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Full modeling of the mooring non-linearity in a two-dimensional floating structure

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Abstract

The dynamic behavior of a two-dimensional model of a small floating structure anchored by chains is analyzed. The structure is first modeled as a two-degrees-of-freedom oscillator with a strongly non-linear stiffness and subjected to a harmonic wave force. This type of structure is sometimes named Catenary Anchor Leg Mooring (CALM) system. The prescription of the vertical displacement leads to a simplified SDOF equation. An algebraic recurrence algorithm is employed to obtain a non-truncated differential equation that may be solved with the desired accuracy. Other authors have solved similar problems with approximate formulations of the geometric non-linearities. A numerical example is presented as an illustration. The time integration is carried out with a standard integration scheme and a power series approach. It is found that the response obtained after considering the strong non-linearity without previous truncations is qualitative different from the one found with a few terms of the expansions.

Keywords: CALM structure; Non-linear mooring; Floating structure; Non-linear differential equation; Power series

1. Introduction

Mooring structures are important to the oil and gas industry and to the river and sea navigation. In the first case, rather large systems are set as loading and off-loading terminals. Smaller-size buoy with catenary chains are employed as navigation aids (flashing lantern, radar reflector, foghorn, etc). They are sometimes referred to as Catenary Anchor Leg Mooring (CALM) system. Frequently, a number of four to eight anchor chains are moored to the seabed. As is known cable structures are load adaptive. These structures undergo a change in the geometry with the application of loads rather than a change in stress. This feature introduces strong nonlinearities in the system regardless of the elastic and linear properties of the material and the linearity of the load. The analysis should be able to handle this complexity and sometimes strong

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simplifications in the non-linear terms are made in order for the solution to converge. As the system is excited by sea waves the response may undergo qualitative changes in its behavior (bifurcations) under changes in the involved parameters.

A recent study by Esmailzadeh and Goodarzi [1] addresses a CALM system in a two-dimensional model of a buoy with mass m moored with two catenary chains. A differential equation of motion is derived where the nonlinearity was approximated yielding a cubic term. A similar rigid mass-slack mooring lines was studied in a two-dimensional domain by Sannasiraj et al. [2] to model pontoon-type floating breakwaters. Other effects were dealt with among others, by Sarkar and Taylor [3]. On the other hand, the statics of a three-component mooring line was analyzed by Smith and MacFarlane [4].

In this paper a CALM system is studied in the two-dimensional domain. The buoy is represented by a rigid mass m moored to the seabed with two catenary chains. The system is thus reduced to a mass–spring–damper oscillator with two degrees of freedom: heave and surge. The sea waves action is simulated with a horizontal harmonic force. Although this is a simplified model, special attention is focused on the nonlinearity that arises from the change of geometry of the chain. A power series approach allows to fully model such a nonlinearity without truncation. The non-linear differential equation is then solved with a systematic algebraic series technique stated before by the authors [5,6]. The resulting algorithm is illustrated with a numerical example. The authors believe that despite the simplicity of the model regarding the mathematical dimension, the hydrodynamics and other effects, the methodology introduced in the present work to address the mooring nonlinearity, may serve as a basis to be extended to other cases.

2. Statement of the problem

The two-dimensional model is described in Fig. 1, referring to a fixed coordinate system X, Y. Its main parts are the platform and the chains (or cables).

The chosen two D.O.F. are the horizontal and vertical displacement of the mass center of the rigid

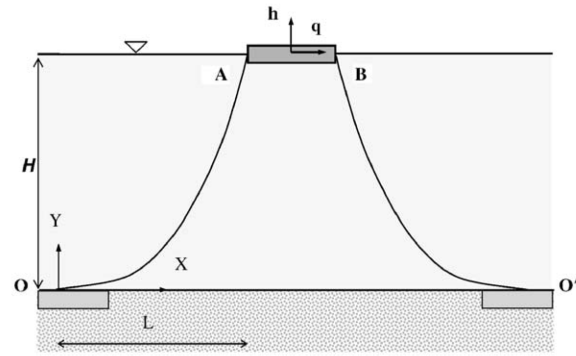


Fig. 1. Geometrical configuration of the two-dimensional CALM system.

platform, $q = q(t)$ and $h = h(t)$, respectively. Let us assume that the fluctuations of the sea water level h are governed by a harmonic function, i.e.

$$h(t) = h_0 \cos \omega t. \tag{1}$$

Thus, the equation that governs the horizontal displacement of the S.D.O.F. system is

$$m\ddot{q} + C_d\dot{q} + (S^l_{Ax} - S^r_{Bx}) = F(t), \tag{2}$$

where m is the platform mass, C_d is the damping, S^l_{Ax} and S^r_{Bx} are the horizontal components of the tension at the chains at points A (l: left) and B (r: right), respectively and $F(t)$ is a dynamic force (e.g. the wave's horizontal excitation). The dot denotes derivative w.r.t. time. Although Eq. (2) is an ordinary-looking ordinary differential equation, the terms between parentheses in the left-hand side are strongly non-linear. Furthermore, their explicit expressions in terms of the variables h and q are not easy to state. For instance, Esmailzadeh and Goodarzi [1] propose several truncations of Taylor expansions in order to be able to have an explicit but approximated Duffing-type equation. Instead, in this paper, a procedure to obtain a non-truncated equation will be proposed. Power series will be the main tool to handle the algebra. Finally, the differential equation will be written in a recurrence form. Next, the necessary algebra to handle the chain geometry is stated.

2.1. Catenary chains

Initially, the geometry of the chain is governed by the following non-linear relationship:

$$Y = \beta_0 \left[\cosh\left(\frac{X}{\beta_0}\right) - 1 \right], \tag{3}$$

where $\beta_0 = S_0/\rho$, S_0 is the tension at the point of null slope (O or O') of the chain and ρ is the weight per unit length of the chain. It is assumed, for the sake of simplicity, that the chain remains with a horizontal tangent at the anchor point. In particular, at point A

$$H = \beta_0 \left[\cosh\left(\frac{L}{\beta_0}\right) - 1 \right] \tag{4}$$

at the initial time $t = t_0$. After some time t, the point $A|_{t_0}$ (resp. $B|_{t_0}$) will move to a new position given by

$$H + h(t) = \beta^{l,r} \left[\cosh\left(\frac{L \pm q(t)}{\beta^{l,r}}\right) - 1 \right], \tag{5}$$

where $\beta = S_t/\rho$ and S_t is the horizontal component of the chain tension at time t; the plus sign corresponds to the left chain and the minus sign to the right one. Also let us introduce the notation

$$\theta^{l,r} = \frac{L \pm q(t)}{\beta^{l,r}}. \tag{6}$$

In order to calculate the horizontal component of the tension at time t, let us state its expression by means of Newton's law:

$$S^l_{Ax} = S^l_0 + m_c \ddot{x}^l_c, \tag{7a}$$

$$S^r_{Ax} = S^r_0 - m_c \ddot{x}^r_c, \tag{7b}$$

in which m_c is the chain mass and $\ddot{x}^{l,r}_c$ are the accelerations of the mass located at the center of gravity of each chain. It may be shown (see Appendix A for details) that the accelerations at the centroid of each of the chains are given by

$$\ddot{x}^{l,r}_c = \ddot{q} \mp \ddot{\beta}^{l,r} \left(\frac{H+h}{l^*}\right) \mp 2\dot{\beta}^{l,r} \frac{\dot{h}}{l^*} \mp \beta^{l,r} \left(\frac{\ddot{h}}{l^*}\right), \tag{8}$$

where l^* is the chain length.

2.2. Non-dimensionalization

Let us introduce the non-dimensional time variable

$$\tau = t/T \tag{9}$$

with T being an interval of interest (to be chosen at will). Now we denote the derivatives w.r.t. τ with bar, i.e. $\mathrm{d}(\bullet)/\mathrm{d}\tau = \overline{(\bullet)}$. Now the differential equation (2) is written as

$$m\bar{\bar{q}} + C_d T \bar{q} + T^2 (S^l_{Ax} - S^r_{Bx}) = F(\tau T) T^2. \tag{10}$$

The term related to the chain tension now yields

$$(S^l_{Ax} - S^r_{Bx}) = \frac{m_c}{T^2} (\bar{\bar{x}}^l_c + \bar{\bar{x}}^r_c). \tag{11}$$

3. Non-truncated expression for the chain tension

Our aim is to obtain a full and explicit expression of the type $\beta = \beta(q, h)$. In other words, this goal might be attained by means of several truncations of Taylor expansion. Here an alternative is chosen so as not to truncate the expressions in the derivation of the governing equations. At the stage of finding the numerical results and after the desired precision is set, naturally a practical truncation will be done. As will be seen, all the involved variables will be expanded in power series of q and h. From Eqs. (5) and (6) one may write

$$H + h = \beta^{l,r} [\cosh \theta^{l,r} - 1] \tag{12}$$

in which $\beta^{l,r}$ is implicit in $\theta^{l,r}$ and in turn the latter is the argument of a hyperbolic cosine. Furthermore,

$$\frac{H}{L} \left(\frac{1 + h/H}{1 \pm q/L} \right) = \frac{H+h}{L \pm q} = \frac{\cosh\theta - 1}{\theta} \equiv \hat{f}(\theta). \tag{13}$$

In what follows the superscripts l and r in θ will be suppressed and used only when necessary. As is known from the Taylor expansion of the hyperbolic cosine

$$\hat{f}(\theta) = \frac{\cosh\theta - 1}{\theta} = \sum_k \gamma_k \theta^k; \quad \text{with}$$

$$\gamma_k = \frac{1}{(k+1)!} (k = 1, 3, 5, \ldots). \tag{14}$$

Let us expand the leftmost term of Eq. (13) in a power series (see Appendix B) as follows:

$$\frac{H}{L}\left(\frac{1+h/H}{1\pm q/L}\right)=\sum_i\sum_j A_{ij}^{l,r}q^ih^j. \tag{15}$$

The values of coefficients $A_{ij}^{l,r}$ are known

$$A_{ij}^{l}=\begin{cases}A_{i0}^{l}=\dfrac{H}{L}\left(\dfrac{-1}{L}\right)^i,\\ A_{i1}^{l}=A_{i0}^{l}/H,\\ \end{cases}\quad A_{ij}^{r}=\begin{cases}A_{i0}^{r}=\dfrac{H}{L}\left(\dfrac{1}{L}\right)^i,\\ A_{i1}^{r}=A_{i0}^{r}/H,\end{cases}$$
$$\text{the rest are null.} \tag{16}$$

As is observed from Eq. (14) the following series of powers of the function θ are required:

$$\theta^k=\sum_i\sum_j R_{kij}q^ih^j. \tag{17}$$

Now if we combine Eqs. (14) and (17)

$$\hat{f}(\theta)=\hat{f}[\theta(q,h)]=f(q,h)=\sum_{k=1,3,\ldots}\gamma_k\sum_i\sum_j R_{kij}q^ih^j \tag{18}$$

the following next result is obtained:

$$f(q,h)=\sum_{i_0}\sum_{j_0}F_{ij}q^ih^j\Rightarrow F_{ij}=\sum_{k=1,3,5,\ldots}\gamma_kR_{kij}, \tag{19}$$

where $()_0$ indicates that the series start from $()=0$. From (13) and (15) one infers that the following equality should hold:

$$F_{ij}=A_{ij}. \tag{20}$$

Evidently,

$$\theta^{k+1}=\theta^k\theta. \tag{21}$$

This obvious statement allows finding a converging solution by means of a recurrence algorithm. Now, from the product of series definition

$$R_{(k+1)ij}=\sum_{n_0}^{i}\sum_{p_0}^{j}R_{knp}R_{1(i-n)(j-p)},$$
$$k=1,2,3,\ldots\ R_{0ij}=\delta_{0i}\delta_{0j}. \tag{22}$$

The δ_{ij} are the Kronecker's delta. After the expansion of (19), from (20)

$$\gamma_1R_{1ij}+\sum_{k=3,5,\ldots}\gamma_kR_{kij}=A_{ij}. \tag{23}$$

Then the R_{1ij}'s may be obtained from an iteration procedure that uses the following recurrent relationship:

$$R_{1ij}=\frac{A_{ij}-\sum_{k=3,5,\ldots}\gamma_kR_{kij}}{\gamma_1},\ \gamma_1=\frac{1}{2}, \tag{24}$$

and expression (22). Thus θ is fully determined for each value of q and h (see expansion (17)). Additionally, let us expand the function

$$\beta^{l,r}=\sum_i\sum_j B_{ij}^{l,r}q^ih^j, \tag{25}$$

where the $B_{ij}^{l,r}$ coefficients are unknowns at this stage and the aim of this section. Now, Eq. (6) may be written as

$$\theta^{l,r}\beta^{l,r}=L\pm q, \tag{26}$$

which in terms of Eqs. (17) and (25) reads

$$\left(\sum_i\sum_j R_{1ij}^{l,r}q^ih^j\right)\left(\sum_i\sum_j B_{ij}^{l,r}q^ih^j\right)=\sum_i\sum_j C_{ij}^{l,r}q^ih^j, \tag{27}$$

where from Eq. (26) $C_{ij}^{l,r}=\delta_{0i}\delta_{0j}L\pm\delta_{1i}\delta_{0j}$. (28)

The expansion of Eq. (27) is the way to find a recurrence relationship for the coefficients $B_{ij}^{l,r}$. The product of series yields

$$C_{ij}^{l,r}=\sum_{k_0}\sum_{n_0}R_{1kn}B_{(i-k)(j-n)}^{l,r}. \tag{29}$$

After rearranging the terms the following recurrence equation yields:

$$B_{ij}^{l,r} = \frac{C_{ij}^{l,r} - (Z_{1ij} + Z_{2ij} + Z_{3ij})}{R_{100}},$$

$$Z_{1ij} = \sum_{k_1}^{i} R_{1k0} B_{(i-k)j}^{l,r}, \quad Z_{2ij} = \sum_{s_1}^{j} R_{10s} B_{i(j-s)}^{l,r},$$

$$Z_{3ij} = \sum_{k_1}^{i} \sum_{n_1}^{j} R_{1kn} B_{(i-k)(j-n)}^{l,r}, \tag{30}$$

where k_1 indicates $k = 1$, etc. Finally, by means of Eq. (25), the function $\beta = \beta(q, h)$ (directly proportional to the chain tension S_t) is fully defined. Given the data of the problem its value is completely determined for any value of q and h. The algorithm steps to find the $B_{ij}^{l,r}$ may be summarized as follows:

- *Step 1*: Coefficients $A_{ij}^{l,r}$ are obtained from expressions (16).
- *Step* 2: Eqs. (24) and (22) allow for the calculation of the R_{kij}'s.
- *Step* 3: Coefficients $C_{ij}^{l,r}$ are found with Eq. (28).
- *Step* 4: Once the sums Z_{1ij}, Z_{2ij} and Z_{3ij} are calculated, the $B_{ij}^{l,r}$'s yield from expression (30).
- *Step* 5: The tension for each q and h may be found after obtaining β with (25). In this way we are able to obtain an explicit expression of β (proportional to the chain tension) which retains the nonlinearity under study (that generated by the change of geometry of the chain).

The convergence behavior of β is shown below in a table included in Section 5: Numerical Applications. The results exhibit an excellent rate of convergence.

4. Solution by means of time algebraic series

In this section the power series are used as a time integration tool. This tool has been used successfully by the authors previously to solve strongly non-linear differential equations. For more details of the fundamentals of the technique the interested reader may refer to Appendix B and Ref. [5,6]. Above its application allowed the statement of a non-truncated differential equation and at this stage finding its numerical solution with arbitrary precision. Numerical results will be found with this technique and also contrasted with a solution found using a Runge–Kutta integration scheme applied to the non-truncated non-linear differential equation stated with the above-introduced procedure.

Let us introduce the algebraic series in τ (re. $\tau = t/T$, T is an arbitrary time interval of interest) for the following time functions:

$$[q^k] = \sum_{i_0}^{M} Q_{ki}\tau^i, \quad [\beta^{l,r}] = \sum_{i_0}^{M} V_i^{l,r}\tau^i,$$

$$[h^k] = \sum_{i_0}^{M} h_{ki}\tau^i, \quad [F] = \sum_{i_0}^{M} G_i\tau^i,$$

$$[(H+h)\beta^{l,r}] = \sum_{i_0}^{M} U_i^{l,r}\tau^i. \tag{31}$$

Note that in this case the unknowns are the Q_{ki}'s and the coefficients h_{1i} may be found from (1) by means of the expansion of the cosine function with a Taylor series in τ. Also the coefficients G_i will be known once the excitation force is introduced. The fulfillment of the consistence condition (Appendix B) yields

$$Q_{ki} = \sum_{p_0}^{i} Q_{(k-1)p} Q_{1(i-p)},$$

$$h_{ki} = \sum_{p_0}^{i} h_{(k-1)p} h_{1(i-p)}. \tag{32}$$

Eq. (25) combined with Eq. (31) and (32) yields

$$\begin{aligned}[\beta^{l,r}] &= \sum_{i_0}\sum_{j_0} B_{ij}^{l,r} \sum_{k_0} Q_{ik}\tau^k \sum_{n_0} h_{jn}\tau^n \\ &= \sum_{i_0}\sum_{j_0}\sum_{k_0}\sum_{n_0} B_{ij}^{l,r} Q_{ik} h_{jn}\tau^{k+n} \\ &= \sum_{p_0} V_p^{l,r}\tau^p, \quad k+n=p\end{aligned} \tag{33}$$

and

$$V_i^{l,r} = \sum_{I_0}\sum_{J_0} B_{IJ}^{l,r} \sum_{p_0}^{i} Q_{Ip} h_{J(I-p)},$$

$$U_i^{l,r} = \sum_{p_0}^{i} V_p^{l,r}[\delta_{0(i-p)}H + h_{1(i-p)}]. \tag{34}$$

Finally, the differential equation (10) may be rewritten as a function of the algebraic series coefficients, i.e.,

$$m_{\text{eq}}\varphi_{2i}Q_{1(i+2)} + C_d T\varphi_{1i}Q_{1(i+1)} + \frac{m_{\text{c}}}{l^*}\varphi_{2i}[U^r_{i+2} - U^l_{i+2}] = T^2 G_i \tag{35}$$

in which $\varphi_{1i} = i+1$; $\varphi_{2i} = \varphi_{1i}(i+2)$. The recurrence equation to find the coefficients $Q_{1(i+2)}$ yields

$$Q_{1(i+2)} = \frac{1}{m_{\text{eq}}}\left\{\frac{T^2 G_i}{\varphi_{2i}} - \left[\frac{C_d T Q_{1(i+1)}}{(i+2)} + \frac{m_{\text{c}}}{l^*}[U^r_{i+2} - U^l_{i+2}]\right]\right\}. \tag{36}$$

Once these coefficients are found, the other Q_{ki}'s may be obtained from (32). Thus the values of q are determined for each t. The results are of arbitrary precision. That is, the number of exact digits is fixed and then the number of terms in the series is increased until these digits remain unchanged.

5. Numerical applications

The performance of the recurrence algorithm is illustrated by a numerical example. A rigid two-dimensional platform is modeled elastically supported with catenary mooring as two chains or cables. Referring to Fig. 1 the assumed values are $L = 40\,\text{m}$, $H = 20\,\text{m}$, $h_0 = 1.5\,\text{m}$, $C_d = 100\,\text{N/m}$, $m = 1000\,\text{kg}$, $\omega = 0.25\,\text{rad/s}$, $F_0 = 0.0005\,\text{H}m_{\text{eq}}\omega^2$, $\rho = 50\,\text{N/m}$. Here $m_{\text{eq}} = m + 2m_{\text{c}}$ and m_{c} is the chain mass.

In Section 3 an algorithm with power series was constructed to find the values of β (proportional to the cable tension) for each position of the rigid platform (i.e. given h and q). Table 1 shows the convergence behavior of that algorithm with $q = 0.1\,\text{m}$ and $h = 0.1\,\text{m}$. It may be observed that it exhibits an excellent convergence behavior.

Table 1
Convergence study of values of β found with the algorithm of Section 3

M	β	M	β
5	42.9852849824673	13	42.9864402630189
6	42.9852849824680	14	42.9864402631089
7	42.9864291286825	15	42.9864402630202
8	42.9864291286825	16	42.9864402630202
9	42.9864401897501	17	42.9864402630202
10	42.9864401897501	18	42.9864402630202
11	42.9864402626703	19	42.9864402630202
12	42.9864402626703	20	42.9864402630202

$q = 0.1\,\text{m}$, $h = 0.1$ m. M is the number of terms of the series and β is proportional to the chain tension (Eq. (25)).

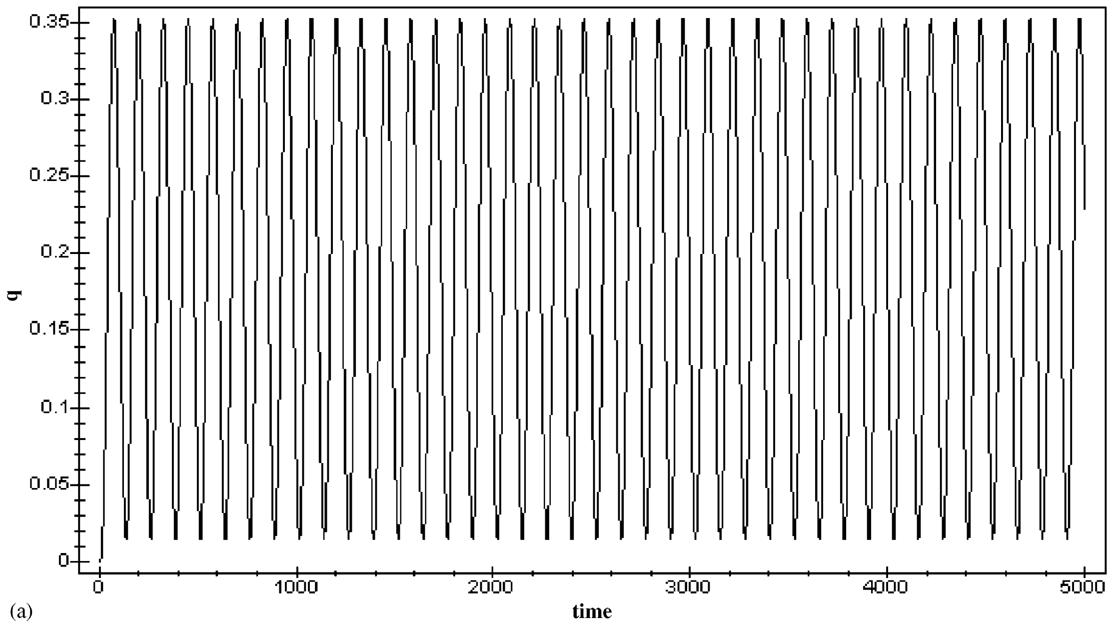
On the other hand, a characteristic parameter for the system can be found after a linearization of the previous algorithm. In effect, if one assumes $h = 0$ and $q = 1$, the values of $\beta_i = 42.025$ and $\beta_i = 38.025$ are obtained (linear approximation). The difference between these values multiplied by the unit weight yields the approximated stiffness $k = 200\,\text{N/m}$ and furthermore a representative magnitude is $\omega_0 = \sqrt{k/m} = 0.369\,\text{rad/s}$ and $f_0 = 0.059\,\text{Hz}$. This is the fundamental (linear) natural frequency.

The exciting horizontal force is assumed as $F(t) = F_0 \sin \omega_f t$ and ω_f is the parameter chosen for this study. A wide range was studied and some distinctive results are shown. Previously it was found that 20 terms in the summations suffice to attain convergent values. For the sake of simplicity the same number of terms were taken to solve both the first algorithm (finding the coefficients of the series that are involved in the function β) and the time integration algorithm. In all the examples, the arbitrary interval of interest (equivalent to a time step) was taken to be $T = 5\,\text{s}$. The motion was investigated in the first 5000 s (time of the experiment).

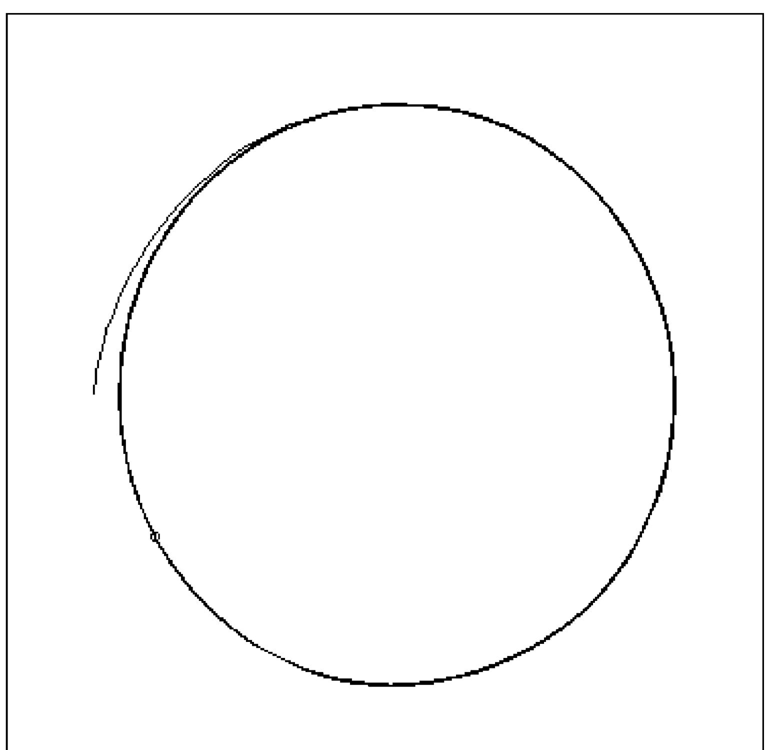
The system was released from rest. For the exciting frequency $\omega_f = 0.2\,\text{rad/s}$ the motion of the horizontal variable $q(t)$ is depicted in several plots, say trajectory (Fig. 2a), phase diagram (Fig. 2b) and the Fourier spectrum in Fig. 2c. Additionally, the $q - h$ trajectory is shown in Fig. 2d. Fig. 2b also shows the Poincaré map which reduces to a point denoting the period-1 behavior.

In Fig. 3 the variation of the trajectories as the forced frequency varies is shown. Although not reported, the other characteristic studies indicate the existence of a periodic attractor. From the reported results and others not shown here, a more tangled trajectory is observed when the frequency is a multiple of the natural one.

Finally the results found with forcing frequency $\omega_f = 1.344$ (a fractional value of the natural frequency) are shown in Fig. 4.

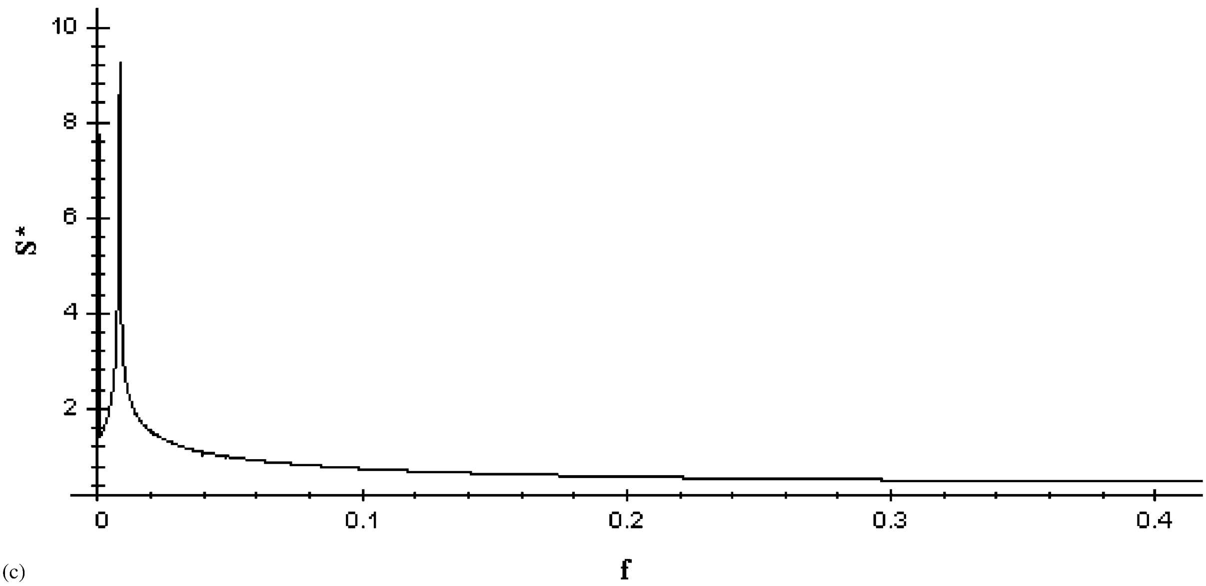


(a)

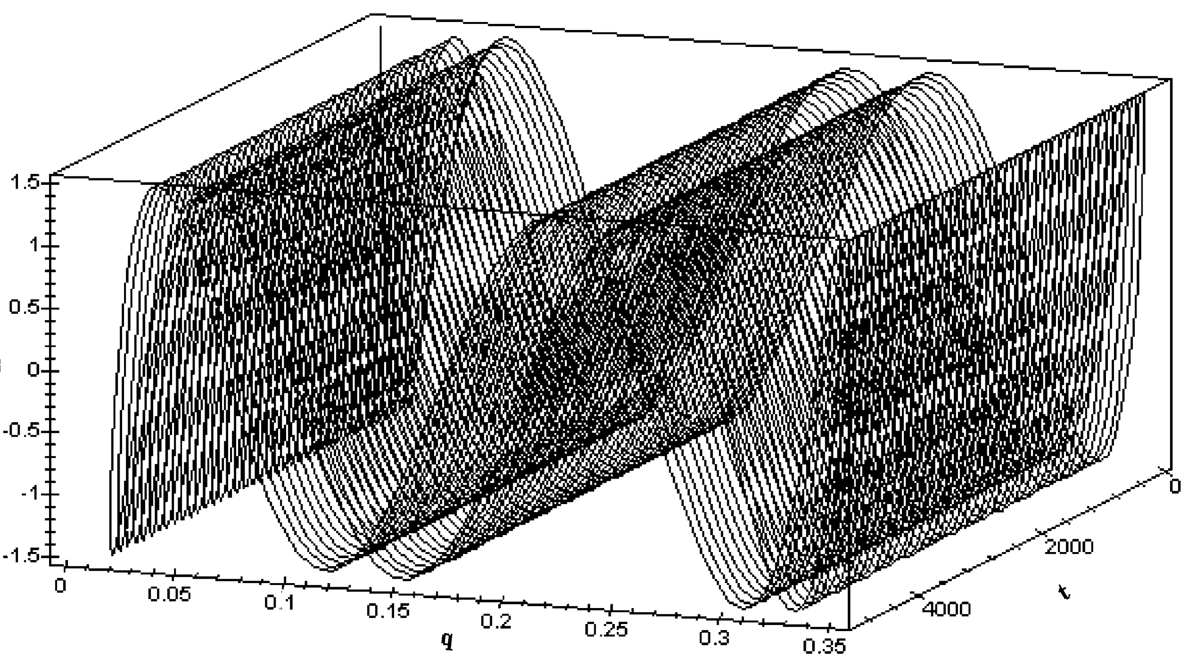


(b)

Fig. 2. Dynamic behavior of the platform with exciting frequency $\omega_f = 0.2\,\mathrm{rad/s}$. (a) $q(t)$ trajectory; (b) phase diagram with Poincaré point; (c) Power spectrum; and (d) spatial trajectory $q - h - t$. q and h are in meters, and time in seconds.



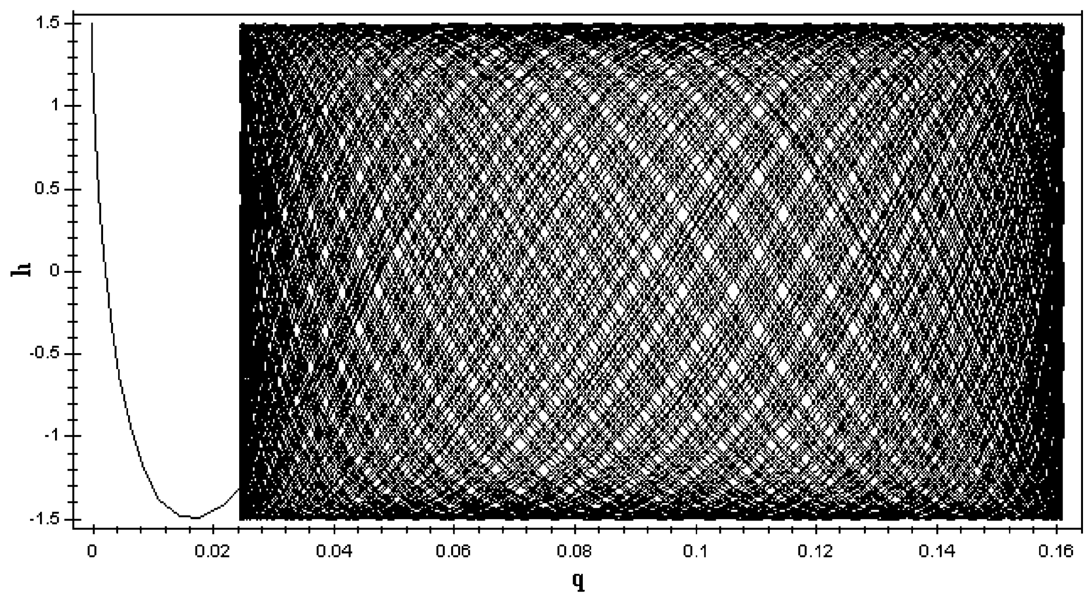
(c)



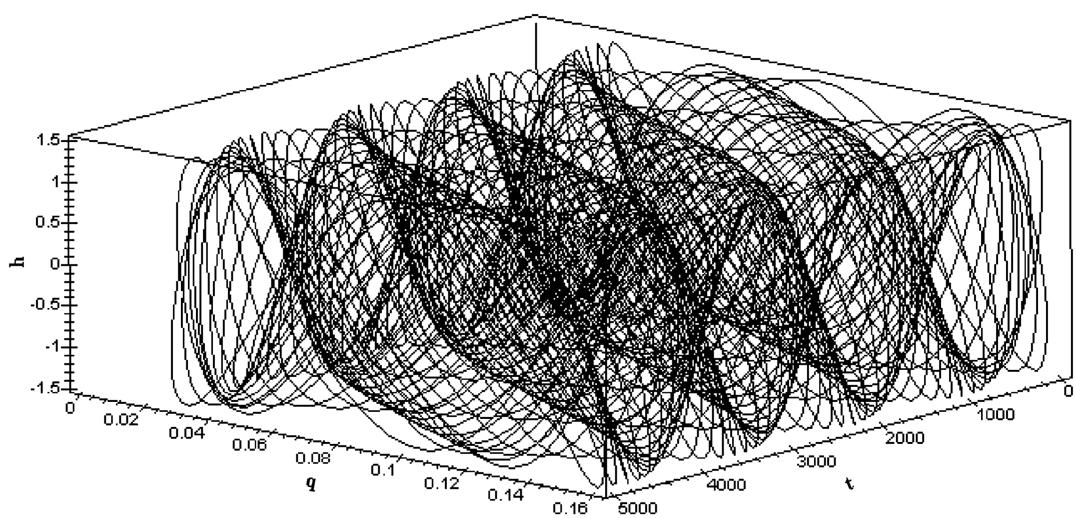
(d)

Fig. 2. (*continued*).

Also, the same ordinary equation was integrated using a standard Runge–Kutta (R–K) integration scheme. The respective phase diagram is shown in Fig. 5. That is, Figs. 4b and 5 represent solutions of the same ODE with the non-linear term stated with the algorithm proposed in this work (in this particular numerical example, the function β was expanded in 20 terms). The second stage, i.e. the time integration



Trajectory $q-h$

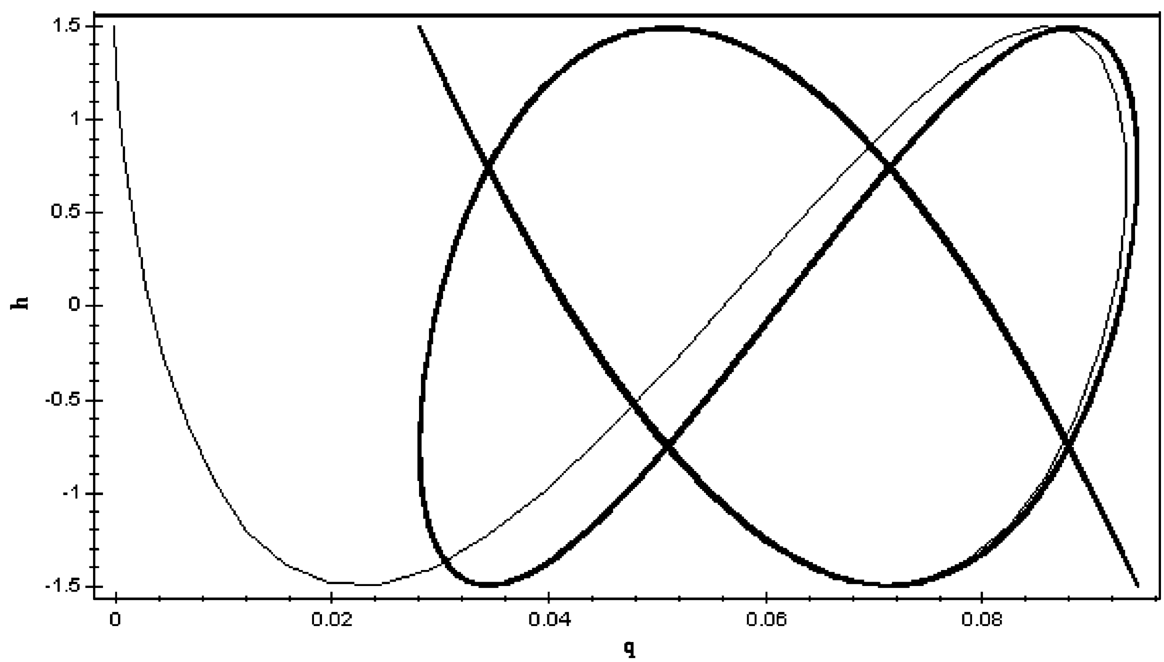


(a) Trajectory $q-h-t$

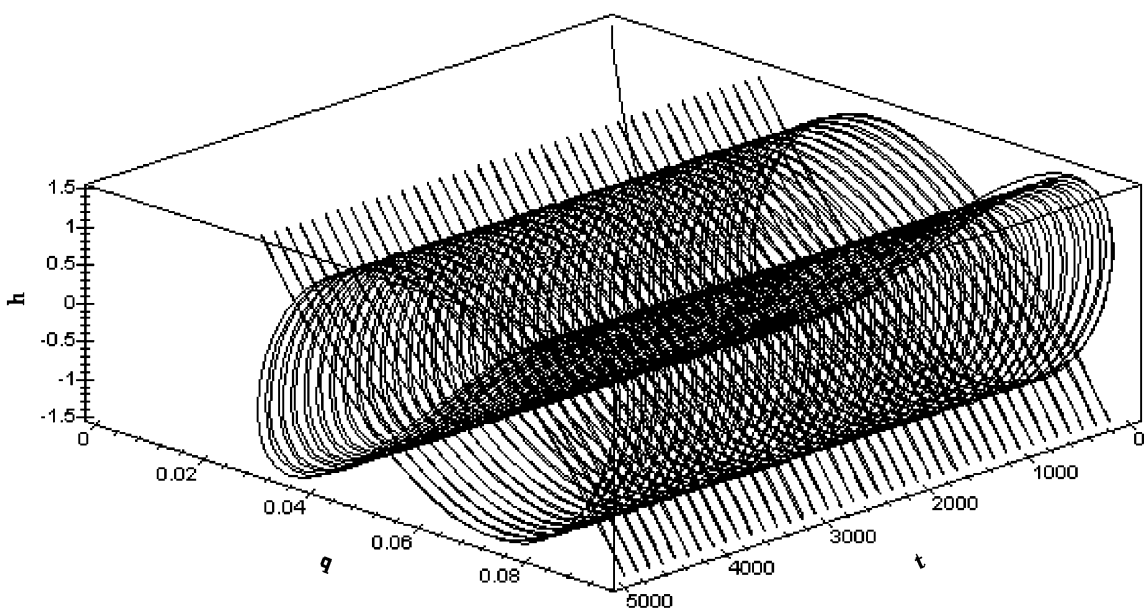
Fig. 3. Trajectories $q-h$ found with different forcing frequencies. (natural frequency $\omega_0 = 0.396\,\mathrm{rad/s}$). q and h are in meters, and time is in seconds. (a) $\omega_f = 0.936\,\mathrm{rad/s}$; (b) $\omega_f = 0.6\,\mathrm{rad/s}$; (c) $\omega_f = 1\,\mathrm{rad/s}$; (d) $\omega_f = 3 \times 0.396\,\mathrm{rad/s}$; and (e) $\omega_f = 2\,\mathrm{rad/s}$.

scheme, is carried out with the power series algorithm in the case of Fig. 4 and with an R–K routine in Fig. 5. The graphs are identical which would allow to infer the excellence of the proposed time integration technique.

Finally, it should be noted that, as mentioned before, the system was reduced to a simple oscillation with a non-linear spring and a mass. In this sense the motions are theoretically unlimited. However, the physical existence of the chain imposes a geometrical



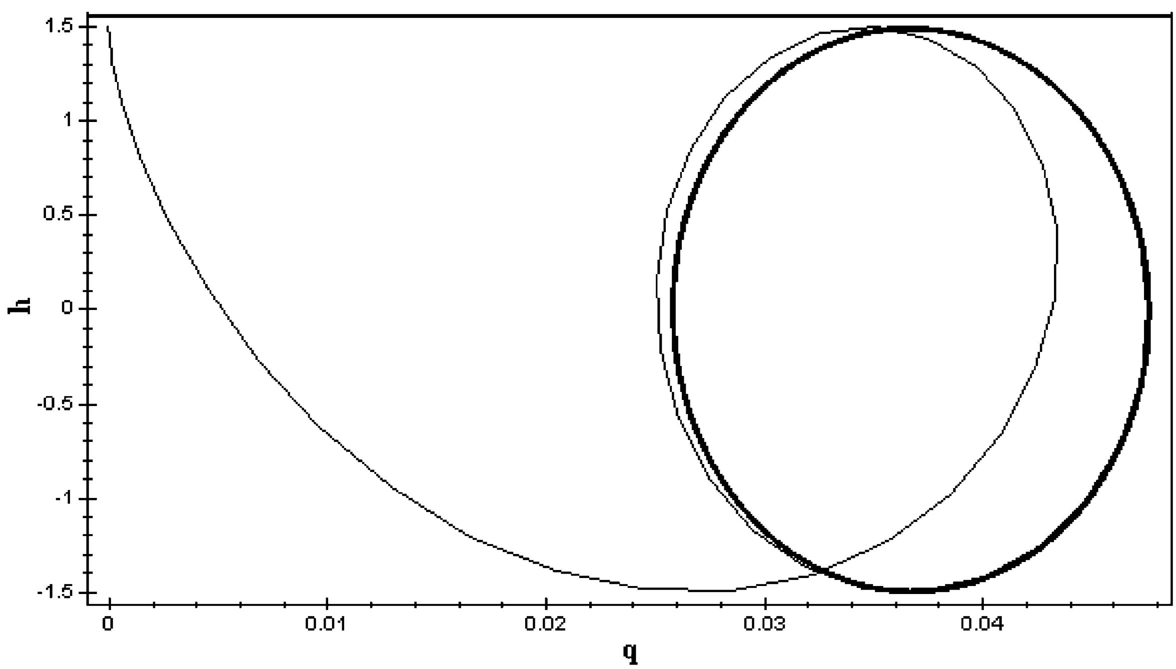
Trajectory $q-h$



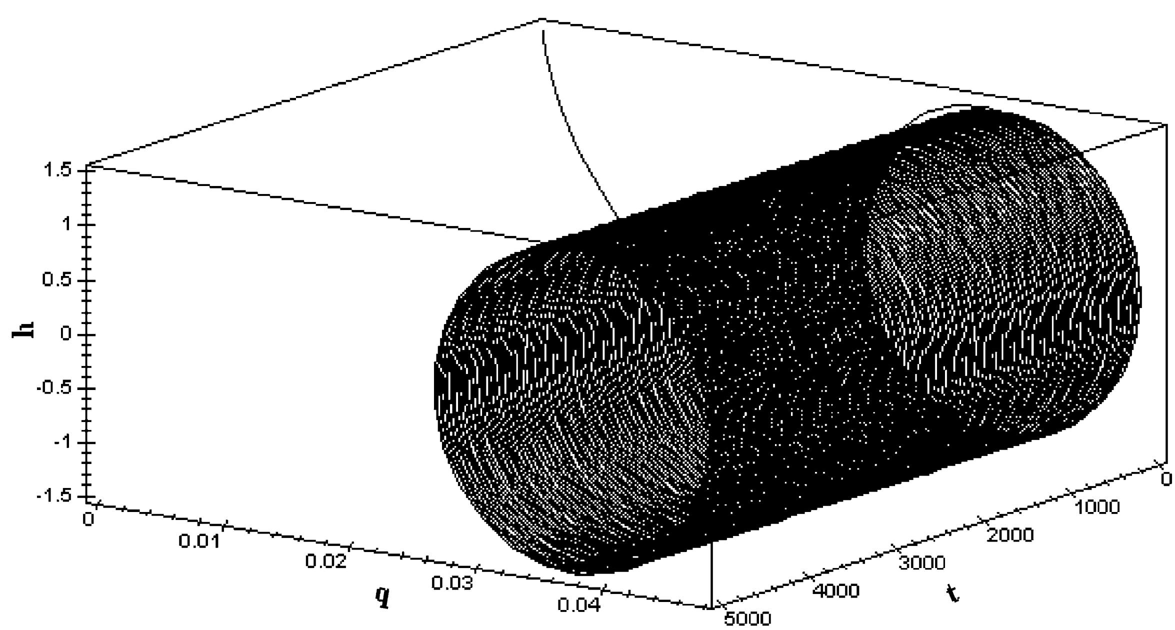
(b)

Trajectory $q-h-t$

Fig. 3. (*continued*).

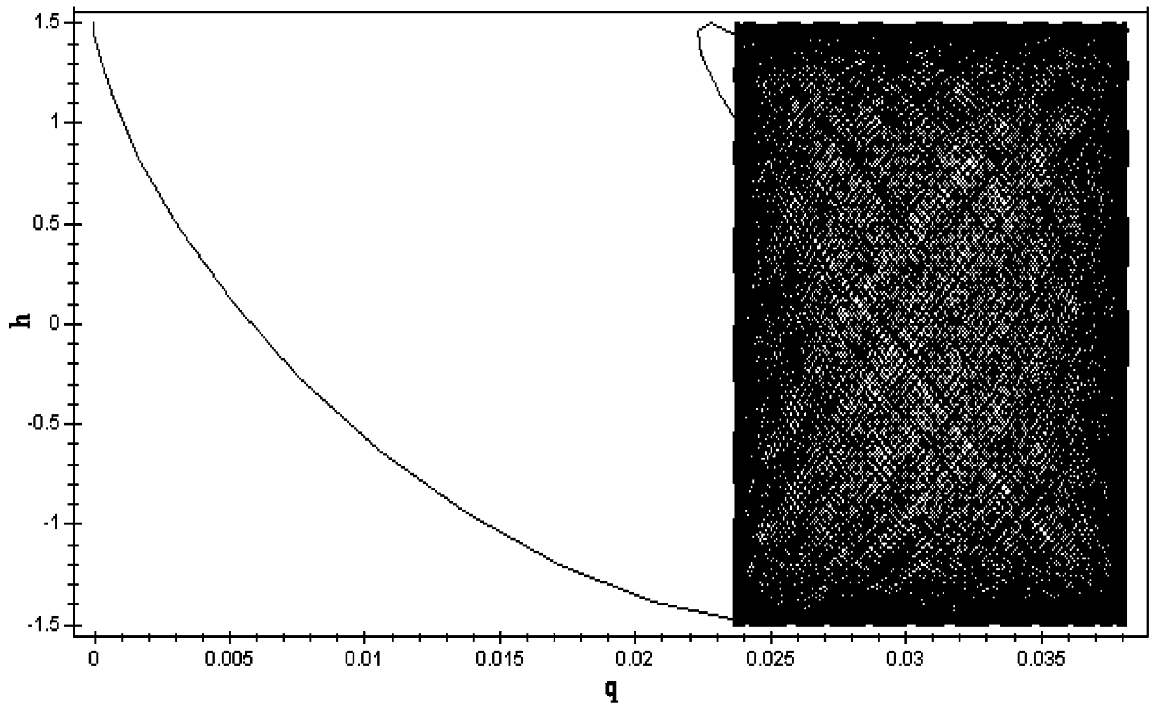


Trajectory $q-h$

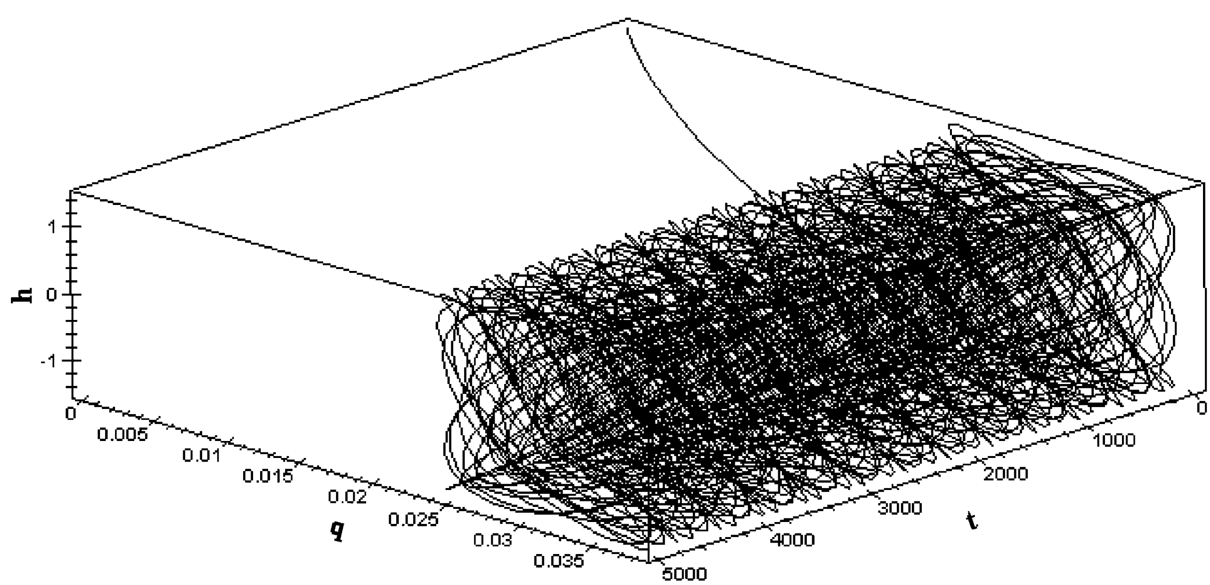


(c) Trajectory $q-h-t$

Fig. 3. (*continued*).

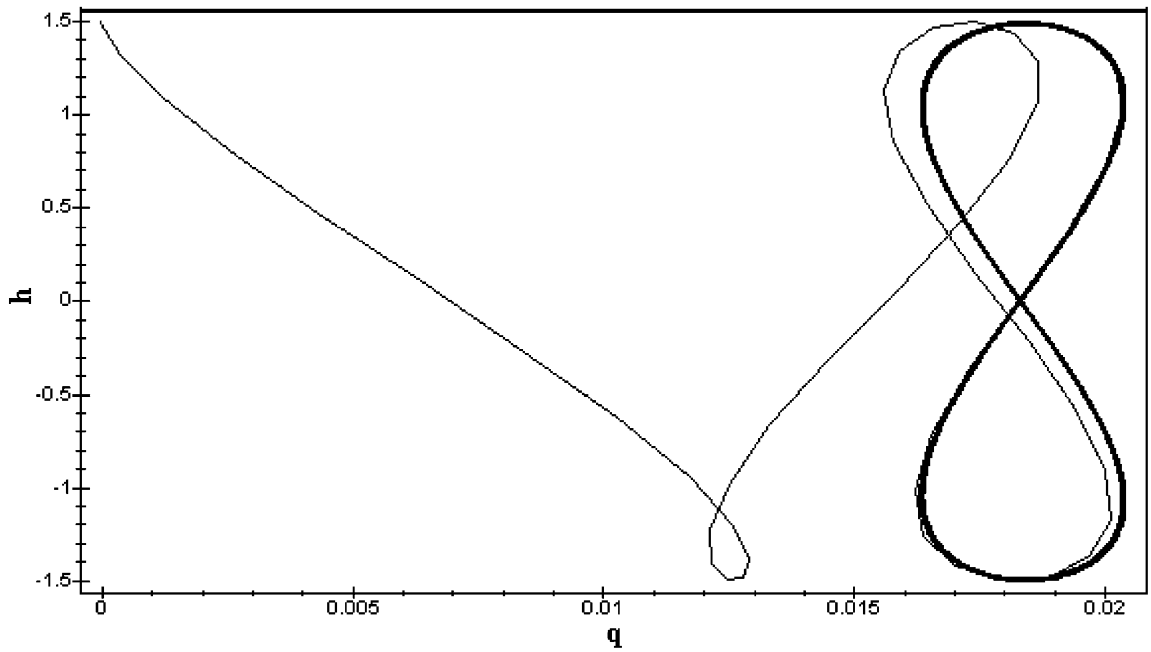


Trajectory $q-h$

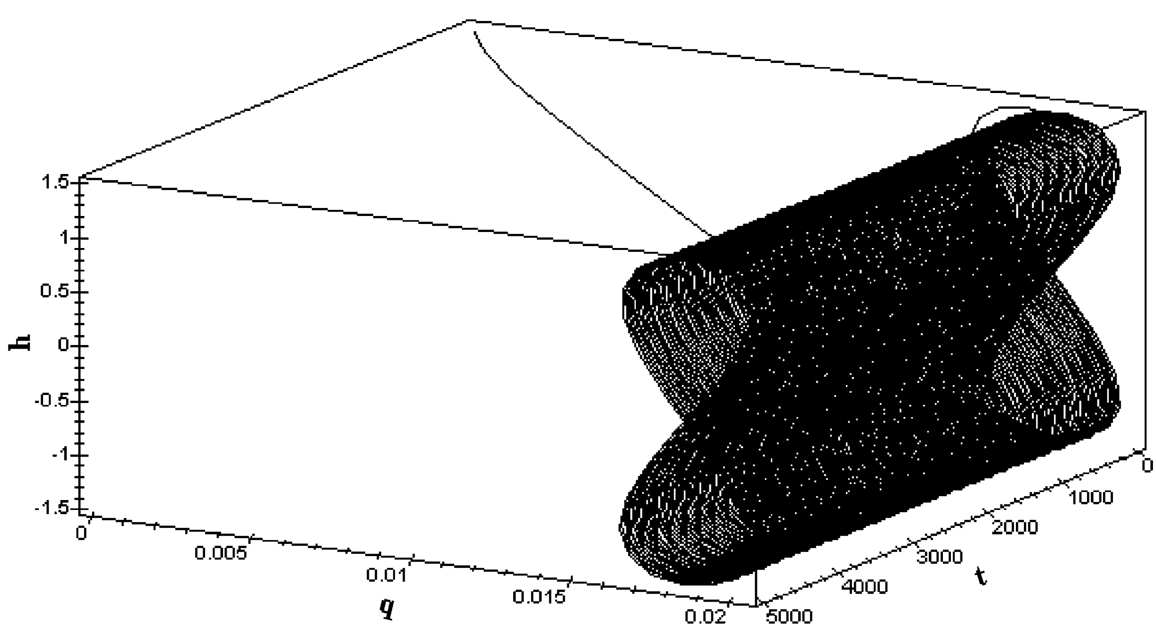


(d) Trajectory $q-h-t$

Fig. 3. (*continued*).

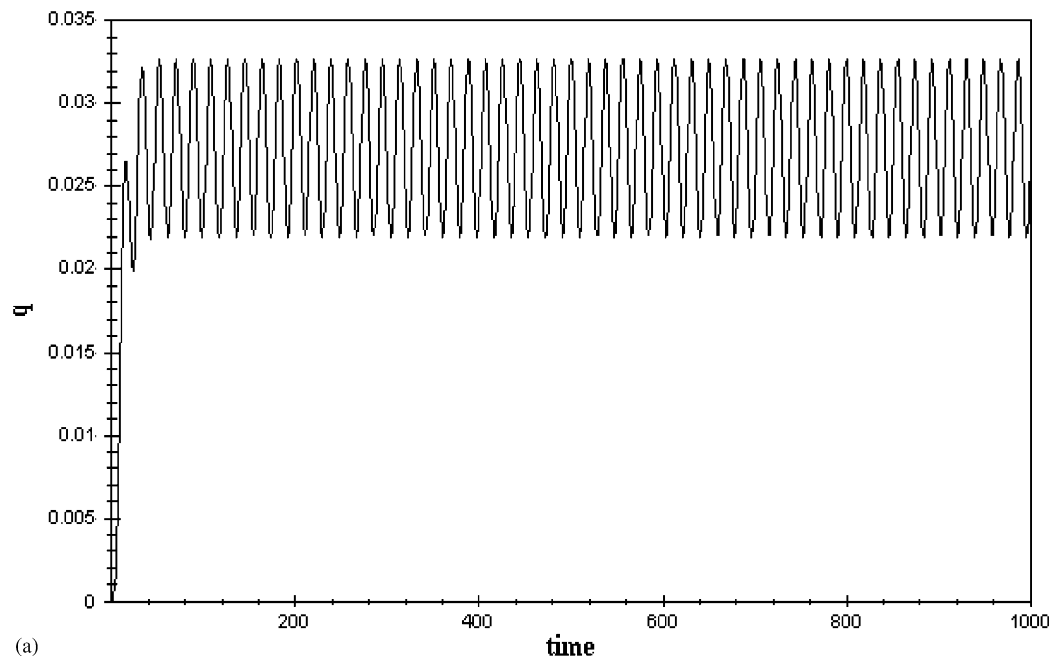


Trajectory $q-h$

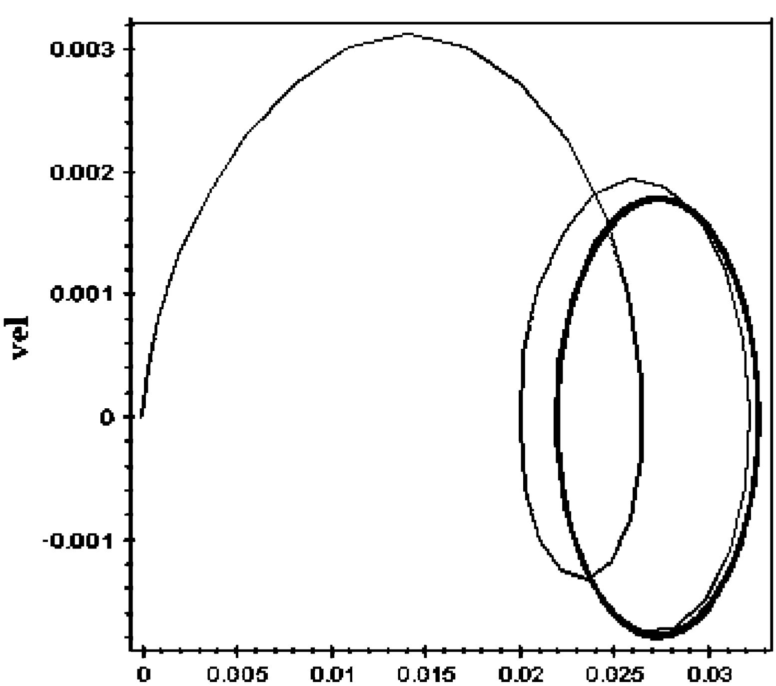


(e) Trajectory $q-h-t$

Fig. 3. (*continued*).

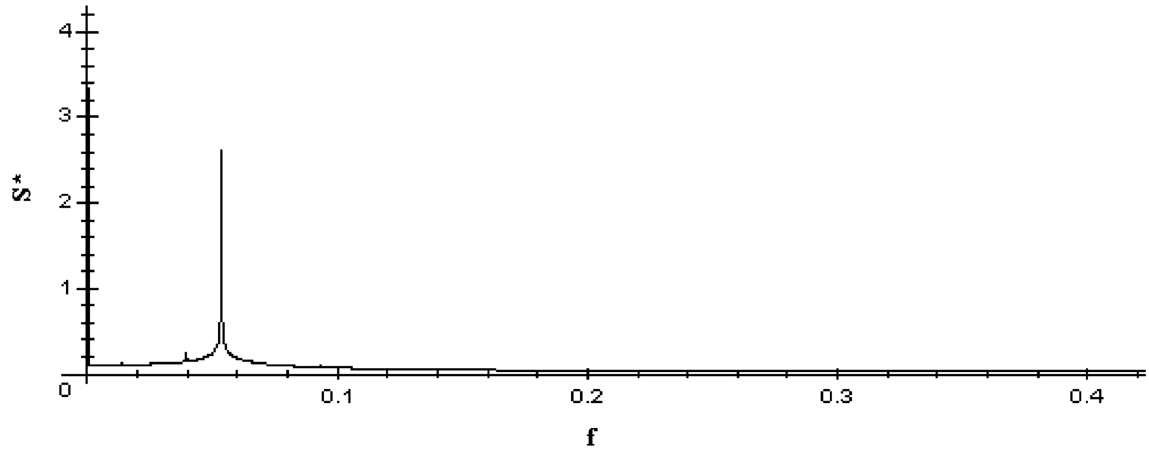


(a)

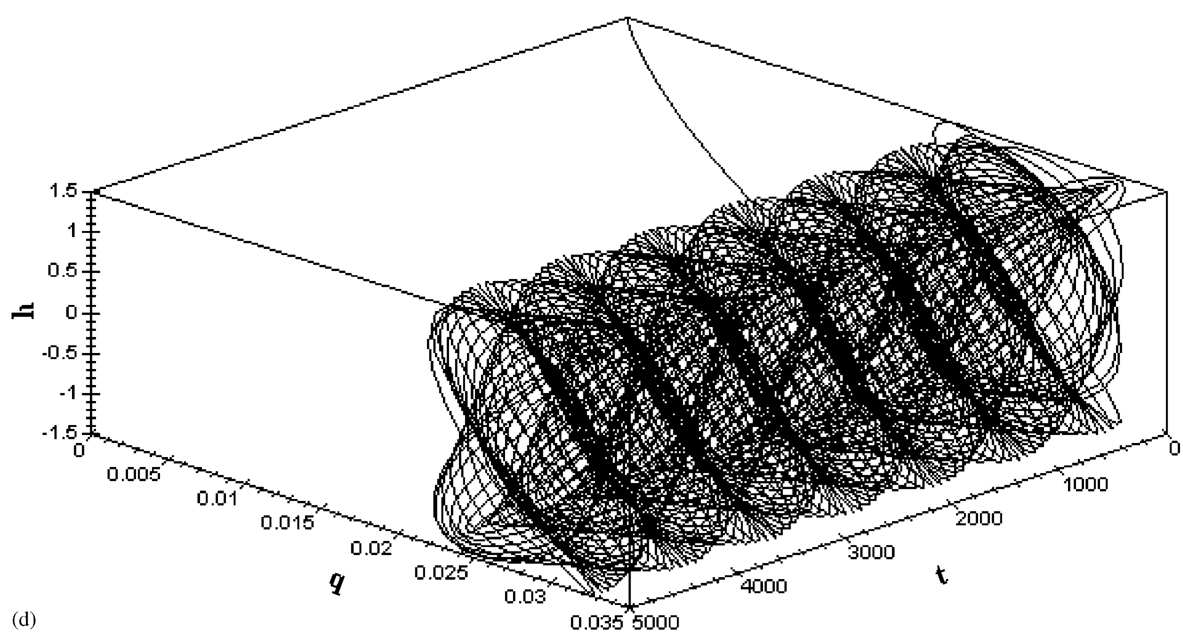


(b)

Fig. 4. Dynamic behavior of the platform with exciting frequency $\omega_f = 1.344\,\mathrm{rad/s}$: (a) $q(t)$ trajectory; (b) phase diagram; (c) Power spectrum; (d) spatial trajectory $q - h - t$. q and h are in meters, and time is in seconds.



(c)



(d)

Fig. 4. (*continued*).

restriction that must not be surpassed in order for the present model to be valid. Consequently, a limit value of q exists, for each value of H and L, considering that the vertical motion h fluctuates between -1.5 and 1.5 m. If this value is exceeded other effects should be included, say the tight behavior of the chain. In this example $H = 20$ and $L = 40$ m the limit value for q is approximately 0.7 m and in all the numerical examples presented here this requirement was fulfilled. In other cases a piecewise non-linear stiffness should be considered (see Raghothama and Narayanan [7]). This limitation was not taken into account in Ref. [1].

6. Final comments

The analysis of a two-dimensional model of a small floating structure with chain mooring was carried out

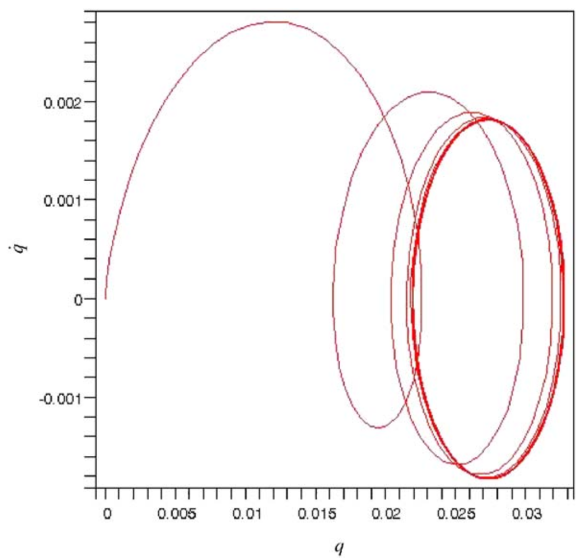


Fig. 5. Phase diagram found with non-linear ODE solved with a standard Runge–Kutta scheme $\omega_f = 1.344\,\text{rad/s}$. Displacement q is in meters, and velocity $\dot{q}$ is in m/s.

with the systematic use of algebraic series. The structure was modeled as a rigid mass with a strongly nonlinear spring and a damper and the vertical displacement was assumed to be coupled with an assumed harmonic motion of the sea surface. Also a harmonic forcing load was applied horizontally. The nonlinearity, which arises from the chain response, is resolved with an algebraic recurrence algorithm. This tool allows obtaining a non-truncated formulation unlike the usual approach of handling a Duffing-type nonlinearity. The present approach leads to qualitatively different responses due to the algorithm developed. Then a comparison would lack significance. At a second stage the algebraic series are also employed as a time integration technique, which had proved successful in previous studies. Numerical simulations were carried out taking the forcing frequency ω_f as a varying parameter. The solutions found with the parameter ranging between 0.2 and 1.5 rad/s are periodic unlike the ones reported in [1] where truncation is used to derive the non-linear equation. Future studies can take into account a three-dimensional model, rotational degrees of freedom, modeling of fluid forces, etc. together with the full modeling of the mooring nonlinearity shown above.

Appendix A.

The accelerations of the mass center of the chains (Eq. (8) in Section 2) may be found as follows: the horizontal coordinate of the mass center of the left (superscript l) and right (superscript r) chains may be written, respectively, as

$$x_c^l = (L+q) - \beta^l \left(\frac{H+h}{l^*}\right), \tag{A.1}$$

$$X_c^r = (L-q) - \beta^r \left(\frac{H+h}{l^*}\right), \tag{A.2}$$

where x_c^l is measured from the left (point O in Fig. 1) and X_c^R is taken from point O'. If the mass center of the right chain is measured from the left (i.e. from point O) its position is

$$x_c^r = (L+q) + \beta^r \left(\frac{H+h}{l^*}\right) + 2z \tag{A.3}$$

in which z is the width of the rigid platform. Applying the derivation twice to expressions (A.1) and (A.2) the acceleration of the mass center of the chains is obtained

$$\ddot{x}_c^{l,r} = \ddot{q} \mp \ddot{\beta}^{l,r} \left(\frac{H+h}{l^*}\right) \mp 2\dot{\beta}^{l,r} \frac{\dot{h}}{l^*} \mp \beta^{l,r} \left(\frac{\ddot{h}}{l^*}\right),$$

which is reported as Eq. (8) in Section 2.

Appendix B.

In this section, well-known aspects of the power series approach are stated. Let us consider a continuous function $x = x(\tau)$ with $0 \leqslant \tau \leqslant 1$. We will denote the expansion in algebraic series as

$$[x] \equiv \sum_{k=0}^{N} a_{1k}\tau^k \tag{B.1}$$

and for powers m

$$[x^m] \equiv \sum_{k=0}^{N} a_{mk}\tau^k (m = 1, 2, \ldots). \tag{B.2}$$

In order to fulfill an *consistence condition* the following relationships have to be satisfied:

$$[x^m] = [x^{m-1}][x]. \tag{B.3}$$

After replacing the series expressions in each factor of this equation, one obtains the following recurrence expressions (Cauchy products):

$$a_{mk} = \sum_{p=0}^{k} a_{(m-1)p} a_{1(k-p)} \quad \text{or}$$

$$a_{mk} = \sum_{p=0}^{k} a_{(m-1)(k-p)} a_{1p}. \tag{B.4}$$

Now let us expand an analytical function $f = \hat{f}(x) = \hat{f}(x(\tau)) = f(\tau)$ in Taylor series

$$\hat{f}(x) = \sum_{m=0}^{M} \alpha_m x^m, \tag{B.5}$$

where α_m are known and, in particular, we denote

$$[1] = \sum_{k=0}^{N} \delta_{0k} \tau^k, \tag{B.6}$$

where $a_{0k} = \delta_{0k}$ and δ_{0k} are the Kronecker delta's. If we substitute Eq. (B.2) into Eq. (B.5) we may write

$$[f(\tau)] = \sum_{k=0}^{N} \varphi_k \tau^k, \quad \varphi_k = \sum_{m=0}^{M} \alpha_m a_{mk}. \tag{B.7}$$

This expression will be used for any analytical function. Now if we have to deal with a rational function $\hat{F}(x)$:

$$\hat{F}(x) = \frac{\hat{g}(x)}{\hat{f}(x)} = \frac{g(\tau)}{f(\tau)} = F(\tau), \tag{B.8}$$

$\hat{g}(x)$ and $\hat{f}(x)$ being analytical functions and $\hat{f}(0) \neq 0$ and $\hat{g}(x) = \sum_{m=0}^{M} \beta_m x^m$ and β_m are known. Then it is possible to write

$$[g(\tau)] = \sum_{k=0}^{N} \varepsilon_k \tau^k \text{(a)} \Rightarrow \varepsilon_k = \sum_{m=0}^{M} \beta_m a_{mk} \text{ (b)}. \tag{B.9}$$

If we denote

$$[F(\tau)] = \sum_{k=0}^{N} \lambda_k \tau^k. \tag{B.10}$$

Now the consistence condition must be applied:

$$[F(\tau)][f(\tau)] = [g(\tau)], \tag{B.11}$$

$$\left(\sum_{k=0}^{N} \lambda_k \tau^k\right)\left(\sum_{k=0}^{N} \varphi_k \tau^k\right) = \sum_{k=0}^{N} \varepsilon_k \tau^k, \quad \text{where}$$

$$\varepsilon_k = \sum_{p=0}^{k} \varphi_p \lambda_{(k-p)}. \tag{B.12}$$

The λ_k's are unknowns and the sets φ_k and ε_k are known. It is apparent that $\lambda_0 = \varepsilon_0/\varphi_0$. Now the recurrence relationship for λ_k is

$$\lambda_k = \frac{\varepsilon_k - \sum_{p=1}^{k} \varphi_p \lambda_{(k-p)}}{\varphi_0}, \tag{B.13}$$

where $\varphi_0 \neq 0$ and $k = 1, 2, \ldots, N$. It should be noted that $\varphi_0 \neq 0$ in order for $F(0)$ to exist. Also all the expansions can be made around τ_0. Here $\tau_0 = 0$.

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