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A Trigonometric Recurrence Algorithm for Solving Nonlinear Problems in Structural Dynamics

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Abstract: An analytical-numerical methodology for solving nonlinear problems governed by partial differential equations (PDE) is presented. The authors have previously used a method named WEM that consists essentially of the statement of extended trigonometric series of uniform convergence (UC). Theorems, demonstrated previously, ensure the UC of the essential functions and the exactness of the eigenvalues. WEM has been applied to nonlinear dynamic problems in two different ways: As a direct variational method and as a solution in the classical sense. The present tool is applied in the second fashion and starts from the statement of the extended trigonometric series for all the unknown functions and derivatives involved in the PDE. The nonlinearities are treated in the same way. The application of consistence conditions leads to recurrence relationships among the coefficients of the extended series. For the sake of comparison an initial conditions problem (the well-known Duffing oscillator) is numerically solved. Then a beam example is solved in detail: A linear (both material and geometrical) supported beam, with end springs of nonlinear analytical response, under the action of a dynamic distributed load and/or prescribed initial conditions, is studied. Two numerical examples are included.

Key words: Nonlinear boundary conditions, nonlinear vibrations, partial differential equations, trigonometric recurrence

1. INTRODUCTION

In previous works, Filipich and Rosales (1994, 1995) and Filipich et al. (1996) have made use of a methodology called the *Whole Element Method* (WEM) based essentially on the use of trigonometric extended series to solve one-dimensional structural problems. Such series always yield uniform convergence (UC) of the function expansions, which results in the exceptional property of securing convergence towards exact solutions—which can consequently be calculated with arbitrary precision—in both boundary-value problems (BVPs) and initial conditions problems (ICPs).

These series are achieved with the simple addition of a linear support to the sines series when dealing with one-dimensional problems ($\in [0, 1]$). An infinite number of series are obtained when this approach is extended to two or more dimensions (Filipich et al. 1998; Filipich and Rosales, 2000), none of them a priori apparent.

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Figures 1–11 appear in color online: http://jvc.sagepub.com

On the other hand, WEM has been employed in two different ways, one (named WEM1) as a direct variational method—for instance, a thorough treatment of direct methods may be found in work by Mikhlin (1963), Rektorys (1975) and Reddy (1991)—finding the stationary condition of an ad-hoc function that leads in all cases to a pseudo theorem of virtual work, as may be found in Rosales (1997). ICPs have been handled successfully after an appropriate change of variables (Filipich and Rosales, 1997). Additionally, the method has been applied to the solution of an initial-boundary value problem (Rosales and Filipich, 1999). The other use of WEM is as a classical method to solve the governing differential equations (Filipich, 1999; Filipich and Rosales, 2001) requiring the UC of all the derivatives involved in the differential system and is known as WEM2. Both have in common the type of sequence.

In the present study WEM is used in its second version, WEM2. The dynamics of a linear beam with nonlinear boundary conditions (BC) effected by means of Duffing type end springs is analyzed. The beam is under a deterministic load with arbitrary time and space variations and arbitrary initial conditions (IC). The calculation of the exact motion of the beam with nonlinear BC is not elementary. Related papers report the solution of a similar problem using Galerkin techniques, harmonic balance, and finite element algorithms. See, for instance, Paslay and Gurtin (1960), Porter and Billet (1965), Nayfeh and Asfar (1986), Pillai and Rao (1992), and Venkateswara Rao and Rajasekhara Naidu (1994). They yield approximate solutions and cannot always be regarded as precise results. A problem similar to the present one but with another type of drive forcing was tackled by Filipich and Rosales (2001). However, the nonlinearities were addressed in a different manner. Rosales and Filipich (2002a, 2002b) also presented an approach using WEM1 in a similar case.

The use of WEM in this type of problem demonstrates its importance since, to the authors' knowledge, no exact solution is available or widely included in a standard finite element code. In addition to the presentation of the WEM solution, we introduce the technique of *trigonometric recurrence*, which permits the calculation of integer powers of functions in a systematic expansion in extended series. The great advantage and power of this concept is the avoidance of the cumbersome numerical integrations. Such integrals are naturally involved when the coefficients of the series are calculated.

The methodology is first illustrated with the well-known Duffing oscillator (ICP) and then with two numerical examples of the beam with and without load (IC-BVP). The results are presented in graphs of trajectories and phase plots.

2. TRIGONOMETRIC RECURRENCE

The initial problem is to find the solution for beam vibrations in the time domain. Evidently our domain is bidimensional: $\{D : 0 \leq x \leq 1, 0 \leq t\}$ after non-dimensionalization. As will be seen below, the semi-infinite domain is reduced to a unitary square domain. Nonlinearities arise from the springs located at the ends of the beams and, consequently the response is only time-dependent. Thus the trigonometric recurrence approach will be shown in one domain in what follows. Higher dimensions may be dealt with an elementary extension.

The extended UC series in $\{D : 0 \leq x \leq 1\}$ to expand a known function $f(x)$ will be one of the following

$$f(x) = \sum_{i=1}^{M} a_{1i} \sin i\pi x + a_{10}x + \alpha_1 \tag{1}$$

$$f(x) = \sum_{i=1}^{M} b_{1i} \cos i\pi x + b_{10} \tag{2}$$

For the sake of simplicity the trigonometric recurrence will be developed starting from base (2). Since $f(x)$ is continuous in D it may be expanded as in (2) with

$$b_{10} = \int_0^1 f(\varepsilon) d\varepsilon, \quad b_{1i} = 2\int_0^1 f(\varepsilon) \cos i\pi x d\varepsilon \tag{3}$$

Our aim is to find integer powers of $f(x)$, i.e., $f^n(x)$, $n = 2, 3, \cdots$, after knowing b_{10} and b_{1i} and applying a simple recurrence algorithm. The following notation is now introduced

$$f^n(x) = \sum_{i=1}^{M} b_{ni} \cos i\pi x + b_{n0}; \quad n = 1, 2, 3, \cdots \tag{4}$$

It is evident that

$$f^n(x) = f^{n-1}(x) f(x) \tag{5}$$

and consequently

$$f^n(x) = \left[\sum_{i=1}^{M} b_{(n-1)i} \cos i\pi x + b_{(n-1)0}\right] \left[\sum_{j=1}^{M} b_{1j} \cos j\pi x + b_{10}\right] \tag{6}$$

that is,

$$\begin{aligned} f^n(x) &= b_{(n-1)0}b_{10} + b_{(n-1)0} \sum_{j=1}^{M} b_{1j} \cos j\pi x + b_{10} \sum_{i=1}^{M} b_{(n-1)i} \cos i\pi x \\ &+ \sum_{i=1}^{M} \sum_{j=1}^{M} b_{(n-1)i} b_{1j} \cos i\pi x \cos j\pi x \end{aligned} \tag{7}$$

From condition (5), equation (4) should be equal to equation (7), and after definition (3) of orthogonality properties, we obtain

$$b_{n0} = b_{(n-1)0}b_{10} + \frac{1}{2} \sum_{k=1}^{M} b_{(n-1)k} b_{1k} \tag{8}$$

$$
\begin{aligned}
b_{ni} &= b_{(n-1)0}b_{1i} + b_{(n-1)i}b_{10} \\
&+ \frac{1}{2}\left[\sum_{k=1}^{i-1} b_{(n-1)k}b_{1(i-k)} + \sum_{k=1+i}^{M} b_{(n-1)k}b_{1(k-i)} + \sum_{k=1}^{M-i} b_{(n-1)k}b_{1(k+i)}\right] \\
n &= 2, 3, 4, \cdots \quad i = 1, 2, \cdots, M
\end{aligned}
\tag{9}
$$

One of the main advantages of this approach is the fact that $f(x)$ may be the sum of several functions, not necessarily algebraic polynomials, the n^{th} power of which may be difficult to integrate. In this way, instead,

$$
\int_0^1 f^n(x)dx = b_{n0} \tag{10}
$$

There is no need to calculate the b_{ni} ‘s to solve these integrals.

For the sake of brevity, the diverse possibilities of this methodology are not included and neither are the applications to the calculation of products and quotients of functions.

3. WHOLE ELEMENT METHOD (WEM)

As was mentioned in the introduction, WEM was developed in two alternative forms which, as shown by the authors in previous work (Filipich 1999 and Filipich and Rosales 2001), lead to the same results. The first alternative—WEM1—is a direct variational method that, after the sequence of a certain function, yields the desired solution. The other—WEM2—is a classical solution of the differential problem. The statement of a particular type of sequence is the common basis of both approaches. Here, WEM2 will be used to solve the motion of a linear beam with nonlinear BC.

3.1. Sequences

The interested reader may consult definitions and properties in greater detail in the Appendix of Filipich and Rosales (2001). If a continuous function $F(x)$is considered, one may state UC sequences for a one-dimensional domain that satisfy the following conditions

$$
|F(x) - \phi_M| \to 0 \quad M \to \infty, \quad \forall x \in D : \{0 \leq x \leq 1\} \tag{11}
$$

where the extended trigonometric series may be any of the following

$$
\phi_M(x) = \sum_{i=1}^{M} A_i s_i(x) + xA_0 + a_0 \tag{12}
$$

$$
\phi_M(x) = \sum_{i=1}^{M} B_i c_i(x) + B_0 \tag{13}
$$

When dealing with a two-dimensional domain the sequence should be such that it ensures the following condition for a continuous function $G(x,y)$

$$|G(x, y) - \varphi_M| \to 0 \quad M \to \infty \quad \forall x, y \in D : \{0 \le x \le 1, 0 \le y \le 1\} \tag{14}$$

where the extended trigonometric series may be written as

$$\varphi_M(x, y) = \sum_{i=1}^{M} A_i(x)s_i(y) + yA_0(x) + a_0(x) \tag{15}$$

$$\varphi_M(x, y) = \sum_{i=1}^{M} B_i(x)c_i(y) + B_0(x) \tag{16}$$

The coefficients $A_i(x)$ and $B_i(x)$ may, in turn, be expanded by extended series as (12) and (13). In the previous equations the following notation was introduced:

$$\begin{aligned} \alpha_m &\equiv m\pi, \quad s_m(x) \equiv \sin \alpha_m x, \quad s_m(y) \equiv \sin \alpha_m y, \\ c_m(x) &\equiv \cos \alpha_m x, \quad c_m(y) \equiv \cos \alpha_m y \end{aligned} \tag{17}$$

(m-integer, $m > 0$, $m = i, j$). As may be observed, depending on the further chosen expansions (one of (12) or (13) series) of the functions $A_i(x)$, $A_0(x)$, $a_0(x)$, $B_i(x)$, $B_0(x)$ an infinite number of possible valid sequences arise. Evidently in expressions (15) or (16) x and y may be permuted. An extension to three-dimensional domains is analogous and was applied by Filipich et al. (1998).

The main feature of these sequences is to ensure the theoretical exactness of the results when dealing with eigenvalues and the UC of the solutions, and the aim is to find essential functions (displacement, slope function, and so on).

We mentioned before that the results can be calculated with arbitrary precision. The user selects the desired accuracy by choosing the required number of digits, and the number of terms in the summations is increased until those digits do not change further.

4. STATEMENT OF THE PROBLEM

Let us consider a uniform straight beam under loads which are variable in time and space, with nonlinear BC and arbitrary IC. Let $w = w(X, t)$ $(t \ge 0)$, $(0 \le X \le L)$ be the sought response (the transverse displacement). The governing equations in a non-dimensional domain ($D = \{0 \le x \le 1, 0 \le y \le 1\}$, $x = X/L$; $y = t/T$, T is an arbitrary time interval of interest), with $u = u(x, y) = w(xL, yT)$ are

$$u'''' + a^2\bar{\bar{u}} = q(x, y) \tag{18}$$

$$BC : \begin{cases} u(0, y) = u(1, y) = 0 \\ u''(0, y) - \beta_0 u'^3(0, y) = 0 \\ u''(1, y) + \beta_1 u'^3(1, y) = 0 \end{cases} \tag{19}$$

$$IC : \begin{cases} u(x, 0) = h(x) \\ \dot{u}(x, 0) = g(x) \end{cases} \tag{20}$$

where $a^2 = \rho F L^4/EJT^2$, $\beta_0 = \beta_0^*/LEJ$, $\beta_1 = \beta_1^*/LEJ$, β_0^* and β_1^* are the rotational constants of the Duffing-type end springs, $h(x)$ and $g(x)$ are known functions of x that satisfy the four BC (19) for $t = 0$.

4.1. WEM2 Alternative

WEM2 is not a direct variational method but a classical approach to solving the differential problem. All the derivatives involved in the differential equation, and initial and boundary conditions are expanded in series taking the form of (12) or (13). Let us represent u'''' by v''''_1 and $\bar{\bar{u}}$ by $\bar{\bar{v}}_2$. Now, v''''_1 is expanded (unknowns A_{ij}, A_{i0}, A_{0j}, etc) and v'''_1, v''_1, v'_1, v_1 are found by successive integrations. $\bar{\bar{v}}_2$ is also represented by this type of series (unknowns B_{ij}, B_{i0}, B_{0j}, etc) and $\bar{v}_2$, v_2 can again found by integration. The following consistence condition should stand

$$v_{1MN} = v_{2MN} \quad \forall x, y \tag{21}$$

Obviously the functions and derivatives should satisfy the differential equation as well as the IC and BC.

5. SOLUTION OF THE PROBLEM

Due to the complexity of the algebra involved in this problem, the solution of the differential problem presented in equations (18) to (20) will be separated into three parts, i.e., $u(x, y) = u_0(x, y) + u_1(x, y) + u_2(x, y)$. These are as follows:

I. $\boldsymbol{u_0(x, y)}$: Satisfies only the differential equation (18) (particular solution);
II. $\boldsymbol{u_1(x, y)}$: Satisfies only the homogeneous differential equation $u''''_1 + a^2\bar{\bar{u}}_1 = 0$ Additionally, $u_0 + u_1$ are required to fulfill the nonlinear BC (19). It is important to note that in order to state such conditions u'_2 should also be involved.
III. $\boldsymbol{u_2(x, y)}$: Satisfies the homogeneous differential equation $u''''_2 + a^2\bar{\bar{u}}_2 = 0$+BC corresponding to a simply supported beam, i.e., $u_2(0, y) = u_2(1, y) = u''_2(0, y) = u''_2(1, y) = 0$. Additionally, $u_0(x, 0) + u_1(x, 0) + u_2(x, 0) = h(x)$ and $\bar{u}_0(x, 0) + \bar{u}_1(x, 0) + \bar{u}_2(x, 0) = g(x)$, which are the IC (20).

The general solution will be

$$u(x, y) = u_0(x, y) + u_1(x, y) + u_2(x, y) \tag{22}$$

Each of the solutions can be stated as follows:

- Finding $u_0(x, y)$ is always possible and simple. In particular, let us assume a linearly varying load $q(x, y) = q_0 x \cos \omega T y$.
- For $u_1(x, y)$ we propose an extended sine series with UC as follows

$$u_1(x, y) = \sum_{k=1} \varphi_k(x) \sin k\pi y + \varphi_0(x)y + \gamma(x) \tag{23}$$

It is not difficult to find that each set of terms is

$$\begin{aligned}
\varphi_k(x) &= A_k \sin\sqrt{k\pi ax} + B_k \cos\sqrt{k\pi ax} + C_k \sinh\sqrt{k\pi ax} + D_k \cosh\sqrt{k\pi ax} \\
\varphi_0(x) &= a_1x^3 + b_1x^2 + c_1x^2 + d_1 \\
\gamma(x) &= a_0x^3 + b_0x^2 + c_0x^2 + d_0
\end{aligned} \tag{24}$$

where $a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, A_k, B_k, C_k, C_k$ are unknown constants. Thus $u_1(x, y)$ satisfies the differential equation with UC.

On the other hand $u_1(x, y)$ should, together with $u_0(x, y)$, satisfy the BC (20). Here the BC may be expressed as

$$\begin{aligned}
F_0(y) &\equiv \beta_0\Delta_0^3(y) \\
F_1(y) &\equiv -\beta_1\Delta_1^3(y)
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
\Delta_0(y) &\equiv u_0'(0, y) + u_1'(0, y) + u_2'(0, y) \\
\Delta_1(y) &\equiv u_0'(1, y) + u_1'(1, y) + u_2'(1, y)
\end{aligned} \tag{26}$$

Here trigonometric recurrence will be used to deal with the nonlinear functions $F_0(y)$ and $F_1(y)$. It will then be necessary to introduce $u_2(x, y)$ as follows:

- The following extended series is adopted to represent $u_2(x, y)$

$$u_2(x, y) = \sum_{j=1} \varepsilon_j(y) \sin j\pi x + x\varepsilon_0(y) + \sigma(y) \tag{27}$$

Finally $u_2(x, y)$ with UC results

$$u_2(x, y) = \sum_{j=1} \left(G_j \sin \frac{j^2\pi^2}{a} y + H_j \cos \frac{j^2\pi^2}{a} y \right) \sin j\pi x \tag{28}$$

It is true that $u_2(0, y) = 0$ and $u_2(1, y) = 0$. For the sake of brevity the proof of uniform convergence of this function and its derivatives $u''''_2(x, y)$ and $\bar{\bar{u}}_2(x, y)$ are not shown herein.

$$\begin{aligned} u_2(x, 0) &= \sum_{j=1} H_j \sin j\pi x = f(x) - u_0(x, 0) - \gamma(x) \\ \bar{u}_2(x, 0) &= \sum_{j=1} \frac{j^2\pi^2}{a} G_j \sin j\pi x = g(x) - \bar{u}_0(x, 0) - \sum_k (k\pi)\,\varphi_k - \varphi_0(x) \end{aligned} \tag{29}$$

6. SOLUTION FOR THE FORCED DUFFING OSCILLATOR

To assist understanding, the differential equation of the forced Duffing oscillator will now be tackled using the trigonometric recurrence method proposed. The motion $u = u(t)$ is governed by the following differential problem

$$\begin{cases} \ddot{u} + \alpha^* u + \beta^* u^3 = F_0^* \cos \omega t \\ u(0) = u_0; \quad \dot{u}(0) = v_0 \end{cases} \tag{30}$$

Let us introduce the interval of interest T $(0 \leq t \leq T)$ and $\tau = t/T$. Consequently $\overset{\bullet}{(\bullet)} = (\bullet)'\,\dot{\tau} = (\bullet)'/T$. The differential problem now becomes

$$\begin{cases} u'' + \alpha u + \beta u^3 = F_0 \cos(\omega T \tau) \\ u(0) = u_0; \quad u'(0) = T v_0 \end{cases} \tag{31}$$

where $\alpha \equiv \alpha^* T^2$, $\beta \equiv \beta^* T^2$ and $F_0 \equiv F_0^* T^2$. Now we introduce the following UC expansion for the n integer power of $u(\tau)$

$$u_1^n(\tau) = \sum_{i_1}^{M} B_{ni} \cos i\pi\tau + B_{n0} \tag{32}$$

where i_1 stands for $i = 1$. Let us introduce the next UC expansions to expand the first and second derivatives respectively,

$$u_2'(\tau) = \sum_{i_1}^{M} P_i \sin i\pi\tau + \tau P_0 + p_0$$

$$u_2''(\tau) = \sum_{i_1}^{M} i\pi\, P_i \cos i\pi\,\tau + P_0 \tag{33}$$

After integration of $u_2'(\tau)$ another expression for $u(\tau)$ may be obtained

$$u_2(\tau) = \sum_{i_1}^{M} \frac{P_i}{i\pi} \cos i\pi\,\tau + \frac{\tau^2}{2} P_0 + \tau\, p_0 + q_0 \tag{34}$$

which should satisfy the consistency condition $u_1^1(\tau) = u_2(\tau)$. This condition is expressed by the equality

$$\sum \left(B_{1i} + \frac{P_i}{i\pi} \right) c_i + (B_{10} - q_0) = P_0 \frac{\tau^2}{2} + p_0\tau \tag{35}$$

From the theory of uniform convergent series, two conditions derive from Eq. (35). They are

$$\begin{aligned} \frac{P_0}{6} + \frac{p_0}{2} &= B_{10} - q_0 \\ \frac{1}{2}\left(B_{1i} + \frac{P_i}{i\pi} \right) &= P_0 \frac{I_{2i}}{2} + p_0 \end{aligned} \tag{36}$$

$$\text{with } I_{mi} = \int_0^1 \tau^m c_i\, d\tau \tag{37}$$

The values of the $P_i's$ and P_0 are needed to state the differential equation. First let us state the initial conditions as follows

$$\text{from IC} \quad u'(0) = T v_0 \quad \Rightarrow \quad p_0 = T v_0 \tag{38}$$

$$\text{from IC} \quad u(0) = u_0 \quad \Rightarrow \quad \begin{cases} B_{10} + \sum_{i_1} B_{1i} = u_0 \\ \text{or} \\ q_0 + \sum_{i_1} \dfrac{P_i}{i\pi} = u_0 \end{cases} \tag{39}$$

Now the differential equation can be written in terms of the expansion as

$$\begin{aligned} (i\pi)\, P_i + \alpha B_{1i} + \beta B_{3i} &= Q_i \\ P_0 + \alpha B_{10} + \beta B_{30} &= Q_0 \end{aligned}$$

$$\text{where} \quad Q_i \equiv 2\int_0^1 \cos(\omega T\tau)\cos(i\pi\tau)\,d\tau \quad \text{and} \quad Q_0 \equiv \int_0^1 \cos(\omega T\tau)\,d\tau \tag{40}$$

The coefficients B_{20} and B_{30} may be found from equation (8) and B_{2i} and B_{3i} from equation (9). In order to solve the problem an iterative procedure is proposed:

1. First, a set B_{1i}^0 is given.
2. B_{10}^0 is found using one of equations (39).
3. Equations (8) and (9) are used to determine B_{30}^0, B_{3i}^0.
4. P_0^0 and P_i^0obtained from Equations (40).
5. Equation (37) yields a new set B_{1i}^1 and the process is repeated until convergence is achieved.

A numerical example was solved using this procedure and the results compared with those produced by using a standard Runge-Kutta integration scheme (the Fehlberg fourth-fifth order Runge-Kutta method) as implemented in Maplesoft. The results are included in Section 8, below.

7. APPLICATION OF THE TRIGONOMETRIC RECURRENCE TO THE PROBLEM STATED IN EQUATIONS (18) TO (20)

The next goal is to be able to write $\Delta_0^3(y)$ and $\Delta_1^3(y)$, and with them the functions $F_0(y)$ and $F_1(y)$ of equation (25), as a cosine series calculated by trigonometric recurrence (as described in Section 2). We start by writing the following expansions:

$$\begin{aligned} \Delta_0(y) &= \sum_{i=1} p_{1i}\cos i\pi y + p_{10} \\ \Delta_1(y) &= \sum_{i=1} q_{1i}\cos i\pi y + q_{10} \end{aligned} \tag{41}$$

Due to the space occupied by the expanded algebra, the details have been omitted. The following description of the algorithm gives the necessary information to proceed.

If solutions $u_0(x, y)$, $u_1(x, y)$ and $u_2(x, y)$ are known, the application of equations (26) and (3) allows us to find p_{10}, p_{1i}, q_{10} and q_{1i} by more or less long direct integration. But once found, it is elementary (using expressions (8) and (9) from section 2 and setting $n = 3$), to find p_{30}, p_{3i}, q_{30} and q_{3i}. Then

$$\begin{aligned} \Delta_0^3(y) &= \sum_{i=1} p_{3i}\cos i\pi y + p_{30} \\ \Delta_1^3(y) &= \sum_{i=1} q_{3i}\cos i\pi y + q_{30} \end{aligned} \tag{42}$$

Once the shapes of $\Delta_0^3(y)$ and $\Delta_1^3(y)$ are known, $F_0(y)$ and $F_1(y)$ can be obtained by means of equation (25). The calculations made using trigonometric recurrences show the potential of the idea.

8. NUMERICAL EXAMPLES

As explained, a Duffing oscillator was solved as a reference example. First Eq. (30) was numerically solved for $\alpha^* = -0.5$, $\beta^* = 0.5$, $F_0 = 7.5$, a forcing frequency $\omega = 0.5$ rad/s and initial conditions $u_0^* = 0$ $v_0^* = 0.5$ m/s. The time of the experiment was taken 35 s. A standard Runge Kutta integration scheme (Maplesoft) yielded the trajectory shown in Figure 1. A convergence study of the trajectories found using the trigonometric recurrence is depicted in Figure 2; it may be observed that as the number of terms increase the results converge to the ones of Figure 1. It should be noted that the value of M is related to the forcing frequency, as the number of waves in the proposed solution should be larger than in the oscillator response. The time step T used in the trigonometric recurrence algorithm is rather large (0.1 s) compared to the other standard integration schemes.

Another numerical illustration of the Duffing oscillator is included (Figure 3) in which a larger time is taken for the experiment (200 s). The results are shown as trajectory, projected phase plane and spatial phase plane found with both methods (i.e., the trigonometric recurrence and the RKF45), showing that the two techniques produce the same outcomes.

Now, the problem governed by partial differential equations will be addressed using two numerical examples to validate and illustrate the approach. A standard steel I beam 10 (European) was chosen ($F = 10.6cm^2$, $I = 171cm^4$) with a span $L = 8$ m. The beam is simply supported against vertical translation and elastically restrained against rotation with Duffing-type springs of $\beta_0 = \beta_1 = 0.5\ m^{-2}$. The calculations were performed for a total period of $1s$ that was subdivided into steps of $T = 0.1s$. The series for the algebraic recurrences were determined up to $M = 50$. In the first case the beam is subjected to a load with triangular variation in space and cosine variation in time, i.e.,

$$q(x,t) = \frac{q_0 X}{L}\cos\omega t \tag{43}$$

in which $q_0 = 100N/m$ and $\omega = 200\ rad/s$. The beam was released from the rest position ($h(x) \equiv 0$ and $g(x) \equiv 0$). Figure 4 depicts the trajectory of the beam at $x = 0.75$ (nondimensional location) and Figure 5 displays the corresponding phase diagram.

Figure 6 shows a comparison of the trajectory of Figure 4 and an analogous problem with linear springs. The solution for the latter was found by the modal superposition method.

The second example was carried out with the same beam, but with no load. The initial conditions were given by choosing a fifth degree polynomial for $h(x)$ (initial displacement that fulfills all the BC, including the nonlinear ones), and null velocity ($g(x) \equiv 0$). Such a function is given by

$$h(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex \tag{44}$$

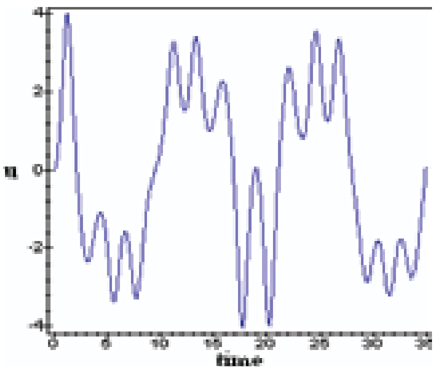

Figure 1. Trajectory of a Duffing oscillator. Time of experiment 35 s. $\alpha^* = -0.5$. $\beta^* = 0.5$. $F_0 = 7.5$. Forcing frequency $\omega = 0.5$ rad/s. $u_0^* = 0$. $v_0^* = 0.5$ m/s. Solution found with the Fehlberg fourth-fifth order Runge-Kutta method.

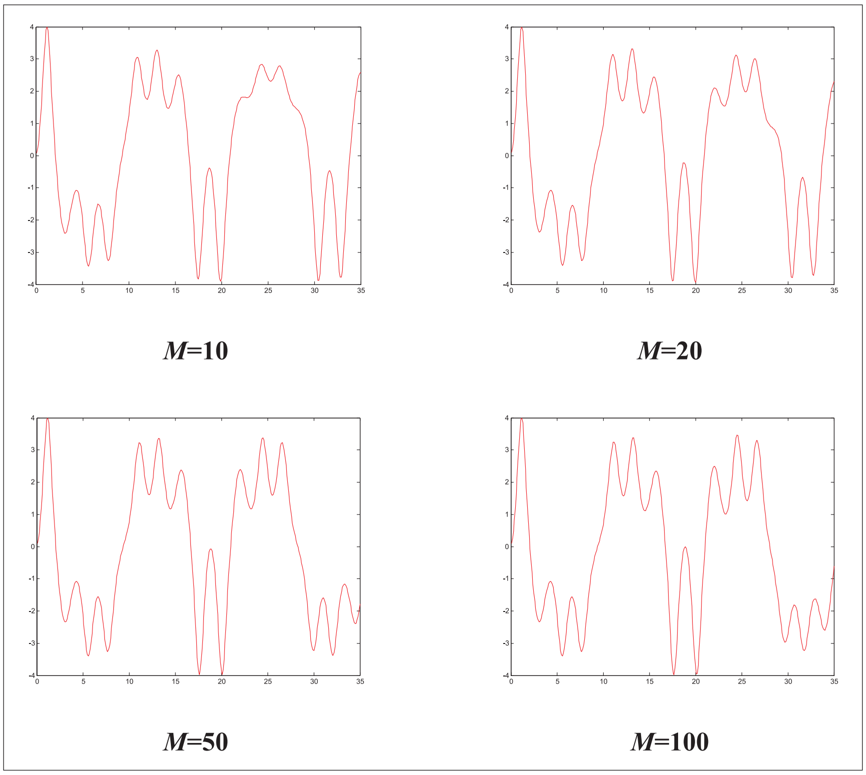

Figure 2. Trajectory of a Duffing oscillator (parameters as Figure 1). Convergence of the solution found with trigonometric recurrence with increment of the number of terms *M*. Time step $T = 0.1$.

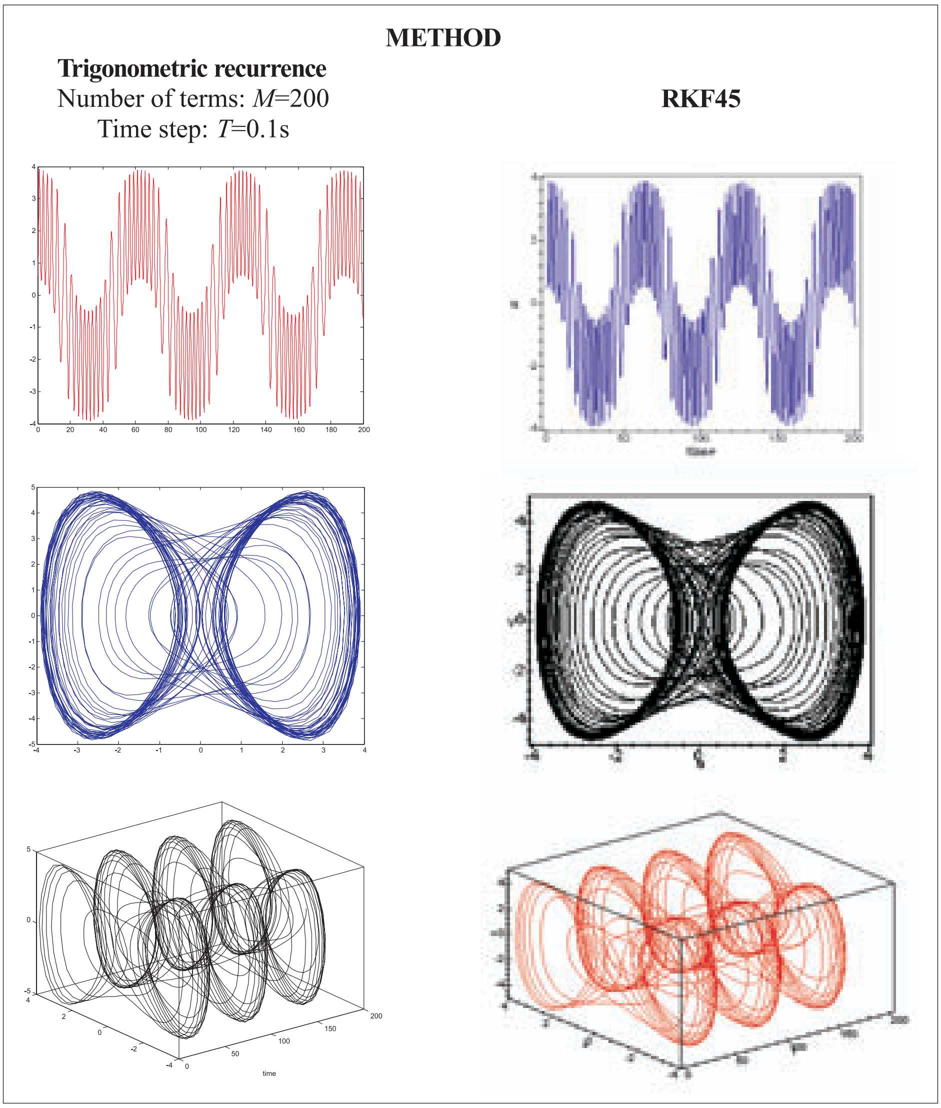

Figure 3. Trajectory, projected phase diagram and spatial phase diagram of a Duffing oscillator. Time of experiment 200 s. $\alpha^* = -0.5$. $\beta^* = 0.5$. $F_0 = 7.5$. Forced frequency $\omega = 0.1$ rad/s. $u_0^* = 0.5$ $v_0^* = 0.5$ m/s. Comparision between solutions found with trigonometric recurrence and RKF45 algorithm.

where the coefficients for the nonlinear Duffing-type springs are

$$\frac{a}{L} = -\frac{\beta_0\theta_0^3}{2} - 3\theta_0 - 3\theta_1 - \frac{\beta_1\theta_1^3}{2}; \quad \frac{b}{L} = \frac{3\beta_0\theta_0^3}{2} + 8\theta_0 + 7\theta_1 + \beta_1\theta_1^3;$$

$$\frac{c}{L} = -\frac{3\beta_0\theta_0^3}{2} - 6\theta_0 - 4\theta_1 - \frac{\beta_1\theta_1^3}{2}; \quad \frac{d}{L} = \frac{\beta_0\theta_0^3}{2}; \quad \frac{e}{L} = \theta_0$$

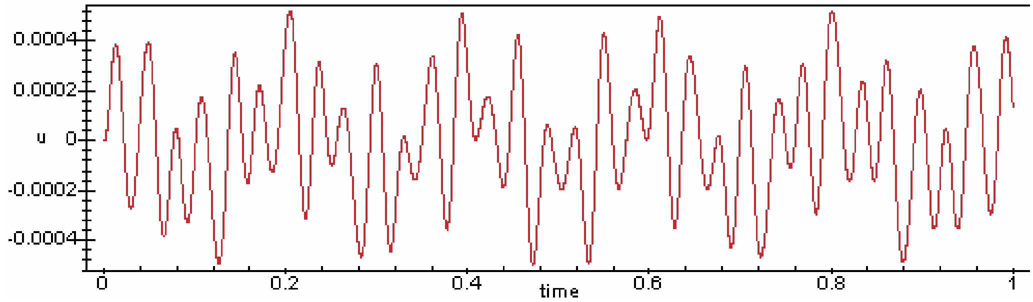

Figure 4. Beam subjected to a triangular load with $q_0 = 100N/m$ and $\omega = 200$ rad/s. End spring constants $\beta_0 = \beta_1 = 0.5$ m^{-2}. Trigonometric recurrence solution with $M = 50$, time step $T = 0.1s$. Time of experiment 1 s. Trajectory at $x = 0.75$.

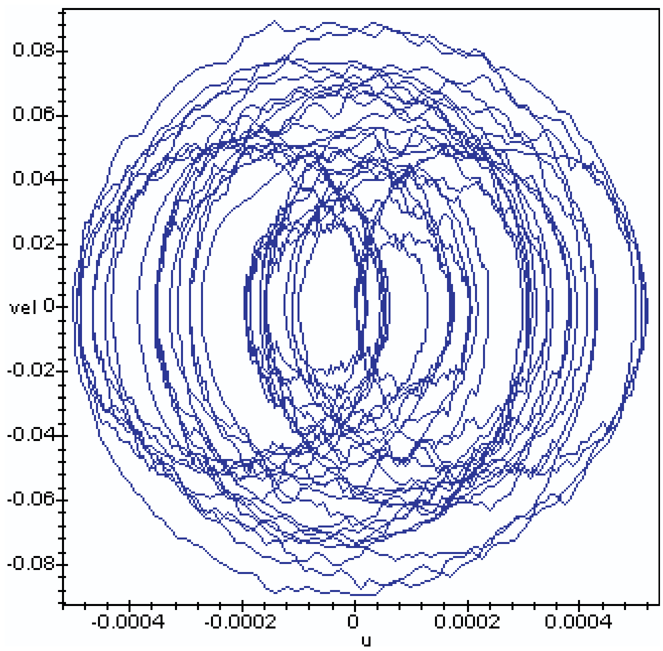

Figure 5. As Figure 4. Phase diagram at point $x = 0.75$.

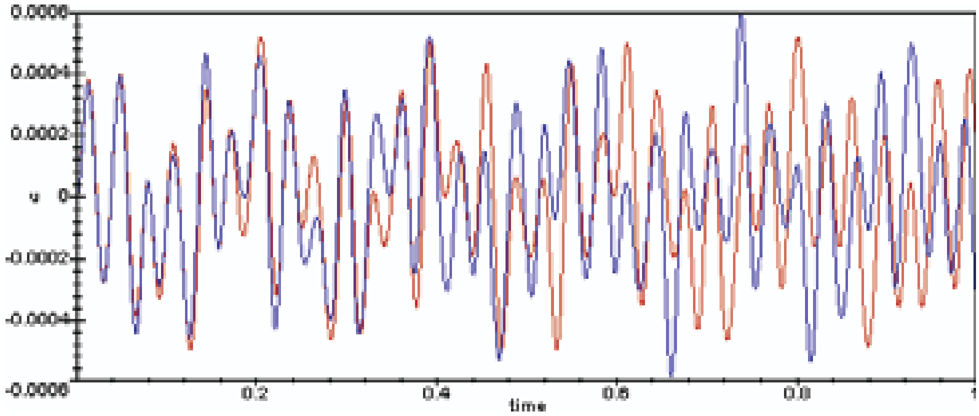

Figure 6. As Figure 4. Comparison of trajectories at point $x = 0.75$. (—) Nonlinear springs (trigonometric recurrence solution). (—) Linear springs (modal superposition solution).

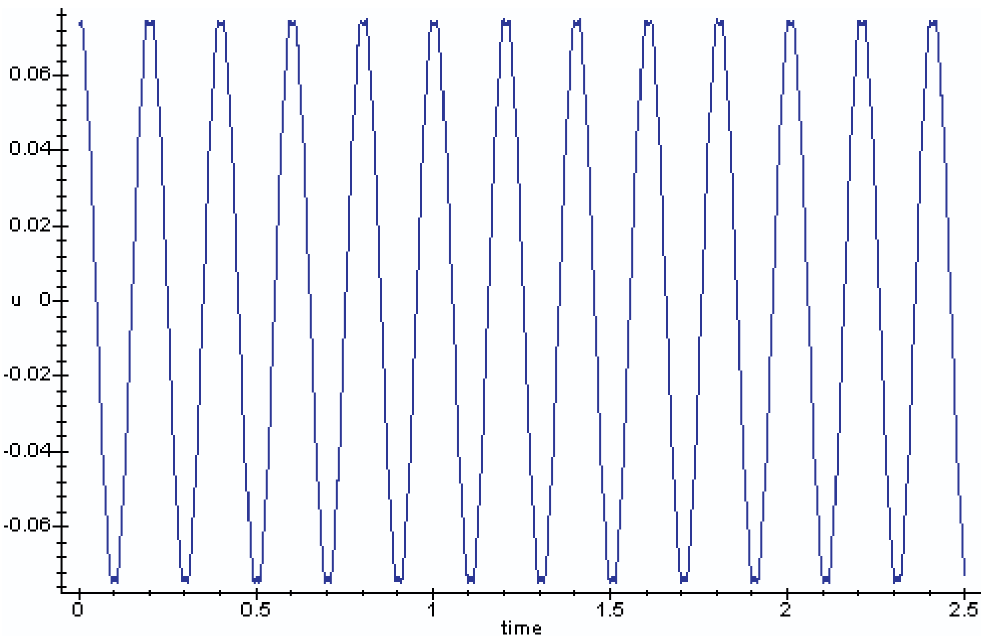

Figure 7. Beam subjected to an initial displacement $f(x)$ imposed as a 5$^{\text{th}}$. degree polynomial with $\theta_0 = 2^\circ; \theta_1 = -2^\circ$. End springs constants $\beta_0 = \beta_1 = 2000/L^2$. Trigonometric recurrence solution with $M = 200$. Time of experiment 2.5 s (One time step, $T = 2.5s$). Trajectory at $x = 0.75$.

The initial rotations at the ends of the beam were taken to be $\theta_0 = 2^\circ; \theta_1 = -2^\circ$. The springs' constants were taken as $\beta_0 = \beta_1 = 2000/L^2$. The time of the experiment was $2.5s$ with only one step, i.e., $T = 2.5s$. Again the displacements and velocities are measured at $x = 0.75$. The trajectory of the point located at $x = 0.75$ is shown in Figure 7.

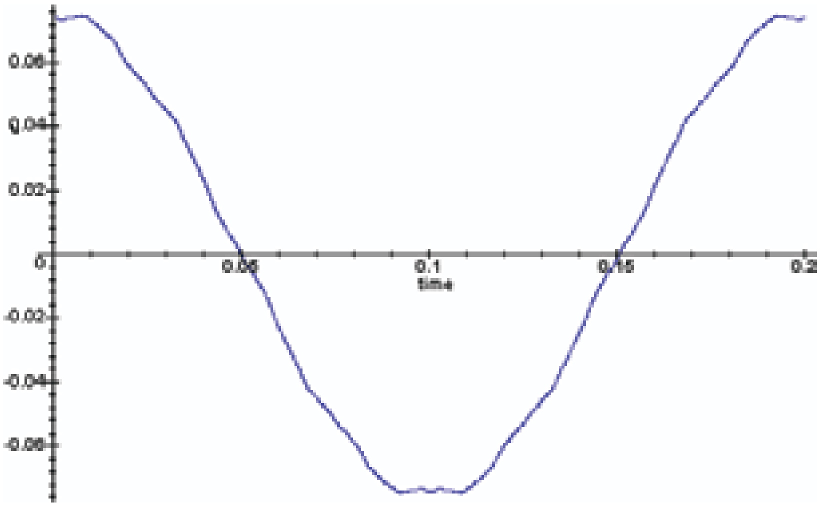

Figure 8. As Figure 7. Time domain [0,0.2s].

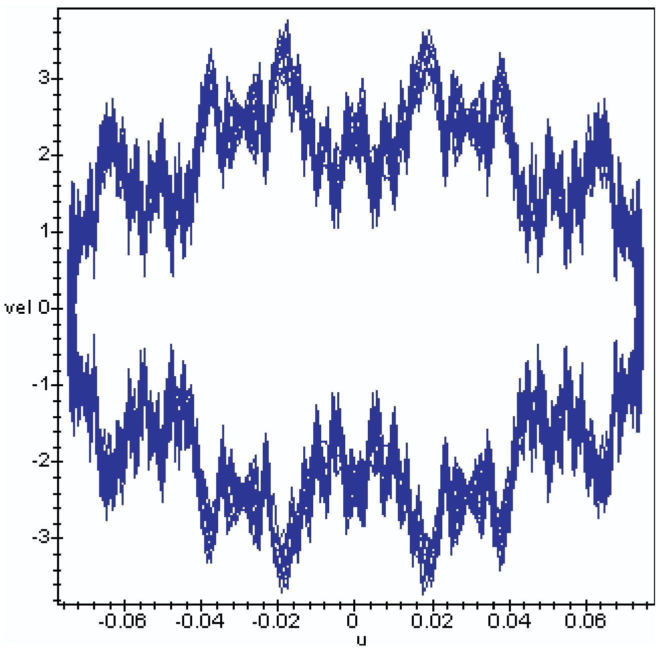

Figure 9. As Figure 7. Total time of experiment 2.5 s. Phase diagram at $x = 0.75$.

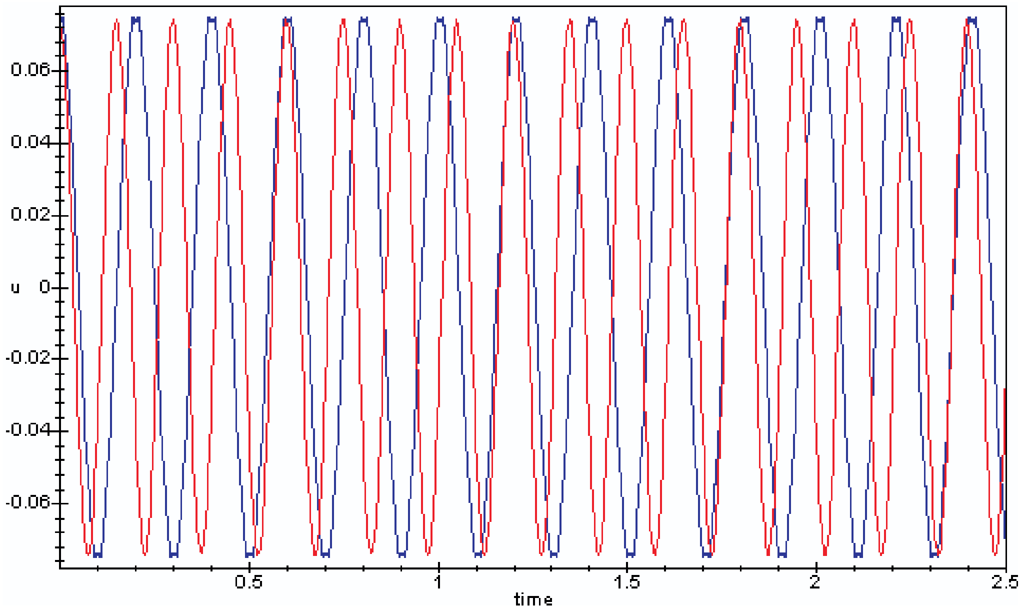

Figure 10. Comparison of trajectory of Figure 7 (—) with a one term cosine function with frequency ω = 41.978 *rad/s* (—).

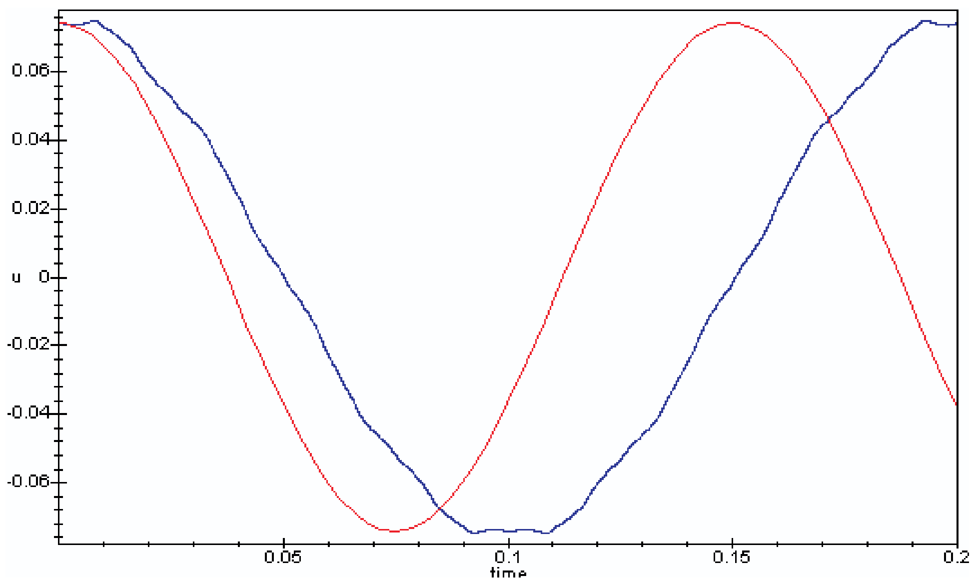

Figure 11. As Figure10. Time domain [0,0.2s].

An enlargement of this curve over a period of 0.2 *s* is shown in Figure 8, where its nonlinear behavior may be observed. The corresponding phase diagram is shown in Figure 9, and was found for a total time of 2.5 s.

Figures 10 and 11 show the comparison between the trajectory found with the present methodology and a one-term temporal cosine function with the frequency reported by Venkateswara Rao and Rajasekhara Naidu (1994). They solved a similar problem using a standard finite element technique after adopting harmonic oscillations (i.e., $w(x, t) = A(x)$ cos ωt). A nondimensional frequency of 175.4 for a cubic spring constant of $\beta_0 = 1000$ was reported. For our numerical example one obtains $\omega = 41.98$ rad/s. The authors of the present work found that the governing function was erroneously stated and consequently the real spring constant should be $\beta_0 = 2000$, as in this example. When a cosine solution with $\omega = 41.98$ rad/s is plotted against the nonlinear trajectory (with a compatible initial displacement function and without external force) solved with the present technique, a phase lag is evident, and obviously the cosine solution is smooth, unlike the trigonometric recurrence, which yields some "bumps" characteristics of the nonlinear behavior.

9. DISCUSSION AND CONCLUSIONS

A systematic methodology for the solution of initial-boundary value problems has been derived. First the well-known Duffing oscillator (initial conditions problem) was tackled and numerically solved with this technique for the sake of reference and comparison. Then an application to the dynamics of a beam with nonlinear Duffing-type end springs was briefly presented and some numerical results depicted in plots.

The problem is governed by a linear PDE with initial conditions and linear and nonlinear BC. The solution is based on the superposition of three functions, each of them satisfying a complementary problem. The novelty is the handling of the nonlinear function, which in this case is cubic. The technique of the so-called *trigonometric recurrence* appears very convenient for avoiding cumbersome nonlinear integrals. Almost every type of load may be addressed, as well as other types of nonlinearities, such as *sine, tan*, etc. as functions of the displacement or slope. The algorithm also handles arbitrary (though compatible) initial conditions. Other types of nonlinear boundary condition such as translational springs (alone or in combination with nonlinear rotational springs) could be considered. Nowadays resilient couplings are used in many applications in industry, such as main rolling mill drives, marine reverse gears and dredge pump drives, among others. They may be modelled as nonlinear springs since the high strength elastomer blocks are designed to provide nonlineal torsional stiffness curves.

The authors believe that other analytical techniques, such as the harmonic balance method, though simplifying the approach, could lead to strong approximations. On the other hand these approximations are adequate when displacements are not too large. The present solution stands for linear displacements of the beam and then it will be suited for end springs with relatively large constants.

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