanish for any $i(i \leq n-2)$. For $\beta \neq 2$, this determinant vanishes if

$$
\lambda_+^{n-i} - \lambda_-^{n-i} = 0 \ \ \Rightarrow \left(\frac{\lambda_+}{\lambda_-}\right)^{n-i} = 1 \ \ (\beta \neq 0), \tag{5.10}
$$

[8]The same determinant appears in the calculation of the harmonic oscillator transition probability using the path integral (see [17], p. 431).

which has no solution for $\beta > 2$. For the case $\beta < 2$, the ADE case, the eigenvalues are complex conjugate of each other, i.e. $\lambda_- = \lambda_+^*$. From eq. (5.9) it is obtained

$$\frac{\lambda_+}{\lambda_-} = e^{i\phi} \quad ,\phi \text{ such that } \beta = 2\cos\phi,$$

but on the other hand for $\beta < 2, \beta = 2\cos\frac{\pi}{N}$ where N is the Coxeter number of G. However it is well known that the maximum length of essential paths L is related to the Coxeter number by $L = N - 2$, thus $\phi = \frac{\pi}{N} = \frac{\pi}{L+2}$, therefore eq. (5.10) is the same as

$$e^{i\frac{\pi(n-i)}{L+2}} = 1,$$

which can never be satisfied. This is so because under the assumptions of this lemma the following inequality should hold $n - i - 1 \leq L$. If it were not so then the path obtained by reversing η and including the first $n - i - 1$ steps would be an essential path of length greater than L in the graph G which is impossible by definition of L. □

Using this lemma the following algorithm can be employed to obtain a unique decomposition of an arbitrary path $\eta \in \mathcal{P}_n$ as in the r.h.s. of (5.6). Decompose η as in (5.7). Then decompose every $c_i(\eta)$ appearing in the first term of the r.h.s. of (5.7) using (5.7) and do the same with $\xi^{(i)}$. At each step of this process the resulting paths are annihilated by one more c_i operator, since the number of these operators that can act on an element of $\mathcal{P}_n$ is $n - 2$ then this process necessarily converges to something belonging to the r.h.s. of (5.6). The ordering of the indices of the $c^\dagger$ operators in (5.6) follows using proposition 13. □

The following result is a simple consequence of the decomposition (5.6).

Proposition 17. The subspaces of $\mathcal{P}_n$ given by

$$P_n^{(l)} = \bigoplus_{i_1 < i_2 < \cdots \leq n-2} c_{i_1}^\dagger c_{i_2}^\dagger \cdots c_{i_l}^\dagger (\mathcal{E}_{n-2l}) \quad ,P^{(0)}(n) = \mathcal{E}_n, \ \ l = 0, \cdots, [n/2],$$

are mutually orthogonal.

Proof. This proposition is proved if we show that

$$M_{lm} = (c_{i_1}^\dagger c_{i_2}^\dagger \cdots c_{i_l}^\dagger (\xi^{(n-2l)}), c_{j_1}^\dagger c_{j_2}^\dagger \cdots c_{j_m}^\dagger (\xi^{(n-2m)})) \propto \delta_{lm},$$
$$\forall \xi^{(n-2l)} \in \mathcal{E}_{n-2l}\ , \xi^{(n-2m)} \in \mathcal{E}_{n-2m},$$

this in turn follows[9] from the relations in proposition 14. □

Thus there exist orthogonal projections on each of the subspaces $P^{(l)}$, $l = 0, \cdots, [n/2]$ that we denote by $\Pi_n^{(l)}$ and satisfy $\Pi_n^{(l)} = \Pi_n^{(l)2} = \Pi_n^{(l)\dagger}$. In particular $\Pi_n^{(0)}$ is a orthogonal projector over essential paths of length n.

[9]See the proof of proposition 20 for a similar argument.

Example 18. *Decomposition of non-essential paths in A_3.* Using the algorithm of the previous theorem the following decompositions of non-essential paths of a given length coming from the concatenation of essential paths are obtained:
Length two:

$$\begin{aligned}
(01)\star(10)=(010) &= \frac{1}{2^{1/4}}c_0^\dagger(0)\\
(21)\star(12)=(212) &= \frac{1}{2^{1/4}}c_0^\dagger(2)\\
(10)\star(01)=(101) &= \frac{1}{\sqrt{2}}(\frac{1}{2^{1/4}}c_0^\dagger(1)-\gamma)\\
(12)\star(21)=(121) &= \frac{1}{\sqrt{2}}(\frac{1}{2^{1/4}}c_0^\dagger(1)+\gamma)
\end{aligned}$$

Length three[10]:

$$\begin{aligned}
(01)\star\gamma &= (\frac{1}{2^{1/4}}c_1^\dagger-2^{1/4}c_0^\dagger)(01)\\
(21)\star\gamma &= -(\frac{1}{2^{1/4}}c_1^\dagger-2^{1/4}c_0^\dagger)(21)\\
\gamma\star(10) &= (\frac{1}{2^{1/4}}c_0^\dagger-2^{1/4}c_1^\dagger)(10)\\
\gamma\star(12) &= -(\frac{1}{2^{1/4}}c_0^\dagger-2^{1/4}c_1^\dagger)(12)\\
(10)\star(012) &= (2^{1/4}c_0^\dagger-\frac{1}{2^{1/4}}c_1^\dagger)(12)\\
(012)\star(21) &= (2^{1/4}c_1^\dagger-\frac{1}{2^{1/4}}c_0^\dagger)(01)\\
(12)\star(210) &= (2^{1/4}c_0^\dagger-\frac{1}{2^{1/4}}c_1^\dagger)(10)\\
(210)\star(01) &= (2^{1/4}c_1^\dagger-\frac{1}{2^{1/4}}c_0^\dagger)(21)
\end{aligned}$$

Length four:

$$(012)\star(210) = (c_1^\dagger-\frac{1}{\sqrt{2}}c_2^\dagger)\,c_0^\dagger((0)) \tag{5.11}$$

$$(210)\star(012) = (c_1^\dagger-\frac{1}{\sqrt{2}}c_2^\dagger)\,c_0^\dagger((2)) \tag{5.12}$$

$$\gamma\star\gamma = (c_1^\dagger-\frac{1}{\sqrt{2}}c_2^\dagger)c_0^\dagger((1)) \tag{5.13}$$

[10] It is recalled that $\gamma=\frac{1}{\sqrt{2}}[(121)-(101)]$ as defined in example 10.

Example 19. *Decomposition of non-essential paths in $A_{[2]}$.* Using the algorithm of the previous theorem the following decompositions of non-essential paths of a given length coming from the concatenation of essential paths are obtained:
Length two:

$$(01)\star(10)=\frac{1}{2}c_0^\dagger((0))+\frac{1}{2}\xi_{010},\quad (02)\star(20)=\tfrac{1}{2}c_0^\dagger((0))-\tfrac{1}{2}\xi_{020},\quad \xi_{010}=\xi_{020}=(010)-(020)\in\mathcal{E},$$
$$(12)\star(21)=\frac{1}{2}c_0^\dagger((1))+\frac{1}{2}\xi_{121},\quad (10)\star(01)=\tfrac{1}{2}c_0^\dagger((1))-\tfrac{1}{2}\xi_{101},\quad \xi_{121}=\xi_{101}=(121)-(101)\in\mathcal{E},$$
$$(20)\star(02)=\frac{1}{2}c_0^\dagger((2))+\frac{1}{2}\xi_{202},\quad (21)\star(12)=\tfrac{1}{2}c_0^\dagger((2))-\tfrac{1}{2}\xi_{212},\quad \xi_{202}=\xi_{212}=(202)-(212)\in\mathcal{E}.$$

It is worth noting that the last two lines above can be obtained from the first one by making cyclic permutations of the vertices $0,1$ and 2 (not for the indices of the $c^\dagger$ operators), i.e. by applying the rotations contained in the symmetry group C_{3v} of the graph $A_{[2]}$.
Length three:

$$\begin{aligned}
(10)\star(012) &= (1012) = &&(\frac{2}{3}c_0^\dagger-\frac{1}{3}c_1^\dagger)(12)+\xi_{1012}\\
(012)\star(21) &= (0121) = &&(\frac{2}{3}c_1^\dagger-\frac{1}{3}c_0^\dagger)(01)+\xi_{0121}\\
(10)\star\xi_{010} &= (1010)-(1020) &&=(\frac{2}{3}c_0^\dagger-\frac{1}{3}c_1^\dagger)(10)+\xi_{(10)\star\xi_{010}}\\
\xi_{010}\star(01) &= (0101)-(0201) &&=(\frac{2}{3}c_1^\dagger-\frac{1}{3}c_0^\dagger)(01)+\xi_{\xi_{010}\star(01)}
\end{aligned}$$

where

$$\begin{aligned}
\xi_{1012} &= \frac{1}{3}[(1012)-(1212)+(1202)]\ \in\mathcal{E}\\
\xi_{0121} &= \frac{1}{3}[(0121)-(0101)+(0201)]\ \in\mathcal{E}\\
\xi_{(10)\star\xi_{010}} &= \frac{2}{3}[(1010)-(1020)-(1210)]\ \in\mathcal{E}\\
\xi_{\xi_{010}\star(01)} &= \frac{2}{3}[(0101)-(0201)-(0121)]\ \in\mathcal{E}
\end{aligned}$$

From these four decompositions and applying the elements of the symmetry group C_{3v} of the graph $A_{[2]}$ the other twenty decompositions can be readily obtained.
Length four:

$$(01210)=(\frac{2}{3}c_1^\dagger-\frac{1}{3}c_2^\dagger)\,[\frac{1}{2}c_0^\dagger((0))+\xi^{(2)}_{01210}]-(\frac{1}{2}c_0^\dagger-\frac{1}{3}c_1^\dagger+\frac{1}{6}c_2^\dagger)\xi^{(2)}_{01210}+\xi^{(0)}_{01210}\quad(5.14)$$

where $\xi^{(0)},\xi^{(2)}\in\mathcal{E}$ are given by,

$$\begin{aligned}
\xi^{(0)}_{01210} &= \frac{1}{6}[(01210)+(02120)+(01020)-(02020)-(01010)+(02010)]\\
\xi^{(2)}_{01210} &= \frac{1}{2}[(010)-(020)].
\end{aligned}$$

Also

$$\begin{aligned} \xi_{010}\star(012) &= [c_1^\dagger - \frac{1}{2}(c_2^\dagger + c_0^\dagger)](012) + \xi_{\xi_{010}\star(012)} \\ (210)\star\xi_{010} &= [c_1^\dagger - \frac{1}{2}(c_2^\dagger + c_0^\dagger)](210) + \xi_{(210)\star\xi_{010}}, \end{aligned}$$

where $\xi_{\xi_{010}\star(012)}, \xi_{(210)\star\xi_{010}} \in \mathcal{E}$ are given by

$$\begin{aligned} \xi_{\xi_{010}\star(012)} &= \frac{1}{2}[(01012) - (02012) - (01212) + (01202)] \\ \xi_{(210)\star\xi_{010}} &= \frac{1}{2}[(21010) - (21020) - (21210) + (20210)]. \end{aligned}$$

Applying the elements of C_{3v} to these decompositions the others can be readily obtained. Finally,

$$\xi_{121}\star\xi_{121} = \frac{2}{3}c_1^\dagger c_1^\dagger(1) - \frac{1}{3}c_2^\dagger c_1^\dagger(1) + \xi_{\xi_{121}\star\xi_{121}},$$

where $\xi_{\xi_{121}\star\xi_{121}} \in \mathcal{E}$ is given by

$$\xi_{\xi_{121}\star\xi_{121}} = \frac{2}{3}[(12121) + (10101) - (10121) - (12101) - (12021) - (10201)].$$

6. The projection

A posteriori motivation for the definition of the projection $P : End^{gr}(\mathcal{P}) \to End^{gr}(\mathcal{E})$ appearing below is given by its properties with respect to the concatenation product of paths (see propositions 27, 29, 36, 37, 39, 43). However it was proposed based on its relation with the representation theory of these weak *-Hopf algebras, representation theory that is not considered in this paper. Before giving this definition a useful result is given:

Proposition 20.

$$(c_{i_1}c_{i_2}\cdots c_{i_n}c_{j_n}^\dagger c_{j_{n-1}}^\dagger\cdots c_{j_1}^\dagger\xi_a, \xi_b) = \delta_{ab}C(i_1,\cdots;i_n;j_n,\cdots;j_1), \qquad (6.1)$$

where ξ_a, ξ_b are elements of an orthonormal basis of $\mathcal{E}$ such that $\#\xi_a = \#\xi_b \geq j_1$ and $\#\xi_a = \#\xi_b \geq i_1$.

Proof. The evaluation of the matrix element in the l.h.s. is considered. By means of relation (5.5) the product of operators $c_{i_n}c_{j_n}^\dagger$ is either replaced by a number β or 1 (which we call a contraction), or they are interchanged with a change in the index of one of them. In any case for the matrix element to be non-vanishing, all the i indices should be contracted with j indices. If this is not the case the matrix element vanishes because necessarily a c operator will be applied to ξ_a or ξ_b which gives zero because they are essential.

□

Let $End(\mathcal{P}_n)$ $(End(\mathcal{E}_n))$ denote the vector space of endomorphism of length n paths (essential paths)[11]. In what follows the vector space of length preserving endomorphism of paths (essential paths) will be considered. They are defined by

$$End^{gr}(\mathcal{P}) = \bigoplus_n End(\mathcal{P}_n) \ , End^{gr}(\mathcal{E}) = \bigoplus_n End(\mathcal{E}_n).$$

Definition 21. A projector[12] $P : End^{gr}(\mathcal{P}) \to End^{gr}(\mathcal{E})$ is defined by its action on the terms appearing in the decomposition (5.6)

$$\begin{aligned} P(c^\dagger_{j_n} \cdots c^\dagger_{j_1} \xi_a \otimes c^\dagger_{i_n} \cdots c^\dagger_{i_1} \xi_b) &= \sum_{\xi_c \in \mathcal{E}} (c_{i_1} c_{i_2} \cdots c_{i_n} c^\dagger_{j_n} c^\dagger_{j_{n-1}} \cdots c^\dagger_{j_1} \xi_a, \xi_c)\ \xi_c \otimes \xi_b \\ &= C(i_1, \cdots, i_n; j_n, \cdots, j_1)\ \xi_a \otimes \xi_b, \end{aligned} \tag{6.2}$$

where $j_1 < j_2 < \cdots < j_n$ and $i_1 < i_2 < \cdots < i_n$.

It is clear that $P^2 = P$ but $P^\dagger \neq P$ which implies that P is not an orthogonal projection.

Remark 22. It should be noted that the projection of an arbitrary element $\eta \otimes \eta'$ of $End(\mathcal{P}_n)$ is obtained by applying definition 21 to each term appearing in the decomposition of $\eta \otimes \eta'$ as in eq. (5.6). Thus in general this projection consists in a summation of elements belonging to

$$\bigoplus_{l=0}^{[n/2]} End(\mathcal{E}_{n-2l}).$$

Therefore it will not in general respect the grading.

Remark 23. Note that because of eq. (6.1) and the orthonormality of the basis $\{\xi_a\}$, the following equality holds

$$P(c^\dagger_{j_n} c^\dagger_{j_{n-1}} \cdots c^\dagger_{j_1} \xi_a \otimes c^\dagger_{i_n} c^\dagger_{i_{n-1}} \cdots c^\dagger_{i_1} \xi_b) =$$
$$= \sum_{\xi_c \in \mathcal{E}} \xi_a \otimes \xi_c\ (\xi_c, c_{j_1} c_{j_2} \cdots c_{j_n} c^\dagger_{i_n} c^\dagger_{i_{n-1}} \cdots c^\dagger_{i_1} \xi_b).$$

Example 24. As an example the projections of the element $(01210) \otimes (21012)$ both for the graph A_3 and $A_{\{2\}}$ are calculated. For A_3:

$$\begin{aligned} P((01210) \otimes (21012))_{A_3} &= \sum_v ((c^\dagger_1 - \frac{1}{\sqrt{2}} c^\dagger_2)\, c^\dagger_0((0)), (c^\dagger_1 - \frac{1}{\sqrt{2}} c^\dagger_2)\, c^\dagger_0((v)))\ v \otimes (2) \\ &= \sum_v (c_0 (c_1 - \frac{1}{\sqrt{2}} c_2)\, (c^\dagger_1 - \frac{1}{\sqrt{2}} c^\dagger_2)\, c^\dagger_0((0)), v)\ v \otimes (2) \\ &= \sum_v \sqrt{2}(\sqrt{2} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{2})\, \delta_{v,0}\ v \otimes (2) = 0 \otimes 2, \end{aligned}$$

[11] In what follows the following equalities $End(\mathcal{P}_n) = \mathcal{P}_n \otimes \mathcal{P}_n$ and $End(\mathcal{E}_n) = \mathcal{E}_n \otimes \mathcal{E}_n$ will be employed. This is so because by means of the scalar product appearing in definition 3 and section 8 it is possible to identify $\mathcal{P}_n$ and $\mathcal{E}_n$ with their duals.

[12] In ref. [18] another projector Q acting on the same vector space is considered. Defining a product as in (7.3) but using Q does not lead to a weak Hopf algebra structure.

where the decompositions (5.11) and (5.12) were employed for the first equality and proposition 14 for the last. For $A_{[2]}$, the decomposition of eq. (5.14) and the following are employed:

$$(21012) \quad = \quad (\frac{2}{3}\, c_1^\dagger - \frac{1}{3} c_2^\dagger)\, [\frac{1}{2} c_0^\dagger((2)) + \xi^{(2)}_{21012}] - (\frac{1}{2} c_0^\dagger - \frac{1}{3} c_1^\dagger + \frac{1}{6} c_2^\dagger)\xi^{(2)}_{21012} + \xi^{(0)}_{21012},$$

where

$$\begin{aligned} \xi^{(2)}_{21012} \quad &= \quad \frac{1}{2}[(212) - (202)], \\ \xi^{(1)}_{21012} \quad &= \tfrac{1}{6}[\quad (21012) + (20102) + (21202) - (20202) - (21212) + (20212)]. \end{aligned}$$

Recalling that the projection kills terms with unequal number of $c^\dagger$ operators applied to essential paths in each factor of the tensor product leads to

$$\begin{aligned} P((01210) \otimes (21012))_{A_{[2]}} = &\sum_{\rho \in \mathcal{E}} (\xi^{(1)}_{01210}, \rho)\, \rho \otimes \xi^{(1)}_{21012} \\ &+ \sum_{\rho \in \mathcal{E}} (([c_1^\dagger - \frac{1}{2}(c_0^\dagger + c_2^\dagger)]\xi^{(2)}_{01210}, [c_1^\dagger - \frac{1}{2}(c_0^\dagger + c_2^\dagger)]\rho)\, \rho \otimes \xi^{(2)}_{21012} \\ &+ \sum_{v \in \mathcal{E}_0} ((\frac{2}{3}\, c_1^\dagger - \frac{1}{3} c_2^\dagger)\, \frac{1}{2} c_0^\dagger(0), (\frac{2}{3}\, c_1^\dagger - \frac{1}{3} c_2^\dagger)\, \frac{1}{2} c_0^\dagger(v))\, v \otimes (2). \end{aligned}$$

Evaluating the scalar products gives

$$P((01210) \otimes (21012))_{A_{[2]}} = \xi^{(1)}_{01210} \otimes \xi^{(1)}_{21012} + \xi^{(2)}_{01210} \otimes \xi^{(2)}_{21012} + \frac{1}{3}\, (0) \otimes (2).$$

7. Star algebra

In the vector space $End^{gr}(\mathcal{P})$ the following involution is considered:

Definition 25. *Star.*

$$(\xi \otimes \xi')^\star = \xi^\star \otimes \xi'^\star, \tag{7.1}$$

where $\xi^\star$ denotes the path obtained from ξ by "time inversion" for elementary paths and extending antilinearly to all $\mathcal{P}$, i.e., by reversing the sense in which the succession of contiguous vertices is followed for elementary paths, i.e.,

$$\xi = (v_o, v_1, \cdots, v_{n-1}, v_n) \Rightarrow \xi^\star = (v_n, v_{n-1}, \cdots, v_1, v_0).$$

From this definition and the one of the scalar product in section 2, it is clear that

$$(\eta, \chi) = \overline{(\eta^\star, \chi^\star)}, \tag{7.2}$$

where the bar indicates the complex conjugate. The underlying vector space of the algebra to be considered is given by the length graded endomorphisms of essential paths[13] $End^{gr}(\mathcal{E})$. The product is defined by:

[13]This choice of the underlying vector space structure doe not mean that the product to be considered is the composition of endomorphisms in $End^{gr}(\mathcal{E})$. It is emphasized that this is *not* the product to be considered but another product that we call $\cdot$ and that will be defined below.

Definition 26. *Product.*

$$(\xi \otimes \xi') \cdot (\rho \otimes \rho') = P(\xi \star \rho \otimes \xi' \star \rho') \quad ; \xi, \rho, \xi', \rho' \in \mathcal{E}. \tag{7.3}$$

This product does not make this algebra a graded one. This is a filtered algebra respect to the length of paths. The product of $\xi \otimes \xi' \in End(\mathcal{E}_{n_1})$ with $\rho \otimes \rho' \in End(\mathcal{E}_{n_2})$ in general belongs to

$$\bigoplus_{l=0}^{[(n_1+n_2)/2]} End(\mathcal{E}_{n_1+n_2-2l}).$$

The identity is

$$\mathbf{1} = \sum_{v,v' \in \mathcal{E}_0} v \otimes v'.$$

The properties to be fulfilled by these definitions are fairly simple to be proved except for the antihomomorphism property of the involution (7.1) and the associativity of the product (7.3). The following result will be employed in the proof of the first of these properties:

Proposition 27.

$$P((\eta \otimes \eta')^\star) = P(\eta \otimes \eta')^\star \quad \forall \eta, \eta' \in \mathcal{P}. \tag{7.4}$$

Proof. Typical contributions to the decomposition (5.6) for η and η' are considered. These contributions should have the same number of $c^\dagger$ operators applied to essential paths in order to have a non-vanishing image when applying the projector. Therefore the following expression is considered

$$\begin{aligned} P(c^\dagger_{j_n} c^\dagger_{j_{n-1}} \cdots c^\dagger_{j_1} \xi_a \otimes c^\dagger_{i_n} c^\dagger_{i_{n-1}} \cdots c^\dagger_{i_1} \xi_b) = \\ \sum_{\xi_c \in \mathcal{E}} (c_{i_1} c_{i_2} \cdots c_{i_n} c^\dagger_{j_n} c^\dagger_{j_{n-1}} \cdots c^\dagger_{j_1} \xi_a, \xi_c)\, \xi_c \otimes \xi_b \\ = \sum_{\xi_c \in \mathcal{E}} (c^\dagger_{j_n} c^\dagger_{j_{n-1}} \cdots c^\dagger_{j_1} \xi_a, c^\dagger_{i_n} c^\dagger_{i_{n-1}} \cdots c^\dagger_{i_1} \xi_c)\, \xi_c \otimes \xi_b, \end{aligned}$$

thus

$$P(c^\dagger_{j_n} c^\dagger_{j_{n-1}} \cdots c^\dagger_{j_1} \xi_a \otimes c^\dagger_{i_n} c^\dagger_{i_{n-1}} \cdots c^\dagger_{i_1} \xi_b)^\star = \sum_{\xi_c \in \mathcal{E}} \overline{(c^\dagger_{j_n} c^\dagger_{j_{n-1}} \cdots c^\dagger_{j_1} \xi_a, c^\dagger_{i_n} c^\dagger_{i_{n-1}} \cdots c^\dagger_{i_1} \xi_c)}\, \xi_c^\star \otimes \xi_b^\star.$$

The time inversion of a path of the form $c^\dagger_{i_n} c^\dagger_{i_{n-1}} \cdots c^\dagger_{i_1} \xi_b$ leads to (the counting starts from the end of this path)

$$(c^\dagger_{i_n} \cdots c^\dagger_{i_2} c^\dagger_{i_1} \xi_b)^\star = c^\dagger_{l+2(n-1)-i_n} c^\dagger_{l+2(n-2)-i_{n-1}} \cdots c^\dagger_{l+2-i_2} c^\dagger_{l-i_1} (\xi_b^\star),$$

where $l = \#\xi_b$. Using this relation leads to

$$\begin{aligned} & P((c^\dagger_{j_n} c^\dagger_{j_{n-1}} \cdots c^\dagger_{j_1} \xi_a)^\star \otimes (c^\dagger_{i_n} c^\dagger_{i_{n-1}} \cdots c^\dagger_{i_1} \xi_b)^\star) = \\ = & \sum_{\xi_d^\star \in \mathcal{E}} ((c^\dagger_{j_n} c^\dagger_{j_{n-1}} \cdots c^\dagger_{j_1} \xi_a)^\star, c^\dagger_{l+2(n-1)-i_n} \cdots c^\dagger_{l+2-i_2} c^\dagger_{l-i_1} \xi_d^\star)\, \xi_d^\star \otimes \xi_b^\star \\ = & \sum_{\xi_d^\star \in \mathcal{E}} ((c^\dagger_{j_n} c^\dagger_{j_{n-1}} \cdots c^\dagger_{j_1} \xi_a)^\star, (c^\dagger_{i_n} c^\dagger_{i_{n-1}} \cdots c^\dagger_{i_1} \xi_d)^\star)\, \xi_d^\star \otimes \xi_b^\star \\ = & \sum_{\xi_d \in \mathcal{E}} \overline{(c^\dagger_{j_n} c^\dagger_{j_{n-1}} \cdots c^\dagger_{j_1} \xi_a, c^\dagger_{i_n} c^\dagger_{i_{n-1}} \cdots c^\dagger_{i_1} \xi_d)}\, \xi_d^\star \otimes \xi_b^\star, \end{aligned}$$

where in the last equality eq. (7.2) was employed and the fact that when ξ_d runs over all $\mathcal{E}$ then $\xi_d^\star$ also. □

Using (7.4) it follows that

Proposition 28.

$$((\xi \otimes \xi') \cdot (\rho \otimes \rho'))^\star = (\rho \otimes \rho')^\star \cdot (\xi \otimes \xi')^\star \quad \forall \xi, \xi', \rho, \rho' \in \mathcal{E}. \tag{7.5}$$

Proof.

$$\begin{aligned} (\rho \otimes \rho')^\star \cdot (\xi \otimes \xi')^\star &= P((\rho \otimes \rho')^\star \star (\xi \otimes \xi')^\star) = P(((\xi \otimes \xi') \star (\rho \otimes \rho'))^\star) \\ &= P((\xi \otimes \xi') \star (\rho \otimes \rho'))^\star = ((\xi \otimes \xi') \cdot (\rho \otimes \rho'))^\star. \end{aligned}$$

□

In order to prove the associativity of the product (7.3) the following preliminary result is considered

Proposition 29.

$$P((\xi \otimes \xi') \star P(\eta \otimes \eta')) = P((\xi \otimes \xi') \star (\eta \otimes \eta')) \tag{7.6}$$

$$P(P(\eta \otimes \eta') \star (\xi \otimes \xi')) = P((\eta \otimes \eta') \star (\xi \otimes \xi')) \quad \forall \xi, \xi' \in \mathcal{E}, \quad \eta, \eta' \in \mathcal{P}. \tag{7.7}$$

Proof. As in proposition 27, typical contributions to the decomposition of $\eta \otimes \eta'$ are considered. Thus the following expression is dealt with

$$\begin{aligned} & P((\xi \otimes \xi') \star P(c^\dagger_{j_n} c^\dagger_{j_{n-1}} \cdots c^\dagger_{j_1} \xi_a \otimes c^\dagger_{i_n} c^\dagger_{i_{n-1}} \cdots c^\dagger_{i_1} \xi_b)) = \\ & \qquad = C(i_1, \cdots, i_n; j_n, \cdots, j_1)\, P((\xi \otimes \xi') \star (\xi_a \otimes \xi_b)), \end{aligned}$$

where it was assumed that $P(c^\dagger_{i_n} c^\dagger_{i_{n-1}} \cdots c^\dagger_{i_1} \xi_a \otimes c^\dagger_{j_n} c^\dagger_{j_{n-1}} \cdots c^\dagger_{j_1} \xi_b)$ does not vanish (if it vanishes it can be easily seen that the r.h.s. of eq.(7.6) also vanishes). Next the r.h.s. of eq.(7.6) is considered

$$\begin{aligned} & P((\xi \otimes \xi') \star (c^\dagger_{j_n} c^\dagger_{j_{n-1}} \cdots c^\dagger_{j_1} \xi_a \otimes c^\dagger_{i_n} c^\dagger_{i_{n-1}} \cdots c^\dagger_{i_1} \xi_b)) = \\ = & P(\xi \star c^\dagger_{j_n} c^\dagger_{j_{n-1}} \cdots c^\dagger_{j_1} \xi_a \otimes \xi' \star c^\dagger_{i_n} c^\dagger_{i_{n-1}} \cdots c^\dagger_{i_1} \xi_b) \\ = & P(c^\dagger_{l+j_n} \cdots c^\dagger_{l+j_1} (\xi \star \xi_a) \otimes c^\dagger_{l+i_n} \cdots c^\dagger_{l+i_1} (\xi' \star \xi_b)) \\ = & C(l+i_1 \cdots, l+i_n; l+j_n, \cdots, l+j_1) P((\xi \otimes \xi') \star (\xi_a \otimes \xi_b)), \end{aligned}$$

where l denotes the length of the path ξ. From its definition (6.1) it follows that

$$C(i_1, \cdots, i_n; j_n, \cdots, j_1) = C(l+i_1 \cdots, l+i_n; l+j_n, \cdots, l+j_1)$$

which completes the proof of the first equality. Eq.(7.7) follows along identical lines.

□

Using this result associativity follows

Proposition 30.
$((\xi_1\otimes\xi_1')\cdot(\xi_2\otimes\xi_2'))\cdot(\xi_3\otimes\xi_3') = (\xi_1\otimes\xi_1')\cdot((\xi_2\otimes\xi_2')\cdot(\xi_3\otimes\xi_3'))\ \ \forall\xi_i,\xi_i'\in\mathcal{E}\ ,i=1,2,3.$

Proof.

$$\begin{aligned}((\xi_1\otimes\xi_1')\cdot(\xi_2\otimes\xi_2'))\cdot(\xi_3\otimes\xi_3') &= P(P((\xi_1\otimes\xi_1')\star(\xi_2\otimes\xi_2'))\star(\xi_3\otimes\xi_3'))\\ &= P((\xi_1\otimes\xi_1')\star(\xi_2\otimes\xi_2')\star(\xi_3\otimes\xi_3'))\\ &= P((\xi_1\otimes\xi_1')\star P((\xi_2\otimes\xi_2')\star(\xi_3\otimes\xi_3')))\\ &= (\xi_1\otimes\xi_1')\cdot((\xi_2\otimes\xi_2')\cdot(\xi_3\otimes\xi_3')).\end{aligned}$$

□

Example 31. *Product for the case of A_3.* It can be explicitly verified that in this case the product coincides with the one of the double triangle algebra[14] described in ref.[11]. For illustrative purposes the calculation of some of these products is given below:

$$\begin{aligned}(21\otimes 12)\cdot(12\otimes 21) &= P(212\otimes 101) = \frac{1}{\sqrt{2}}P\left(\frac{1}{2^{1/4}}c_0^\dagger(2)\otimes(\frac{1}{2^{1/4}}c_0^\dagger(1)+\gamma)\right)\\ &= \frac{1}{\sqrt{2}}(2)\otimes(1)\\ (10\otimes 12)\cdot(01\otimes 21) &= \frac{1}{2}P\left((\frac{1}{2^{1/4}}c_0^\dagger(1)-\gamma)\otimes(\frac{1}{2^{1/4}}c_0^\dagger(1)+\gamma)\right)\\ &= \frac{1}{2}(1\otimes 1-\gamma\otimes\gamma)\\ (12\otimes 12)\cdot(21\otimes 21) &= \frac{1}{2}P\left((\frac{1}{2^{1/4}}c_0^\dagger(1)+\gamma)\otimes(\frac{1}{2^{1/4}}c_0^\dagger(1)+\gamma)\right)\\ &= \frac{1}{2}(1\otimes 1+\gamma\otimes\gamma)\\ (\gamma\otimes 012)\cdot(10\otimes 21) &= P(\gamma\star 10\otimes 0121)\\ &= P\left((\frac{1}{2^{1/4}}c_0^\dagger-2^{1/4}c_1^\dagger)(10)\otimes(2^{1/4}c_1^\dagger-\frac{1}{2^{1/4}}c_0^\dagger)(01)\right)\\ &= -(10)\otimes(01)\\ (\gamma\otimes\gamma)\cdot(\gamma\otimes\gamma) &= P\left((c_1^\dagger-\frac{1}{\sqrt{2}}c_2^\dagger)c_0^\dagger(1)\otimes(c_1^\dagger-\frac{1}{\sqrt{2}}c_2^\dagger)c_0^\dagger(1)\right) = (1)\otimes(1).\end{aligned}$$

[14]In that reference product is calculated using the pairing with the dual algebra, i.e. the 6j-symbols using the dual product (that is the composition of endomorphisms) and coming back with the 6j-symbols. The same construction for the case of $A_{[2]}$ is not known. The extension of 6j-symbols (Ocneanu cells) for this last case is not obvious since for example $A_{[2]}$ is not a bicolorable graph.

Example 32. *Product for the case of $A_{[2]}$.* With the notation of example 19 the following illustrative products can be computed:

$$
\begin{aligned}
(21 \otimes 12) \cdot (12 \otimes 21) &= P(212 \otimes 121) \\
&= P\left(\frac{1}{2}c_0^\dagger(2) - \frac{1}{2}\xi_{212} \otimes \frac{1}{2}c_0^\dagger(1) + \frac{1}{2}\xi_{121}\right) = \frac{1}{2}(2) \otimes (1) - \frac{1}{4}\xi_{212} \otimes \xi_{121} \\
(10 \otimes 12) \cdot (01 \otimes 21) &= P(101 \otimes 121) \\
&= P\left((\frac{1}{2}c_0^\dagger(1) - \frac{1}{2}\xi_{101}) \otimes (\frac{1}{2}c_0^\dagger(1) + \frac{1}{2}\xi_{121})\right) \\
&= \frac{1}{2}(1) \otimes (1) - \frac{1}{4}\xi_{121} \otimes \xi_{121}) \\
(12 \otimes 12) \cdot (21 \otimes 21) &= P(121 \otimes 121) \\
&= P\left((\frac{1}{2}c_0^\dagger(1) + \frac{1}{2}\xi_{121}) \otimes (\frac{1}{2}c_0^\dagger(1) + \frac{1}{2}\xi_{121})\right) \\
&= \frac{1}{2}(1) \otimes (1) + \frac{1}{4}\xi_{121} \otimes \xi_{121}) \\
(\xi_{121} \otimes 012) \cdot (10 \otimes 21) &= P(\xi_{121} \star 10 \otimes 0121) \\
&= P\left((\frac{2}{3}c_1^\dagger - \frac{1}{3}c_0^\dagger)(10) + \xi_{\xi_{121}\star 10} \otimes (\frac{2}{3}c_1^\dagger - \frac{1}{3}c_0^\dagger)(01) + \xi_{0121}\right) \\
&= \frac{2}{3}\,(10) \otimes (01) + \xi_{\xi_{121}\star 10} \otimes \xi_{0121} \\
(\xi_{121} \otimes \xi_{121}) \cdot (\xi_{121} \otimes \xi_{121}) &= \\
&= P\left((\frac{2}{3}c_1^\dagger c_1^\dagger - \frac{1}{3}c_2^\dagger c_1^\dagger)(1) + \xi_{\xi_{121}\star\xi_{121}} \otimes (\frac{2}{3}c_1^\dagger c_1^\dagger - \frac{1}{3}c_2^\dagger c_1^\dagger)(1) + \xi_{\xi_{121}\star\xi_{121}}\right) \\
&= \frac{4}{3}\,(1) \otimes (1) + \xi_{\xi_{121}\star\xi_{121}} \otimes \xi_{\xi_{121}\star\xi_{121}}.
\end{aligned}
$$

8. Weak bialgebra

The definition of a weak $\star$-bialgebra is recalled

Definition 33. A weak $\star$-bialgebra is a $\star$-algebra A together with two linear maps $\Delta : A \to A \otimes A$, the coproduct, and $\epsilon :\ A \to \mathbb{C}$, the counit, satisfying the following axioms

$$
\begin{aligned}
\Delta(ab) &= \Delta(a)\Delta(b) \\
\Delta(a^\star) &= \Delta(a)^\star \\
(\Delta \otimes Id)\Delta &= (Id \otimes \Delta)\Delta,
\end{aligned}
$$

and

$$\begin{array}{rcl} \epsilon(ab) & = & \epsilon(a\mathbf{1_1})\epsilon(\mathbf{1_2}b) \\ (\epsilon\otimes Id)\Delta = & Id & = (Id\otimes\epsilon)\Delta \\ \epsilon(aa^\star) & \geq & 0, \end{array}$$

where in the first equation Sweedler convention is employed and also in the following equation that defines $\mathbf{1_1}$ and $\mathbf{1_2}$

$$\Delta(\mathbf{1}) = \mathbf{1_1}\otimes\mathbf{1_2},$$

with $\mathbf{1}$ being the identity in A.

The definition of coproduct and counit considered for the star algebra of the previous section are

Definition 34. Coproduct[15],

$$\Delta(\xi\otimes\xi') = \sum_{\substack{\xi_a\in\mathcal{E}\\ \#\xi_a=\#\xi}} \xi\otimes\xi_a\boxtimes\xi^a\otimes\xi', \tag{8.1}$$

where the summation runs over a complete orthonormal basis for $\mathcal{E}$.

Definition 35. Counit,

$$\epsilon(\xi\otimes\xi') = (\xi,\xi').$$

The axioms appearing in the definition of a weak bialgebra are fairly simple to prove for the above definitions except for the morphism property for the coproduct and the one involving the counit of a product. For the first property the following preliminary results are useful:

Proposition 36.

$$\Delta P = P^{\otimes 2}\Delta_{\mathcal{P}}, \tag{8.2}$$

where $\Delta_{\mathcal{P}}(\chi\otimes\chi') = \sum_{\eta\in\mathcal{P}}\chi\otimes\eta\boxtimes\eta\otimes\chi'$ with summation over a complete orthonormal basis of $\mathcal{P}$.

Proof. Eq. (8.1) is applied to a generic element $\eta\otimes\eta'$of $End(\mathcal{P})$, typical terms in the decomposition (5.6) with non-vanishing image by the projector are considered

$$\Delta P(c^\dagger_{j_n}c^\dagger_{j_{n-1}}\cdots c^\dagger_{j_1}\xi_a\otimes c^\dagger_{i_n}c^\dagger_{i_{n-1}}\cdots c^\dagger_{i_1}\xi_b) = \sum_{\xi_c\in\mathcal{E}.} C(i_1,\cdots,i_n;j_n,\cdots,j_1)\,\xi_a\otimes\xi_c\boxtimes\xi_c\otimes\xi_b,$$

[15] In the dual weak Hopf algebra to the one considered here, this coproduct maps to the product. Eq. (8.1) implies that this product corresponds to the composition of endomorphisms in the dual weak Hopf algebra.

where it was assumed that $P(c^{\dagger}_{j_n} c^{\dagger}_{j_{n-1}} \cdots c^{\dagger}_{j_1} \xi \otimes c^{\dagger}_{i_n} c^{\dagger}_{i_{n-1}} \cdots c^{\dagger}_{i_1} \xi')$ does not vanish. On the other hand

$$\begin{aligned} & P^{\otimes 2} \Delta_{\mathcal{P}}(c^{\dagger}_{j_n} c^{\dagger}_{j_{n-1}} \cdots c^{\dagger}_{j_1} \xi_a \otimes c^{\dagger}_{i_n} c^{\dagger}_{i_{n-1}} \cdots c^{\dagger}_{i_1} \xi_b) = \\ = \; & P^{\otimes 2}(\textstyle\sum_{\eta \in \mathcal{P}} c^{\dagger}_{j_n} c^{\dagger}_{j_{n-1}} \cdots c^{\dagger}_{j_1} \xi_a \otimes \eta \boxtimes \eta \otimes c^{\dagger}_{i_n} c^{\dagger}_{i_{n-1}} \cdots c^{\dagger}_{i_1} \xi_b) \\ = \; & \textstyle\sum_{\xi_c, \xi_d \in \mathcal{E}} \sum_{\eta \in \mathcal{P}} (\xi_c, c_{j_1} \cdots c_{j_n} \eta)(c_{i_1} \cdots c_{i_n} \eta, \xi_d)\, \xi_a \otimes \xi_c \boxtimes \xi_d \otimes \xi_b \\ = \; & \textstyle\sum_{\xi_c, \xi_d \in \mathcal{E}} \sum_{\eta \in \mathcal{P}} (c^{\dagger}_{j_n} \cdots c^{\dagger}_{j_1} \xi_c, \eta)(\eta, c^{\dagger}_{i_n} \cdots c^{\dagger}_{i_1} \xi_d)\, \xi_a \otimes \xi_c \boxtimes \xi_d \otimes \xi_b \\ = \; & \textstyle\sum_{\xi_c, \xi_d \in \mathcal{E}} (c^{\dagger}_{j_n} \cdots c^{\dagger}_{j_1} \xi_c, c^{\dagger}_{i_n} \cdots c^{\dagger}_{i_1} \xi_d)\, \xi_a \otimes \xi_c \boxtimes \xi_d \otimes \xi_b \\ = \; & \textstyle\sum_{\xi_c, \xi_d \in \mathcal{E}} (\xi_c, c_{j_1} \cdots c_{j_n} c^{\dagger}_{i_n} \cdots c^{\dagger}_{i_1} \xi_d)\, \xi_a \otimes \xi_c \boxtimes \xi_d \otimes \xi_b \\ = \; & \textstyle\sum_{\xi_c \in \mathcal{E}} C(i_1, \cdots, i_n; j_n, \cdots, j_1)\, \xi_a \otimes \xi_c \boxtimes \xi_c \otimes \xi_b. \end{aligned}$$

□

Proposition 37.

$$P^{\otimes 2}(\Delta_{\mathcal{P}}(\xi_a \otimes \xi_b) \star \Delta_{\mathcal{P}}(\xi_c \otimes \xi_d)) = P^{\otimes 2}[P^{\otimes 2} \Delta_{\mathcal{P}}(\xi_a \otimes \xi_b) \star P^{\otimes 2} \Delta_{\mathcal{P}}(\xi_c \otimes \xi_d)].$$

Proof.

$$\begin{aligned} & P^{\otimes 2}(\Delta_{\mathcal{P}}(\xi_a \otimes \xi_b) \star \Delta_{\mathcal{P}}(\xi_c \otimes \xi_d)) = \\ & \quad = P^{\otimes 2}(\sum_{\eta, \chi \in \mathcal{P}} (\xi_a \otimes \eta \boxtimes \eta \otimes \xi_b) \star (\xi_c \otimes \chi \boxtimes \chi \otimes \xi_d)) \\ & \quad = P^{\otimes 2}(\sum_{\eta, \chi \in \mathcal{P}} P(\xi_a \star \xi_c \otimes \eta \star \chi) \boxtimes P(\eta \star \chi \otimes \xi_b \star \xi_d)). \end{aligned}$$

On the other hand,

$$\begin{aligned} & P^{\otimes 2}[P^{\otimes 2} \Delta_{\mathcal{P}}(\xi \otimes \xi') \star P^{\otimes 2} \Delta_{\mathcal{P}}(\rho \otimes \rho')] = \\ & \quad = P^{\otimes 2}[P^{\otimes 2} \sum_{\eta, \chi \in \mathcal{P}} (\xi_a \otimes \eta \boxtimes \eta \otimes \xi_b) \star P^{\otimes 2} \Delta_{\mathcal{P}}(\xi_c \otimes \chi \boxtimes \chi \otimes \xi_d)] \\ & \quad = P^{\otimes 2} \sum_{\eta, \chi \in \mathcal{P}} P(P(\xi_a \otimes \eta) \star P(\xi_c \otimes \chi)) \boxtimes P(P(\eta \otimes \xi_b) \star P(\chi \otimes \xi_d)). \end{aligned} \tag{8.3}$$

Next it is noted that

$$\begin{aligned} P(\xi_a \otimes \eta) \star P(\xi_c \otimes \chi) &= \sum_{\omega, \sigma \in \mathcal{E}} (\omega, \eta)(\sigma, \chi)(\xi_a \otimes \omega) \star (\xi_c \otimes \sigma) \\ &= \sum_{\omega, \sigma \in \mathcal{E}} (\xi_a \star \xi_c) \otimes (\omega \star \sigma)\, \delta_{\omega\eta} \delta_{\sigma\chi} \\ &= (\xi_a \star \xi_c) \otimes (\eta \star \chi). \end{aligned}$$

Replacing in (8.3) leads to

$$P^{\otimes 2}[P^{\otimes 2} \Delta_{\mathcal{P}}(\xi_a \otimes \xi_b) \star P^{\otimes 2} \Delta_{\mathcal{P}}(\xi_c \otimes \xi_d)] = P^{\otimes 2}(\sum_{\eta, \chi \in \mathcal{P}} P(\xi_a \star \xi_c \otimes \eta \star \chi) \boxtimes P(\eta \star \chi \otimes \xi_b \star \xi_d)).$$

□

Using the above results leads to

Proposition 38.
$$\Delta((\xi\otimes\xi')\cdot(\rho\otimes\rho'))=\Delta(\xi\otimes\xi')\cdot\Delta(\rho\otimes\rho').$$

Proof.
$$\begin{aligned}\Delta((\xi\otimes\xi')\cdot(\rho\otimes\rho')) &= \Delta(P((\xi\otimes\xi')\star(\rho\otimes\rho')))\\ &= P^{\otimes 2}\Delta_{\mathcal{P}}((\xi\otimes\xi')\star(\rho\otimes\rho'))\\ &= P^{\otimes 2}(\Delta_{\mathcal{P}}(\xi\otimes\xi')\star\Delta_{\mathcal{P}}(\rho\otimes\rho'))\\ &= P^{\otimes 2}[P^{\otimes 2}\Delta_{\mathcal{P}}(\xi\otimes\xi')\star P^{\otimes 2}\Delta_{\mathcal{P}}(\rho\otimes\rho')]\\ &= P^{\otimes 2}[\Delta(P(\xi\otimes\xi'))\star\Delta(P(\rho\otimes\rho'))]\\ &= \Delta(\xi\otimes\xi')\cdot\Delta(\rho\otimes\rho').\end{aligned}$$

□

Regarding the counit of a product the following result will be employed:

Proposition 39.
$$\epsilon(P(\eta\otimes\eta'))=\epsilon(\eta\otimes\eta')\ \ \forall\,\eta,\eta'\in\mathcal{P}.$$

Proof. A generic element $\eta\otimes\eta'$of $End(\mathcal{P})$ is considered, typical terms in the decomposition (5.6) with non-vanishing image by the projector are considered:

$$\begin{aligned}&\epsilon(P(c^\dagger_{j_n}c^\dagger_{j_{n-1}}\cdots c^\dagger_{j_1}\xi_a\otimes c^\dagger_{i_n}c^\dagger_{i_{n-1}}\cdots c^\dagger_{i_1}\xi_b))=\\ &=\epsilon(\sum_{\xi_c\in\mathcal{E}}(c_{i_1}c_{i_2}\cdots c_{i_n}c^\dagger_{j_n}c^\dagger_{j_{n-1}}\cdots c^\dagger_{j_1}\xi_a,\xi_c)\ \xi_c\otimes\xi_b)\\ &=(c_{i_1}c_{i_2}\cdots c_{i_n}c^\dagger_{j_n}c^\dagger_{j_{n-1}}\cdots c^\dagger_{j_1}\xi_a,\xi_b)\\ &=\epsilon(c^\dagger_{j_n}c^\dagger_{j_{n-1}}\cdots c^\dagger_{j_1}\xi_a\otimes c^\dagger_{i_n}c^\dagger_{i_{n-1}}\cdots c^\dagger_{i_1}\xi_b).\end{aligned}$$

□

Thus,

Proposition 40.
$$\epsilon((\xi_a\otimes\xi_b)\cdot(\xi_c\otimes\xi_d))=\epsilon((\xi_a\otimes\xi_b)\cdot\mathbf{1}_1)\epsilon(\mathbf{1}_2\cdot(\xi_c\otimes\xi_d)).$$

Proof.
$\epsilon((\xi_a\otimes\xi_b)\cdot(\xi_c\otimes\xi_d))=\epsilon(P(\xi_a\star\xi_c\otimes\xi_b\star\xi_d))=\epsilon(\xi_a\star\xi_c\otimes\xi_b\star\xi_d)=(\xi_a\star\xi_c,\xi_b\star\xi_d).$
On the other hand,
$$\begin{aligned}\epsilon((\xi_a\otimes\xi_b)\cdot\mathbf{1}_1)\epsilon(\mathbf{1}_2\cdot(\xi_c\otimes\xi_d)) &= \sum_{v,u,v'}\epsilon((\xi_a\otimes\xi_b)\cdot(v\otimes u))\epsilon((u\otimes v')\cdot(\xi_c\otimes\xi_d))\\ &= \sum_{v,u,v'}\delta_{r(\xi_a)v}\delta_{r(\xi_b)u}\delta_{us(\xi_c)}\delta_{v's(\xi_d)}(\xi_a,\xi_b)(\xi_c,\xi_d)\\ &= (\xi_a\star\xi_c,\xi_b\star\xi_d)=\epsilon((\xi_a\otimes\xi_b)\cdot(\xi_c\otimes\xi_d)).\end{aligned}$$

□

9. The antipode

In general the axioms to be satisfied by the antipode are

$$\begin{aligned} S(ab) &= S(b)S(a) \\ S((S(a^\star)^\star) &= a \\ \Delta(S(a)) &= S\otimes S\,(\Delta^{op}(a)) \\ S(a_1)\cdot a_2\otimes a_3 &= \mathbf{1}_1\otimes a\mathbf{1}_2, \end{aligned} \tag{9.1}$$

where in the last equation Sweedler convention has been employed.

The following ansatz for the antipode is considered:

$$S(\xi\otimes\omega) = F(\xi,\omega)\,\omega^\star\otimes\xi^\star, \tag{9.2}$$

where $F(\xi,\omega)$ is a numerical factor to be determined. It is fairly simple to show that the first three axioms in (9.1) are satisfied by this definition. The proof of the last axiom is more involved. The following preliminary results are considered:

Proposition 41. The following holds,

$$c^\dagger_{n-1}c^\dagger_{n-2}\cdots c^\dagger_0(v_0) = \sum_{\eta\in\mathcal{P}_n/s(\eta)=v_0}\sqrt{\frac{\mu_{r(\eta)}}{\mu_{s(\eta)}}}\,\eta\star\eta^\star, \tag{9.3}$$

where the summation is over the orthonormal basis of elementary paths with starting vertex v_0 and,

$$(\Pi^{(0)}_n\star\Pi^{(0)}_n)c^\dagger_{n-1}c^\dagger_{n-2}\cdots c^\dagger_0(v_0) = \sum_{\xi\in\mathcal{E}_n/s(\eta)=v_0}\sqrt{\frac{\mu_{r(\xi)}}{\mu_{s(\xi)}}}\,\xi\star\xi^\star, \tag{9.4}$$

where $\Pi^{(0)}_n$ is the orthogonal projector over essential paths of length n mentioned after proposition 17. The notation $(\Pi^{(0)}_n\star\Pi^{(0)}_n)$ indicates that when applied to a path of length $2n$ this operator projects over paths that are essential in its first n steps and also essential in its last n steps.

Proof. It follows from definition (3.1). □

Proposition 42. Let $\xi,\rho\in\mathcal{E}_n$, then,

$$c_{i_1}\cdots c_{i_n}\xi^\star\star\rho = \delta_{i_n n-1}\cdots\delta_{i_1 0}\,\delta_{\rho\xi}\sqrt{\frac{\mu_{v_n^{\xi^\star}}}{\mu_{v_0^{\xi^\star}}}}s(\xi^\star).$$

Proof. Since $\xi^\star,\rho\in\mathcal{E}$ then the only c operator that could give a non-zero result when applied the path $\xi^\star\star\rho$ is c_{n-1}(thus $i_m = n-1$), indeed it gives a non-zero result only if given a certain elementary path $\xi^\star_I$ appearing in the expression of $\xi^\star$there is a corresponding elementary path ρ_I appearing in the expression of ρ such that the first step in ξ_I (i.e. the inverse of the last step of $\xi^\star_I$) coincides with the first step of ρ_I. More precisely, if $\xi^\star_I = (v_0^{\xi^\star}, v_1^{\xi^\star}, \cdots, v_n^{\xi^\star})$ and $\rho_I = (v_0^\rho, v_1^\rho, \cdots, v_n^\rho)$, then

$$\begin{aligned} c_{n-1}(\xi^\star_I \star \rho_I) &= \delta_{v_n^{\xi^\star} v_0^\rho}\, \delta_{v_{n-1}^{\xi^\star} v_1^\rho} \sqrt{\frac{\mu_{v_n^{\xi^\star}}}{\mu_{v_{n-1}^{\xi^\star}}}} (v_0^{\xi^\star}, v_1^{\xi^\star}, \cdots, v_{n-1}^{\xi^\star}, v_1^\rho, \cdots, v_n^\rho) \\ &= \delta_{v_n^{\xi^\star} v_0^\rho}\, \delta_{v_{n-1}^{\xi^\star} v_1^\rho} \sqrt{\frac{\mu_{v_n^{\xi^\star}}}{\mu_{v_{n-1}^{\xi^\star}}}}\, \xi^\star_{I,n-1} \star \rho_{I,n-1}. \end{aligned}$$

The first delta function appears because the concatenation $\xi^\star \star \rho$ should not vanish, the second from the definition of the c operator and the last equality is just a definition of the path $\xi^\star_{I,n-1} \star \rho_{I,n-1}$. Next consider the application of a c-operator to $\xi^\star_{I,n-1} \star \rho_{I,n-1}$, in a similar fashion, only c_{n-2} (thus $i_{m-1} = n-2$) gives a non-zero result, which is

$$\begin{aligned} c_{n-2}(\xi^\star_{I,n-1} \star \rho_{I,n-1}) &= \delta_{v_{n-2}^{\xi^\star} v_2^\rho} \sqrt{\frac{\mu_{v_{n-1}^{\xi^\star}}}{\mu_{v_{n-2}^{\xi^\star}}}} (v_0^{\xi^\star}, v_1^{\xi^\star}, \cdots, v_{n-2}^{\xi^\star}, v_2^\rho, \cdots, v_n^\rho) \\ &= \delta_{v_{n-2}^{\xi^\star} v_2^\rho} \sqrt{\frac{\mu_{v_{n-1}^{\xi^\star}}}{\mu_{v_{n-2}^{\xi^\star}}}}\, \xi^\star_{I,n-2} \star \rho_{I,n-2}. \end{aligned}$$

Proceeding in this way and collecting the contribution of each elementary term finally leads to

$$c_{i_1} \cdots c_{i_m} \xi^\star \star \rho = \delta_{i_m n-1} \cdots \delta_{i_1 0}\, \delta_{\rho\xi} \sqrt{\frac{\mu_{v_n^{\xi^\star}}}{\mu_{v_0^{\xi^\star}}}} s(\xi^\star).$$

□

Using the above result leads to

Proposition 43. Definition (9.2) satisfies (9.1) with

$$F(\xi, \omega) = \sqrt{\frac{\mu_{s(\omega)} \mu_{r(\xi)}}{\mu_{r(\omega)} \mu_{s(\xi)}}}.$$

Proof. Replacing the ansatz (9.2) in the last axiom in (9.1) leads to

$$\sum_{\xi_c, \xi_d \in \mathcal{E}} F(\xi, \xi_c)(\xi_c^\star \otimes \xi^\star) \cdot (\xi_c \otimes \xi_d) \boxtimes \xi_d \otimes \omega = \sum_{v,u,v' \in \mathcal{E}_0} v \otimes u \boxtimes (\xi \otimes \omega) \cdot (u \otimes v'). \quad (9.5)$$

Employing the definition of the product and the fact that $(\xi \otimes \omega) \cdot (u \otimes v') = \delta_{r(\xi)u} \delta_{r(\omega)v'} (\xi \otimes \omega)$ shows that (9.5) is equivalent to

$$\sum_{\xi_c \in \mathcal{E}} F(\xi, \xi_c) P(\xi_c^\star \star \xi_c \otimes \xi^\star \star \xi_d) = \delta_{\xi_d \xi} \sum_{v \in \mathcal{E}_0} v \otimes r(\xi). \quad (9.6)$$

Choosing the factor $F(\xi, \xi_c)$ to be of the form

$$F(\xi, \xi_c) = \alpha(\xi) \sqrt{\frac{\mu_{v_0^{\xi_c}}}{\mu_{v_n^{\xi_c}}}},$$

the l.h.s. of this last equation is given by

$$\sum_{\xi_c \in \mathcal{E}} F(\xi,\xi_c) P(\xi_c^\star \star \xi_c \otimes \xi^\star \star \xi_d) =$$

$$= \alpha(\xi) \sum_{v(=r(\xi_c))} P((\Pi_n^{(0)} \star \Pi_n^{(0)}) c_{n-1}^\dagger c_{n-2}^\dagger \cdots c_0^\dagger(v) \otimes \xi^\star \star \xi_d)$$

$$= \alpha(\xi) \sum_{v \in \mathcal{E}_0, \rho \in \mathcal{E}} v \otimes \rho((\Pi_n^{(0)} \star \Pi_n^{(0)}) c_{n-1}^\dagger c_{n-2}^\dagger \cdots c_0^\dagger \rho, \xi^\star \star \xi_d)$$

$$= \alpha(\xi) \sum_{v \in \mathcal{E}_0, \rho \in \mathcal{E}} v \otimes \rho(\rho, c_0 \cdots c_{n-2} c_{n-1} (\Pi_n^{(0)} \star \Pi_n^{(0)}) \xi^\star \star \xi_d)$$

$$= \alpha(\xi) \sum_{v \in \mathcal{E}_0, \rho \in \mathcal{E}} v \otimes \rho(\rho, r(\xi)) \delta_{\xi\xi_d} \sqrt{\frac{\mu_{s(\xi)}}{\mu_{r(\xi)}}}$$

$$= \alpha(\xi) \sqrt{\frac{\mu_{s(\xi)}}{\mu_{r(\xi)}}} \delta_{\xi\xi_d} \sum_{v \in \mathcal{E}_0} v \otimes r(\xi),$$

where in the first equality we have employed (9.4) of proposition 41, the second equality involves the definition of the projector P, the hermiticity of the projector $(\Pi_n^{(0)} \star \Pi_n^{(0)})$ was employed in writing the third equality, the fact that $\xi^\star$ and ξ_d are already essential and proposition 42 were employed in the fourth equality. Thus choosing

$$\alpha(\xi) = \sqrt{\frac{\mu_{r(\xi)}}{\mu_{s(\xi)}}} \Rightarrow F(\xi,\xi_c) = \sqrt{\frac{\mu_{r(\xi)}\mu_{s(\xi_c)}}{\mu_{s(\xi)}\mu_{r(\xi_c)}}}$$

leads to the result.

□

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R. Trinchero
Centro Atómico Bariloche e Instituto Balseiro,
Bariloche, Argentina.
trincher@cab.cnea.gov.ar

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